12. Reynold's stress: governing equations - I

So, welcome you all again. In the last class, we looked into the so called Reynolds stress equations. So, we were looking into the Reynolds stress equations. and the way we started to derive this Reynolds stress equation says starting from the governing equation for a fluctuating fluid motion. So, there is an equation a transport equation was available to us for fluctuation that is u' v' w' using that we multiplied it with another fluctuation and ensemble averaged and then we proceeded some tricks were done and then we established this equation here.

And in this equation of course, we did not proceed to its final form. So, on the left hand side we already saw using the product rule, you can move the fluctuation terms inside the derivative and therefore, it gets into this standard unsteady rate terms. This is your unsteady rate of change of your Reynolds stresses and then your advection rate here. advection rate of change of your Reynolds stresses just like the way you write your any general transport equation.

So, our idea is to come to that form a general transport equation for Reynolds stresses ok. So, we will carefully take term by term to see what will happen to this right. So, the first term is the where you have a pressure and velocity correlation. So, in this now I would like to move this velocity fluctuation inside the derivative term. This is the first term on the right hand side.

So, I will rewrite this as $-\frac{1}{\rho}$ and then I have its ∂ p'u_k'I have moved the u_k' inside the operator. ∂x_i that means there will be minus of another term will come which is -p $\frac{\partial u_k'}{\partial x_i}$, again the product rule. So, I have this term and then there is another term here. So, again I am going to move this inside the derivative ∂x_k - p'$\frac{\partial u_i'}{\partial x_k}$,. So, I have four terms.

This is the first this is going to be the first bracketed term here. Alright So, we will see what happens to that later and what it means. Now, we will go to the viscous term. Again viscous term I am going to do the same trick, I am going to bring this u_k'and u_i' fluctuations inside the differential operator. So, I would get $\frac{\partial}{\partial x_j}$ is a second order derivative.

So, $\frac{\partial}{\partial x_j}$ of I get u_k' $\frac{\partial u_i'}{\partial x_j}$. u_k' has gone inside $\frac{\partial}{\partial x_j}$ of u_k'$\frac{\partial u_i'}{\partial x_j}$ Alright minus of I would get $\frac{\partial u_i'}{\partial x_j}$ and $\frac{\partial u_k'}{\partial x_j}$. Is this clear what we are doing here? I am just using product rule here continuously, using product rule all these terms. I used product rule again I am using here. the product rule.

And then I have of course, there is a, it is a correlation. So, I have another term which can be moved in and out of it. So, I will get $\frac{\partial u_i'}{\partial x_j}$, this goes inside the derivative operator. and then I have $\frac{\partial u_k'}{\partial x_j}$. Minus of I need another term here.

So, minus of let me just make some space for this term. So, I have minus $\frac{\partial u_k'}{\partial x_j}$ and then $\frac{\partial u_i'}{\partial x_j}$ over bar of course there is a bracket. for the whole thing. So, it splits into two will become four different terms inside the viscous term as well as the pressure velocity correlation term all right. So, now I have I go to this particular part the one in the underlined green color.

So, this term need not be done anything no trick is required here it is perfectly fine the way it is. So, this is as I said this leads to what we call the production rate of change of your Reynolds stresses ok. So, that is the production rate term perfectly fine as it is. So we have the last term. This last term I will do some trick.

So let me just copy this and then paste. So I have the production rate term. Now I have the last particular term which is this the triple velocity correlation terms. So, here what I do is I have minus of and then let me take a bracket here. So, I already have this $u_j' u_i' \frac{\partial u_k'}{\partial x_j} + u_j' u_k' \frac{\partial u_i'}{\partial x_j}$ and then I am going to use a trick here.

The objective is I would like to use a product rule here also. So, I will use this continuity equation for fluctuating velocity. So, I will simply write this as I can add 0 here, the 0 trick that we did earlier. this particular term. So, what did we do at that particular part is that basically we looked into let us say I have I will write this in another color.

So, if I take $\frac{\partial u_j'}{\partial x_j}$equal to 0 is the continuity equation for fluctuating velocity, the continuity equation. for the fluctuating fluid motion. So, now I can multiply this with two other fluctuating terms and still be 0. So, I have $u_i'$and u_k', it is still 0 and then I can average it, it is still 0, the same trick as I am doing before if you just recollect and go back to the old class remember this. So, this gives 0 and therefore, I am going to plug this particular term here.

ok So, all I do now is to put them into the 0 term here. So, let me just erase this. So, I get this 0 term which is $u_i' u_k' \frac{\partial u_j'}{\partial x_j}$ average. Right So, we will start to rearrange the terms again left hand side nothing to do. So, these two are as it is equal to the first particular term which is the pressure velocity correlation term.

I see that there are two terms which are similar $p' u_k'$ average. So, it is a perfectly a pressure velocity correlation rather than a correlation of pressure fluctuation and a strain rate strain rate fluctuations. So, these two terms I can write it together. So, I will do $-\frac{1}{\rho}$ and then. So, I would like to write $\frac{\partial p' u_k'}{\partial x_i} + \frac{\partial p' u_i'}{\partial x_k}$ here.

ok So, these two I will write it together and then I have the other two terms. which is so it is negative of negative for that it will be positive here. So, I get basically $\frac{p'}{\rho} \frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}$ average. It is a constant density flow. So, the density can go in and out of the operation.

So, it is basically p'and its correlation here. So, p' is common. So, I have taken p'of the strain rate. So, if you look here, this is actually a strain rate here, the strain rate tensor if you remember. So, now it is a fluctuating part of it.

It is $\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}$. So, there is a meaning why we are writing it like this. So, these two are two terms that has popped out of the pressure velocity correlation. In addition to that, if you look closely, I can write for example, the next viscous term here, I would like to bring this u_k' again inside the differential operator. I would like to bring this u_k' again inside the differential operator.

So, for that all I have to do is again rewrite these two terms and then you will see that this becomes plus I am going to the viscous part. So, if I write this as $\nu \frac{\partial}{\partial x_j}$ of if I take these two parts. I am taking let us say this particular term and this particular term, it is $\frac{\partial}{\partial x_j}$ of this entire thing. So, I can rewrite this as $\frac{\partial}{\partial x_j}$ of I have $u_k' \frac{\partial u_i'}{\partial x_j} + u_i' \frac{\partial u_k'}{\partial x_j}$.

averaging an addition commute. So, I can put either two over bars or a single over bar does not matter and you can see that now I can easily use the product rule here. This is essentially a thing but $\frac{\partial}{\partial x_j}$ of u_i' u_k' it will split into two parts correct. So, it becomes two parts here of that one. So, this is fine and the last term is I have minus nu of this and minus nu of this. So, it will be - 2 ν - 2 $\nu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}$.

So, last one if you see it is $\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}$ or $\frac{\partial u_k'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}$. This is a second rank tensor j is repeated index. So, it will sum over it. So, it is a second rank tensor of i and k and it is symmetric. whether we expand i or k, it does not matter.

Which this term? First term means first of what? This one? Pressure term. i and k, this is you are talking about this one, correct. Yes, so this is i here, that is a typo, i and k and i and k, this is correct.Right ok So, I have got this - 2 ν of this two different terms here. Let us say if I take these two part, this color is not so visible.

So, I have essentially taken these two blue terms and put it together as this minus 2 nu of this. ok And this is the two green terms that has come together here. And then the production rate term that is unchanged. So, I will get - u_i' u_j' $\frac{\partial \bar{u}_k}{\partial x_j}$ - u_i' u_k' $\frac{\partial \bar{u}_i}{\partial x_j}$ as it is. And the last term here you can see it is $\frac{\partial}{\partial x_j}$of the three terms.

So, essentially I can move this u_i' u_j' inside this. So, this essentially becomes minus of $\frac{\partial}{\partial x_j}$ of u_i' u_j', u_k' prime. So, this is your, this particular term here or I can just simply change the color for your, for you to easily recognize, I can make it let us say this particular term. the correlation of three velocity fluctuations. So, in the statistical analysis we saw there is a term with higher order fluctuation fluctuating term.

I mean two signals can have same mean and variance, but I said you may need an higher order term and that term has come already here in the equation. So, I need to do some more simplifications. this particular term is fine the one with the blue tick these two I need to do some more things some more simplifications. So, let us see on the right hand side now. So, this red bracketed term that is the pressure velocity correlation.

Now, if you see you have this in the orange bracket I have this $\frac{\partial}{\partial x_j}$ term. and j is the divergence term here the summing over is about j your nabla dot of whatever term in the vector form is j, j is j indicates the divergence term here sum over it and I also have this green bracketed term where $\frac{\partial}{\partial x_j}$ terms are there ok, but I do not have such a term here. So, I would like to use some trick here with the this $\frac{\partial p'uk'}{\partial x_i}$ over bar term, the one in the red bracket. So, what can I do here to make it $\frac{\partial}{\partial x_j}$? I would like to rewrite this so that, so I would like to do essentially $\frac{\partial}{\partial x_j}$ instead of $\frac{\partial}{\partial x_i}$. What can I do to get that? and still not change mathematically everything should be same, no modeling, no approximation.

I would like to rewrite this equation as $\frac{\partial}{\partial x_j}$ of the $p'u_k'$. So, what should I do here? yes so there is two special tensors that i talked about one is the Kronecker delta or identity tensor delta ij which takes the value of 1 when i is equal to j and the other one is the levi-civitas epsilon that we do not care but right now we can use the Kronecker delta here so that means if i put delta ij it essentially means this when i and j is equal only this term should exist So $\frac{\partial}{\partial x_i}$ and its nature has not changed I would like to do this and then same thing here I would like to write this as $\frac{\partial}{\partial x_j}$ here i and k are free index see here on the left hand side the Reynolds stresses are u_i' u_k'. So, whether you can write u_k' u_i' or if you replace continuously throughout by m and n or any other index does not matter it is a free index. Therefore, I have this p' $\overline{u_i}$ here and I am writing the divergence term $\frac{\partial}{\partial x_j}$ again same thing I need it was k earlier Right So, I need it to be delta kj or jk.

When j equal to k, it is the same. Is this clear? I just use this, I took the help of this identity tensor to get this particular form. you can retain it as it is there, but by writing it this way we can give a special meaning to this. So, one of the red bracketed terms is fine. I would like to bring this other term this green term also into here. So, I can rewrite this term as because that also has a divergence term $\frac{\partial}{\partial x_j}$.

So, it is nabla dot of this entire term here right. So I would like to bring that and there again I can use the product rule, it is $u_k'\frac{\partial u_i'}{\partial x_j}$ + $u_i'u_k'$by ∂xj. Essentially it is nothing but $\frac{\partial}{\partial x_j}$of $u_i'u_k'$. So this is nothing but $\partial u_i'\ u_k'$ that is your Reynolds stresses by ∂xj.

This is the, this green term. I would like to just put a bracket here. Let us see, let me use the same red color because all the $\frac{\partial}{\partial x_j}$ terms I would like to write it together. Now I have the other one, this is this orange one, I would like to write that as well minus of $\frac{\partial}{\partial x_j}$ $u_i'u_j'u_k'$. So, this is your the orange term.

So, I have simply rewrote the three terms. with the divergence term dou by dou xj terms. So, now I have the other forms which is one of this term is the pressure strain rate term which is $\frac{p'}{\rho}$ $\frac{\partial u_i'}{\partial x_k}$ plus $\frac{\partial u_k'}{\partial x_i}$ that is as it is. Now I have the production rate term which is minus of I have this $u_i'\ u_j'\ \frac{\partial \bar{u}_k}{\partial x_j}$ – $u_j'u_k'$. I am simply rewriting what is already there, no more changes required.and the last this blue term - 2 $\nu\ \frac{\partial u_i'}{\partial x_j}\ \frac{\partial u_k'}{\partial x_j}$ over bar. Alright So, each of this gets a meaning here.