11. RANS equations II (II)

Well, at least we have an equation 8, an equation for fluctuating motion. And there is as I said we can use this equation 8 to get Reynolds stress equation. and therefore, try to close the RANS equation that we can do it. But before that, this particular equation 8 forms the basis for what is called a linear stability theory. They do some reduction of terms here.

So, we are not going in that direction, but you can still take a note of that. So, this equation 8 or the last bracketed term here, this is note number 1, the last bracketed term vanishes if fluctuations are infinitesimal. So, because that only this particular term here this term has product of fluctuations. if one is small then small multiplied by small is very small.

So, it is negligible and therefore, if you are taking your fluctuations as infinitesimal then this last term is negligible and if you get rid of it that 12 unknowns are gone. So, this is the foundation for what they call a linear stability theory or stability analysis by omitting the bracketed term assuming that fluctuations are infinitesimal. If you do that you get rid of this particular term, then the equations are closed at least you have now got one closed equation ok. So, that is the basis or foundation for this. Sir.

Yes. Why should it be that the derivative of the perturbation should also be small? Come again. You said $u_j \frac{\partial u_j}{\partial x_j}$ is a small perturbation. But if a perturbation is small, is it the derivative of the perturbation should also be small? May not be, but you are pushing, you can push this u_j' inside the derivative here. You can use the chain rule for products, then it is $u_i'u_j'$.

So, it is essentially multiplication of two small terms. So, and therefore, that forms the foundation for this what is called this is forms the forms the basis or foundation of linear stability theory. So, the last bracketed terms vanishes only when fluctuations are infinitesimal. This is very important for students of turbulence here. That is because first thing you are going into the linear world and we have already seen turbulence is a non-linear process.

So, you will not gain much by going in this direction simply because you are getting rid of the Reynolds stresses. So, by getting rid of the nonlinear terms, you are essentially achieving a closed form of equation, which is fine. But this assumes that fluctuations themselves are infinitesimally small. So, the way you continue in this so-called linear stability theory is basically, of course, we have by omitting this.okay So, the equation 8 becomes closed becomes closed omitting the bracketed term the 12 unknowns are gone.

But then you the way to proceed here is that in the so called linear stability theory or other stability analysis is that you are always assuming a base flow to be a laminar flow. You consider a laminar flow that means you have some analytical expression for it and then you assume that fluctuations are infinitesimally small and then you go ahead and solve this to get some solution. But the only problem is that when fluctuations grow this theory fails. when fluctuations grow that is when the flow will go from a laminar to a transitional and turbulent state. So, transition to turbulence when it occurs when fluctuations grow it is no longer infinitesimally small and when fluctuations are not infinitesimally small your base flow is not laminar and therefore, the theory fails.

And therefore, I said this direction is not very helpful if you are if you are you know interested in turbulence ok. So, for example, if I want to graphically illustrate let us say you have you are measuring the in a turbulent boundary layer not in a turbulent boundary. Let us say in a boundary layer if you are measuring some velocity signal and let us say it is going like this and suddenly you get some burst of information. Let us say this is the signal. Let us say this is inside your boundary layer, in a boundary layer, in a boundary layer problem.

So, then this onset here, this particular junction where a turbulent spot is seen. These are called turbulent spots. and this is intermittent so some turbulence comes and then laminar zone a turbulence and then a laminar zone so this signal here up to here this is intermittent also this is also called intermittency and this is a completely a transitional zone so this particular part is called the flow is experiencing transition and this is the turbulent part when it has gone into the turbulent condition and this particular junction where the turbulence kicks in is where you have your critical Reynolds number starting ok. So, this is your call it subcritical Reynolds number. So, this is a subcritical Reynolds number that means it indicates the Reynolds number at which the flow will go into a transitional regime.

You can get this number using stability theory, linear stability theory in natural transition ok. The natural transition means the base flow is laminar, it is free from any free stream turbulence, no background turbulence is allowed. If background turbulence is allowed then the flow experiences what we call a bypass transition. So, this particular subcritical Reynolds number you can achieve it here. So, this linear stability theory or analysis if you solve that equation 8 omitting the nonlinear term, linear stability analysis can give you this subcritical Reynolds number.

So, Reynolds number at which the flow will go into a transition regime, but it cannot tell you how the transition occurs later, the process later it cannot tell because the base flow is no longer laminar ok. So, once the transition starts, once transition to turbulence starts fluctuations are no longer infinitesimal ok and the base flow is no longer laminar So, stability theory is not useful here. Stability analysis is not useful. So, the takeaway is you can close the equation 8 fluctuating momentum equation by omitting the nonlinear terms. and you can obtain a subcritical Reynolds number for given problems in natural transition.

That means the base flow must be free from any fluctuations at all. It should be completely in laminar state, ok. Ok, you are saying that if I use this particular, if you have access to these three fluctuations and then you are able to compute. If you are, if you have access to it, yes, you can compute. So you can actually say that these are not extra 6 unknowns, you still have

6 and they go away by closing, correct.

Yes. . That will be there, that will be there, yes. So this will be, it is a good question. So basically if you have access to u_i' that means 3 velocity fluctuations, you can compute the velocity gradient here and therefore this does not constitute 6 extra unknowns here. only 6 unknowns are there correct. So, then it will be but still the equations are not closed there is a turbulence closure problem those things remain ok.

So, now one more takeaway is that here that only valid this stability analysis stability analysis is only valid for natural transition that is no no free stream turbulence or background turbulence in the base flow because you are essentially going to start when you take the equation 8 omitting the bracketed term you are going to assume this ui bar ok. taking let us say if it is a boundary layer flow you know the velocity ui bar or if it is a wake or a jet you have to assume ui bar flow and then consider a small fluctuation u_1' u_2' u_3' and solve for the equation to see when the system will give you a subcritical Reynolds number ok that is the idea there so that means you are always considering a base flow ui bar and that ui bar of course, bar does not have any meaning there because it is a laminar flow, but that is the constitutes the base flow. And if the base flow has some turbulence then this natural transition is no longer natural we call it a bypass transition. So, this subcritical or this what we call linear stability analysis has a limited use I would say. So, now we go back to our original aim that is derive an equation for Reynolds stresses right.

So, for that we start with again this equation 8 without dropping the term without dropping that bracketed term that we have to retain. So, we will go ahead and so now we have considered equation 8 or the equation for fluctuating motion and we proceed to derive an equation for Reynolds stresses ok. So, we are not dropping this term here. we need this term bracket is there minus of minus. So, now what we do is the first thing is we multiply this with another fluctuation the way to derive what is called Reynolds Reynolds stress equation.

So, we start with equation 8 and then we multiply throughout by multiply by another fluctuating velocity called u_k', i and j are already used. So, I take u_k' and ensemble average and ensemble average. So, if I do that I get u_k' $\frac{\partial u_i'}{\partial t}$ average plus $\overline{u_j}$is the statistical value that comes out. So, I get u_k' $\frac{\partial u_i'}{\partial \mathrm{x}}$ ensemble average. equal to minus 1 by $\rho u_k' \frac{\partial p'}{\partial x_i}$.

So, now you see you are also seeing correlation of pressure and the velocities. You know changing pressure field somewhere, fluctuating pressure is affecting the velocity fluctuations around. So, we are seeing a correlation of these two plus $\nu u_k' \frac{\partial^2 \overline{u}_i'}{\partial x_j^2}$ minus $u_j' u_k'$. The mean strain that is $\frac{\partial \overline{u}_i}{\partial x_j}$ is not affected by ensemble averaging.

So, it stays outside ok. And then this particular term which is minus $\overline{u_j' u_k' \frac{\partial u_i'}{\partial x_j}}$ and then minus of minus plus u_j' sorry $u_k'\ u_j' \frac{\partial u_i'}{\partial x_j}$. So, we have simply multiplied by u_k' throughout and ensemble averaged ok. Any term that goes away here, the first term is a correlation, also the second term correlation term. The first term on the right hand side pressure and velocity correlation, again velocity correlation, velocity correlation. What about the last term? this is average of just the fluctuating quantity here because the last one under the bra under the over bar is already a Reynolds stress.

So, this term is going away, but no need to panic this was our Reynolds stresses, but upon multiplying by fluctuation and averaging it is going away no need to panic because it has come back in other forms you see here it has already come in a different in a different term ok. So, now we have one such equation here, let us call this I will call this let us say star 1. Now there is a trick that we do here, one is that this i and k are now free indices, i takes value 1, 2 or 3, k takes value 1, 2 or 3. So, I can rewrite this entire equation by interchanging i and k.

So, i comma k are free indices. So, interchanging and rewriting the equation, we get one more equation here, simply rewriting by interchanging the i and k. So, I get $u_i' \frac{\partial u_k'}{\partial t}$. It is not any other equation, it is the same equation whether i k value changes when you expand it, it is going to be the same ok. That is the whole point of a free index plus j is used for divergence, j is repeated ok.

So, that remains the same. Here I get $u_i' \ \frac{\partial u_k'}{\partial x_j}$. average equal to$-\frac{1}{\rho}$, this is $u_i' \frac{\partial p'}{\partial x_k}$, k is replaced by i, i is replaced by k. Let me know if I am making any mistake, i and k xj, $u_i' \frac{\partial p'}{\partial x_k}$, correct. plus $\nu u_i' \frac{\partial^2 u_k'}{\partial x_j^2}$ - $u_j' U_i' \frac{\partial \bar{u}_k}{\partial x_j} - u_j' u_i'$. $\frac{\partial \bar{u}_k'}{\partial x_j}$. Let me call it an equation star 2. Star 1 and star 2 are same if you expand, if you give a value i equal to 1 and k equal to 1, you expand it you will get one equation, you get 6 equations here. So, now all I have to do is add these two equations to get my Reynolds stress equation ok. We will see what happens. If I add star 1 plus star 2, adding the equation will yield.

So, here the equations are getting little intense right. So, make sure that we are following the right indices and all the rules properly. The equations has to be correct, then only you can model them. So, now I have the two terms writing$u_i' \frac{\partial u_k'}{\partial t} + U_k' \frac{\partial u_i'}{\partial t}$ right. That is the first term adding the first two terms and then I have another term which is the convection term, so I am going to write it here itself plus I have $\bar{u}_j u_i' \frac{\partial u_k'}{\partial x_j}$ plus I have taken $\bar{u}_j$ common out of the equation.

So, I get $u'_k \frac{\partial \bar{u}_i'}{\partial x_j}$. The left hand side two parts equal to the right hand side. I get minus $-\frac{1}{\rho}$

I have $u'_k \frac{\partial p'}{\partial x_i} + u'_i \frac{\partial p'}{\partial x_k}$, one of the terms which is the pressure velocity correlation term, adding them together. Now, I add the viscous term which is plus ν, I have k' to square u_i' $\frac{\partial^2 u'_k}{\partial x_j^2}$. The viscous term separates out and then I have addition of these two terms which is - $u'_j u'_i$.

$\frac{\partial \bar{u}_k}{\partial x_j}$ - $u'_j u'_k$ $\frac{\partial \bar{u}_i}{\partial x_j}$ those two terms. And then finally, I have this - $u'_j u'_i$ $\frac{\partial u_k'}{\partial x_j}$ - $u'_j u'_k$ ∂ u_i sorry $\frac{\partial u_i'}{\partial x_j}$ okay. So by adding these two equations, I have got this. So some extra terms have come and each of this term has a physical meaning.

I will come to that one. So this part and then this part. So before we start to rearrange, I have to do some tricks here to rearrange. First thing to note is that a pressure velocity correlation term has come. And you have a term here where mean strain $\frac{\partial \bar{v}_i}{\partial x_j}$, mean strain is interacting with Reynolds stresses ok. And the last term is correlation of three velocity fluctuations ok.

So, the third order term that we discussed in chapter 1 statistical analysis after variance I said right two signals can have same mean and variance, but they have higher order terms the third order fourth order like skewness, flatness. So, you see this term has come third order fluctuation. So now if I rearrange the left hand side this will take a little bit more time, but let us do the only the left hand side part today now. So, I can use the if you look closely this $u'_i u'_k$and then $u'_k u'_i$ in the first term. I can use chain rule for products here and push the u_i' inside the differential operator ok.

So, using product rule I get the left hand side as $\frac{\partial}{\partial t}(u'_i u'_k)$. Some of you were bit upset that we lost the Reynolds stress term, but you see it has come back ok and we got it back in the form that we wanted an unsteady rate of change of Reynolds stresses. That is why I said we are getting a transport equation for Reynolds stresses. and then no need to panic we are getting an advective term also. The $\bar{u}_j$ again product rule push the u_i' inside the differential operator.

So, I get $\frac{\partial}{\partial xj}$ divergence of your $u'_i u'_k$average ok. So, all the CFD enthusiasts should be very happy now. Now, we are getting some general transport equation. the left hand side, left hand side is the easiest part, the right hand side will come up with more complicated terms and each of them has a physical significance ok.