10. RANS equations II (I)

Okay fine, so I welcome you all again and in the last class we looked at derivation of RANS equation. so called Reynolds average Navier-Stokes equation was derived by taking Navier-Stokes equation and then Reynolds decomposition and ensemble averaging. With that we achieved RANS equation. The only thing is in RANS equation we got extra unknowns. So, the RANS equation looked like you had $\frac{\partial \overline{u_i}}{\partial t} + \bar{u}_j$ I am just rewriting what has been already derived right equal to the mean pressure gradient $\frac{\partial \bar{P}}{\partial x_i}$ plus I have $\frac{\partial}{\partial x_j}$of or I can say $\nu\ \partial x_j$. $\overline{u_i}$, the viscous stress tensor and the Reynolds stresses $u_i' u_j'$. I am just rewriting the equation. What was the equation number for this if you remember? 7. So, this was equation number 7.

So, this equation is perfectly fine for those who want mean velocity and pressure in engineering applications. The only problem is this particular term. You have extra unknowns here, 6 extra unknowns due to the Reynolds stresses appearing. So, now there are two ways of going forward. One option is of course to model them.

Option 1, option 1 is to model Reynolds stresses. that is your $\overline{u_i' u_j'}$ model Reynolds stresses to close equation 7 and therefore, solve them once the equations are closed. This we will see this particular option and its methodology is what is called RANS models RANS model that we will see later. The option 2 is of course, to derive a transport equation for the Reynolds stresses ok. To get a transport equation for Reynolds stresses $u_i' u_j'$ that means 6 extra equations.

So, that is one theoretical way of looking into it. And this is the direction Osborne Reynolds took deriving an equation for the Reynolds stresses to see. whether that is closed if that equation is closed then you do not need to model anything ok. For that we have two different options there are two ways of deriving this Reynolds stress equation and one leads more unknowns than the other one. Both are of course derived from first principles we have not made any assumption so far no modeling has been done we have used only Navier Stokes equations.

And if you start from one of the equation, it leads to more unknowns than the other. So the starting point for derivation matters. So we will do the one which gives least number of unknowns because we need to model. So the modeling community chooses derivation of this Reynolds stress equation in a different way. And the starting point is an equation for fluctuating fluid motion.

We have done the equation for mean momentum. Now, we look into fluctuating momentum that is a starting point. So, we had two option. So, we will go ahead and take option 2 first to see what happens. So, that is the equation for equation for fluctuating fluid motion.

or your fluctuating momentum. So, the mean momentum equation is already there which is your RANS equation. So, now all I have to do is take one equation which is Reynolds decomposed Navier-Stokes equation. Take Reynolds decomposed not averaged So, you are essentially taking Navier-Stokes equation and decompose it into mean and fluctuation ok. So, Reynolds decomposed Navier-Stokes and then minus of the RANS equation.

So, basically you take out the RANS equation terms, subtract them from the Navier-Stokes. Navier-Stokes is the instantaneous and the mean to get the fluctuation. So, this is your, so what I do is this is the RANS equation minus RANS equation. So, this is your mean part and this is your instantaneous. This instantaneous minus mean should give you the fluctuations and that is what we want, an equation for fluctuation u_i'.

So, if you recollect the Reynolds decomposed Navier-Stokes equation, what was the equation number there? The momentum equation. Let me write the equation then you will tell me what it looks like. So, I have the decomposed equation of the form. $\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \overline{u_i}}{\partial t}$ + I have the four convective terms here which is $\bar{u}_j \frac{\partial \overline{u_i}}{\partial x_j} + \overline{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \overline{u_i}}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j}$. Only Reynolds decomposed, no averaging here because we want the instantaneous Navier-Stokes equation, right.

And this is equal to we have splitted the pressure term also into mean and fluctuation part. So, $\frac{\partial \bar{P}}{\partial x_i}$. $-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} - \frac{1}{\rho}\frac{\partial P'}{\partial x_i}$ and the two viscous terms which is $+\nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j}$ what was the equation number for this we have there was no number for it ok fine no problem. So, now we have to simply take equation star minus equation 7 that would give us our equation for fluctuating fluid motion ok.

So, if I just copy this part. So, I have, if I have the equation minus of this, so I have just copy pasted equation 7 here. So, let us just write this as these equations. So, I would get So, if I do this, I am going to write it here, what this implies is this equation star minus equation 7. So, we already see certain terms cancel out straight forward, this minus of this and then we have mean pressure term going away.

and the mean viscous stress terms goes away. So, I have, now we will write out the rest of the terms that is available here. So, we have about, so before that I would like to have this equation let me see, yes. So, I would like to have this last term written in a different way. So, this we use the trick to get this term, push the fluctuation inside the differential operator.

Before that it had its own original term. So, I will retain that particular term here which was your $u_j' \frac{\overline{\partial u_i'}}{\partial x_j}$. upon ensemble averaging you got this particular term. Then you added that trick using continuity plus 0 to push this inside using chain rule. So, I am retaining the original form.

So, this minus this. So, what are the terms that is coming out is basically I have

$\frac{\partial u_i'}{\partial t}$ is your equation 7. So, The equation for fluctuating fluid motion, I have one unsteady term. There is a convective term plus + $\bar{u}_j \frac{\partial u_i'}{\partial x_j}$ and then I have on the left hand side, one of this is gone. So, two more extra terms are there, this and this. So, I would take it out onto the right hand side.

So, I will write the pressure term first, fluctuating pressure term, $\frac{\partial p'}{\partial x_i}$ and the viscous term plus $\nu \frac{\partial^2 u_i'}{\partial x_j^2}$. Sometimes I write ∂x_j^2 . This is not the actually it is just my own convention. Many times people write it, but it actually means $\partial x_j\, \partial x_j$. The rest of the terms is this is plus minus of this.

So, I get so this becomes still So, this becomes minus here, coming on to the other side $-u_j' \frac{\partial \overline{u_i}}{\partial x_j}$ - $\overline{u_j' \frac{\partial u_i'}{\partial x_j}}$ $u_j' \frac{\partial u_i'}{\partial x_j}$ and then this minus of this, this will be positive So this takes the positive form here plus $\overline{u_j' \frac{\partial u_i'}{\partial x_j}}$ ok. Or I can simply rewrite this as minus of, I can say minus of this So I got an extra equation for fluctuating fluid motion. That means if you are interested to study turbulence itself, not the mean velocities but fluctuating velocities u1 ‘, u2 ‘, u3 ‘, you have an equation from first principles here. And at least the first four terms looks fine. They look lot similar to your Navier-Stokes equation except that it is ‘ instead of an instantaneous velocity.

An unsteady rate, convective rate, pressure gradient as well as the viscous term. But it has three extra unknown terms that has come on to the right hand side. One of them is of course the Reynolds stresses. the last one. So, one when you derive this equation one easy way of finding out whether the derivation is correct is if you average this all terms should vanish because its average of the fluctuation is 0.

So, you have to check this thing. So, note upon averaging is this equation 8, the last equation was 7 right. So, let call this equation 8. So, upon averaging all terms in equation 8 must vanish that is if the derivation is correct. So, that you can easily see the first one average of the fluctuation is gone.

So, first is ok the second one u j bar is statistical. So, average of the second term is 0 pressure gradient also average of p’ is 0 ensemble averaging of u_i’ is 0 this particular term averaging of u_j’ is also 0 and the last term if I average this is it 0 that is not 0 right if I average the bracketed term the bracketed term the first term in that part if I average it that is not 0 because it is correlation term. But the second one inside the bracket is the Reynolds stresses. So, it is Reynolds stresses minus Reynolds stresses it goes to 0. So, all terms must vanish if you have derived the equation right.

That is an easy way of checking. So, we have an equation 8. So, what? I mean does this solve the turbulence problem? So, we will see here again that how many equations and how many unknowns are there? How many equations and how many unknowns? So, I have got 3 extra unknowns, 3 equations sorry which is for my u_1' u_2', I have the fluctuating continuity equation also, using that I can get an extra equation not a problem, one extra equation to get a fluctuating pressure. So, here I have same thing, it is u_1' u_2' u_3',that is my u_i', three unknowns. I have p' that is one unknown. but p' you can also get an equation for it using continuity equation.

We have continuity will give 3 plus 1. So, p' can be taken care of. In addition to that I have how many unknowns are there now? I have the Reynolds stresses anyway, the $u_j' \frac{\partial u_i'}{\partial x_j}$. This is anyway giving me 6 unknowns. In addition to that, I have the other term, the first term of the bracketed term that is giving me another 6, the $u_j' \frac{\partial u_i'}{\partial x_j}$.

This is not a statistical term. It is not a correlation now. This is another 6 unknowns. or 9. 9 is total, but it is a symmetric tensor.

So, 6. Again you have less equations compared to number of unknowns. And the number of unknowns is growing. In the RANS equation you had only 6 unknowns, but now you got extra 12 unknowns. That is why it is called turbulence closure problem. More you average, more you try to dig deeper, more unknowns pops in.

It is an endless exercise. So, again you have equations less than unknowns, the turbulence closure problem.