Strength of Materials
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Lecture - 05
Stress Tensor

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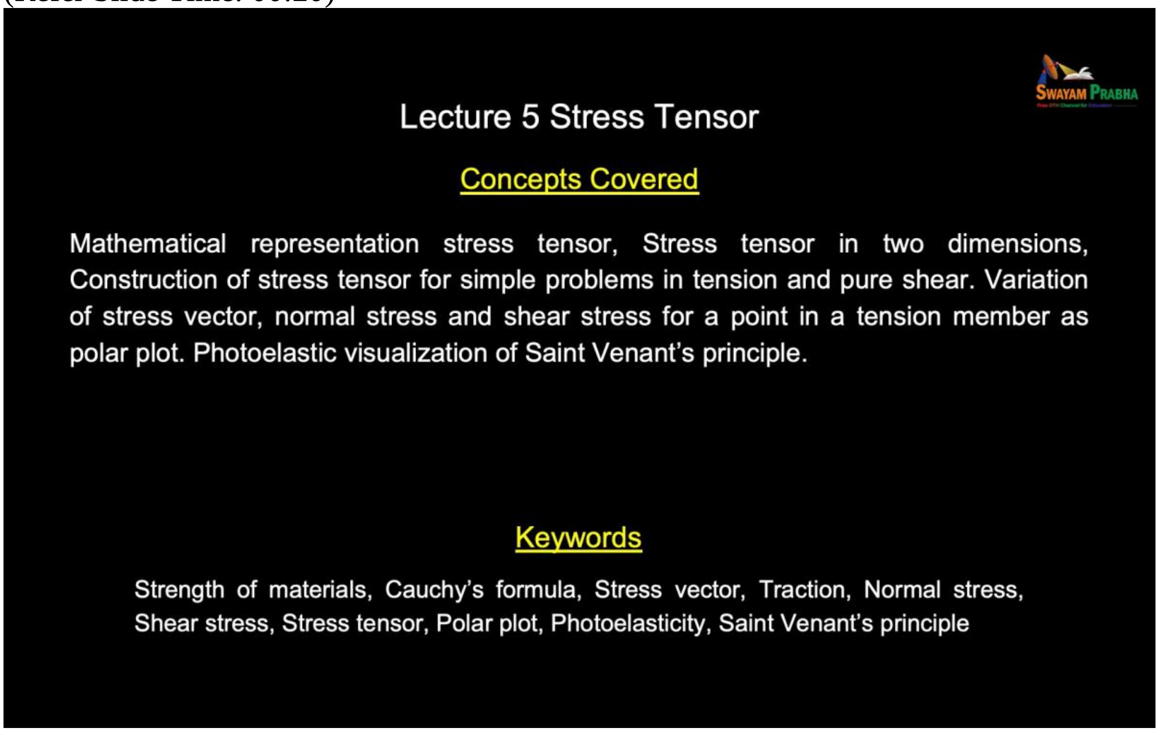

Let us continue our discussion on concepts related to stress.

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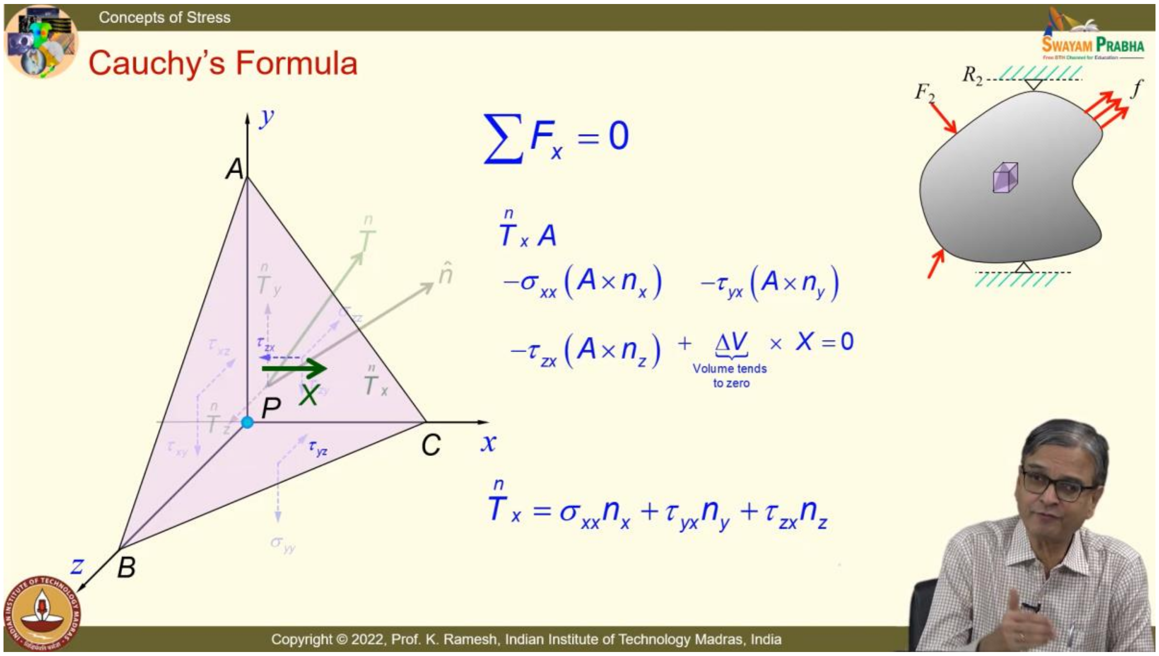

And in the last class, we said that we want to find out what happens in an arbitrary plane. So, we took out a tetrahedron from the three-dimensional model. In the limit, the tetrahedron was shrunk to zero. You already know how to write the forces acting on the sides of the tetrahedron. You know the stress component and you can find out what is the force in the *x*-direction by simply multiplying by the area. And when we come to the negative *x*-plane, you have the stress as σ_{xx} and the area is written as $A \times n_x$. We did this and got the final expression in one fashion. But in the last class, I also wrote the final expression in a different manner; not all of you have noticed it.

Now, let us read this expression, we got this as

$$\overset{n}{T}_x = \sigma_{xx} n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

Like this, we determined it for all the three directions.

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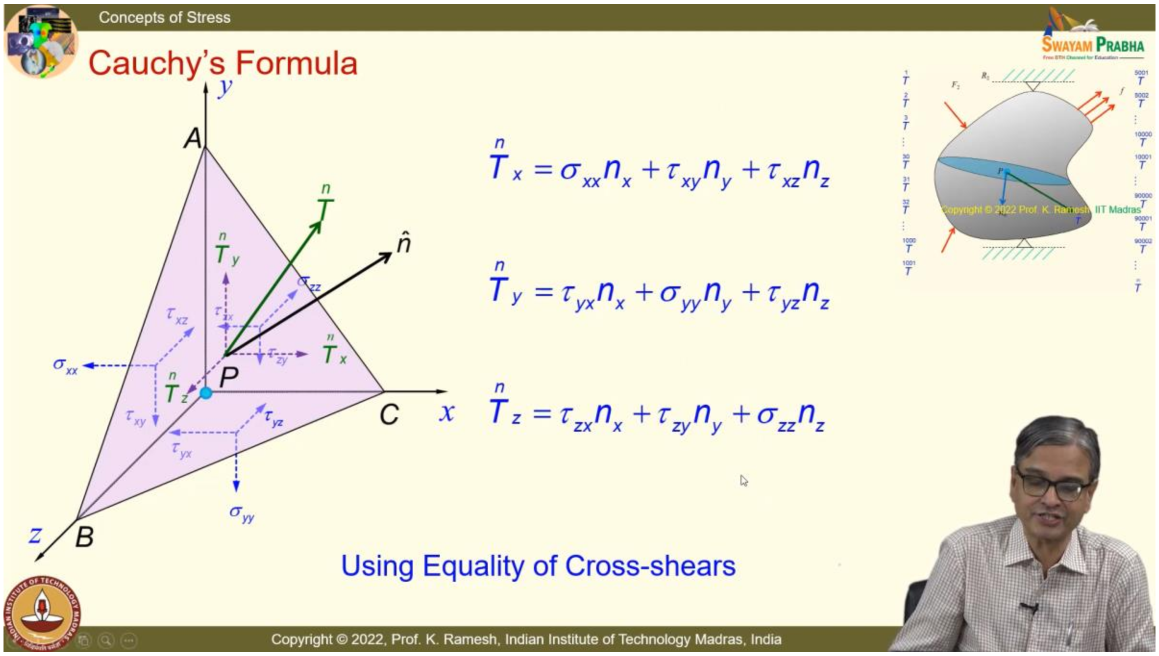

But when I summarized it, I wrote it in a different fashion. I am happy that at least one student noticed it. What I had written is $\overset{n}{T}_x$ equal to $\sigma_{xx}n_x + \tau_{xy}n_y$ instead of τ_{yx}. Similarly, I have put this as $\tau_{xz}n_z$ instead of τ_{zx}.

Writing it in this manner helps me to write it as a matrix notation. I do it on the strength that τ_{xy} equal to τ_{yx}, which we will prove. We have to wait for a class. That principle is known as *equality of cross shears*, fine? Because you know, the concepts related to stress are you know; you have to go back and forth to understand it fully. So, using the equality of cross shears, the equations are rewritten in this fashion, which is convenient for me to write it in a matrix notation.

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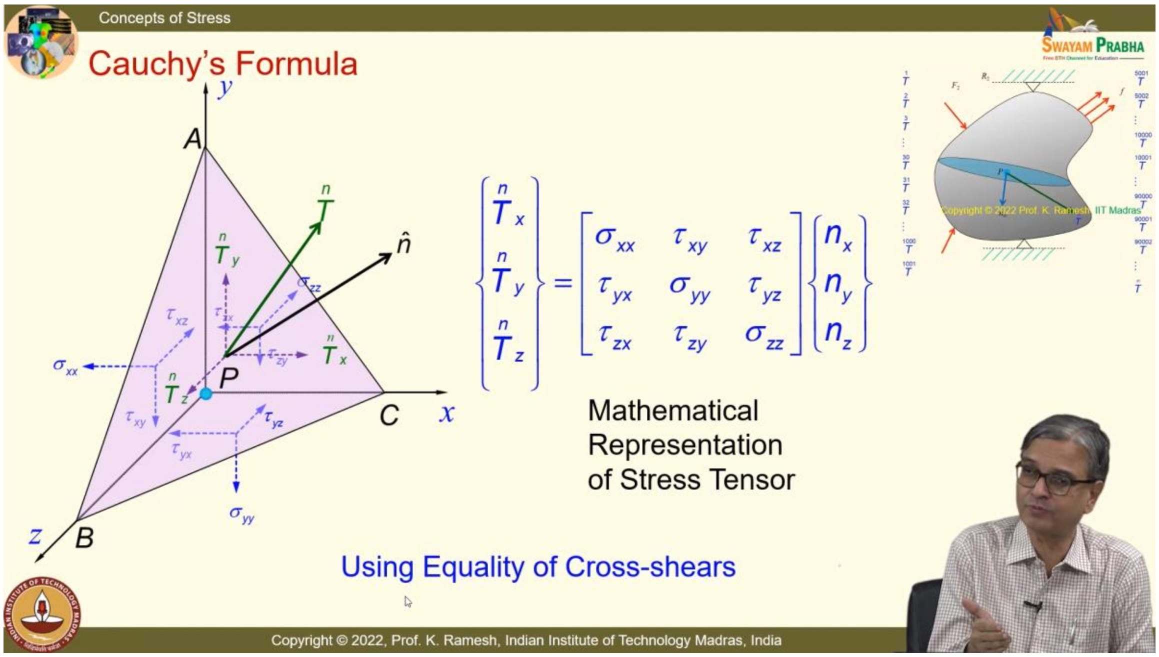

And what we said was, even though my interest is to find out stress vector on any arbitrary plane, what is fundamental at the point of interest is, a matrix of 3 x 3, which I also said, it is representing stress tensor and the rank of this tensor is 2. Here again to emphasize that I am able to write this using the principle - *equality of cross shears* which we will prove in the next class.

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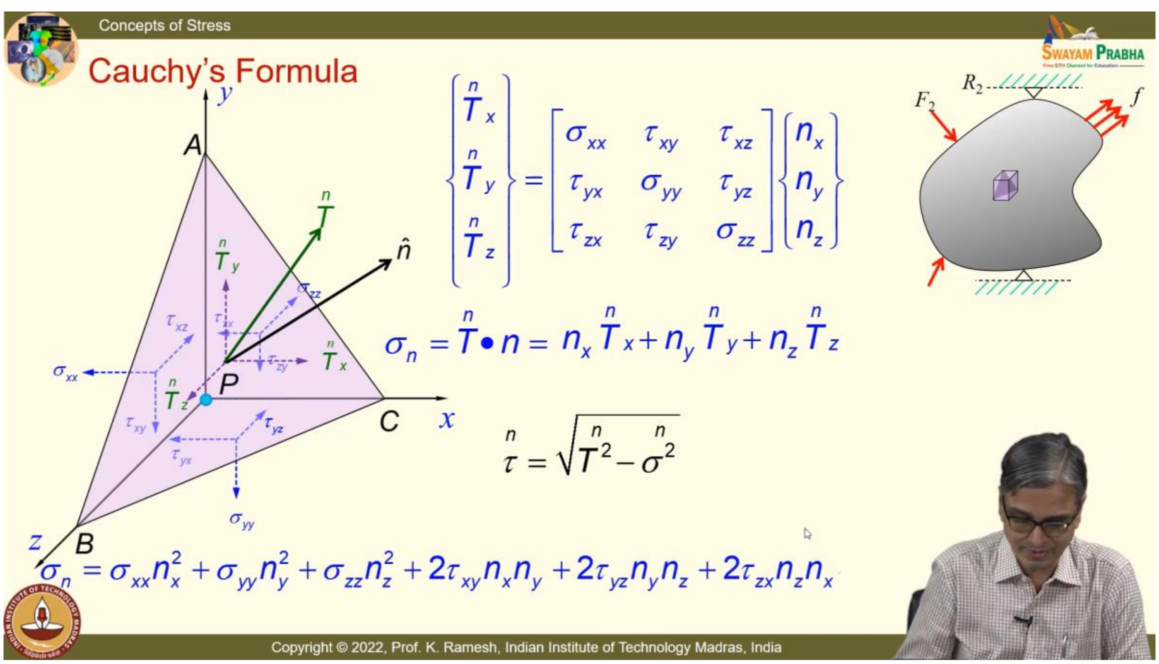

And what is important is, you know when I have this, we are always interested in looking at the stress vector as a composition of a normal stress and a shear stress. Because later on, when you go and look at theories of failure, some theories use the normal stress as a criterion to judge. Theories related to ductile materials wants us to find out what is the shear stress. So, we need to know how to find out these quantities.

I can find out the normal stress as simply a dot product of $\overset{n}{T} \bullet n$ and when I do this, I get this as

$$\sigma_n = \overset{n}{T} \bullet n = n_x \overset{n}{T}_x + n_y \overset{n}{T}_y + n_z \overset{n}{T}_z$$

which you can also write it down in long hand. I will also show it because you know what is $\overset{n}{T}_x$, $\overset{n}{T}_y$, and $\overset{n}{T}_z$. So, when I multiply by this, I get a very nice expression. The normal stress is

$$\sigma_n = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + \sigma_{zz} n_z^2 + 2\tau_{xy} n_x n_y + 2\tau_{yz} n_y n_z + 2\tau_{zx} n_z n_x$$

Now, if I want to find out the shear stress, it is easy for you to write the shear stress on the plane $\hat{n}$ is nothing but

$$\overset{n}{\tau} = \sqrt{\overset{n}{T}^2 - \overset{n}{\sigma}^2}$$

And this is the basic expression and you can coin a variety of problems. At a point of interest, you will know what is the stress tensor and for any given arbitrary plane, you would be in a position to find out the stress vector. Once you find out the stress vector, on that plane, you can find out the normal stress or the shear stress. You can also have a class of problems wherein I want to find out the orientation of the plane. That means, I want to find out what are the direction cosines n_x, n_y, and n_z in a manner that on a plane, you have only normal stress.

So, you can have two categories of problems. For a plane, find out the stress vector using the stress tensor or by dictating what should be the stress vector on the plane, determine the plane. You have both these classes of problems and you should be quite comfortable when handling these equations. You know, you can have equally inclined plane that is very important from further studies. That is also part of your tutorial problem, fine?

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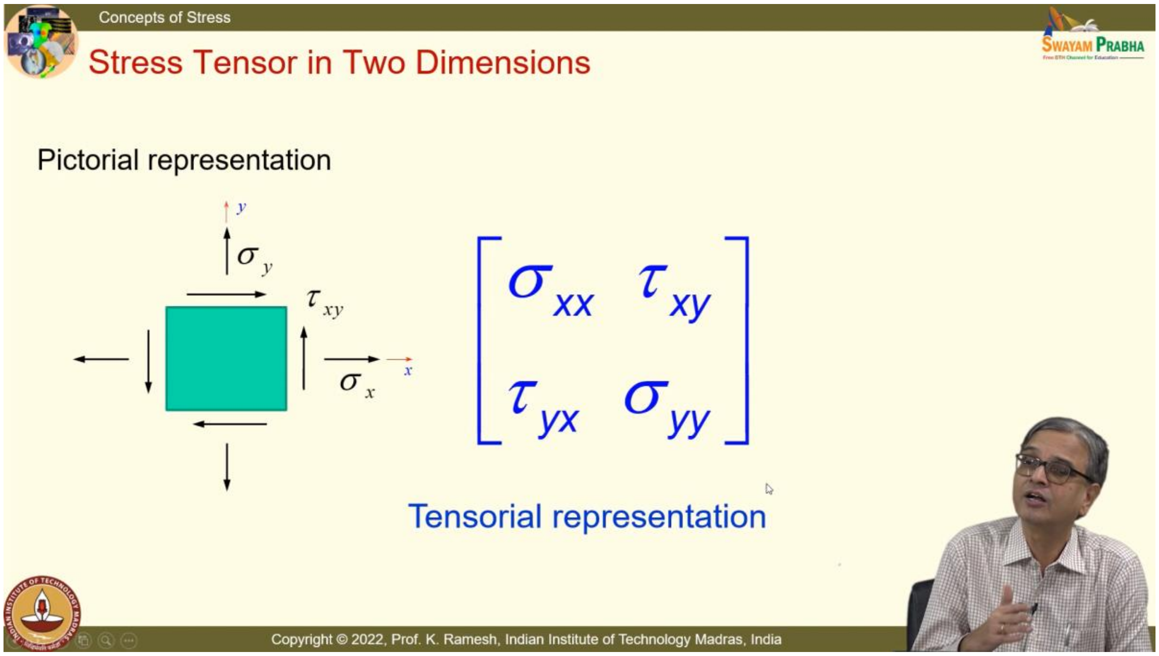

And now, we will restrict ourselves to two dimensions and the moment I go to two dimensions, I have a pictorial representation. I want to show what is the *x*-plane, I want to show what is the *y*-plane, I want to write what are the stress components acting on it. And what you should keep in mind is, this is only a representation of what happens in mutually perpendicular planes.

It is not a rectangle or square taken out from the model. It has zero dimensions. So, this is a pictorial representation of state of stress at a point and when I write it in two dimensions, this is a tensorial representation of what is the state of stress at the point. See, in those days, people were very very comfortable with graphical representations. And, can you think of what kind of a graphical representation that we can do? In fact, if you look at the history, people have tried various kinds of graphical representations. You have Cauchy stress quadric, you have Lame's stress ellipsoid, later you also have what is known as a Mohr's circle. We will get on to Mohr's circle later, because you will also appreciate the elegance in that.

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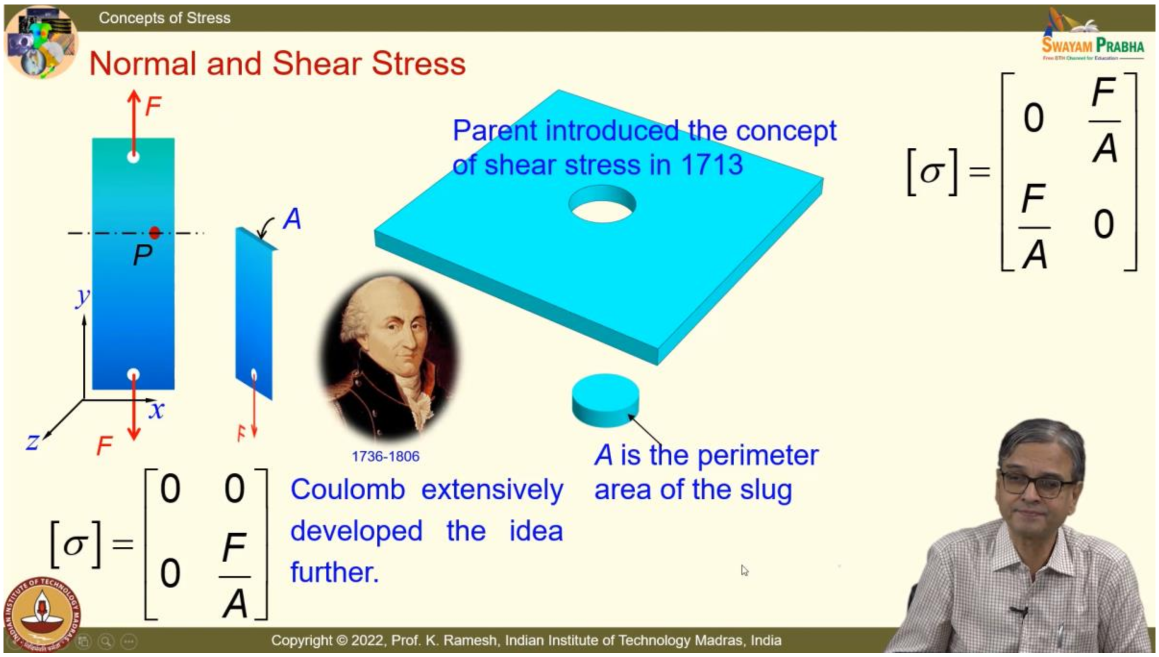

Now let us go back to our simple axial tension problem and understand what is normal stress in little more detail. Fine? So, we have taken a point *P* and then we have also looked at the cross section is taken as *A* and we have simply said the stress at this point is *P*/*A*. But *P*/*A* appears like a scalar, instead of *P*/*A*, here I have put the force as *F*, so it is *F*/*A*. My suggestion is, always try to represent the stress as a tensorial quantity.

How do I write for this? I have determined from my mathematics, from a simple appreciation of what happens to a member subjected to tension, I have determined the stress as *F*/*A*. If I go back with the reference axis, actually, this corresponds to what happens in the y-plane. There is no stress in the x-plane, so I can write this as a tensorial notation as

$$[\sigma]=\begin{bmatrix} 0 & 0 \\ 0 & \frac{F}{A} \end{bmatrix}$$

Is the idea clear? And we will also look at another interesting simple problem. You know, you have a punching operation. In many of the manufacturing processes, people do it by punching. And this is again a very interesting problem, where I have a punch which pierces and I get a disc like this. And I now define the area as the complete perimeter surface of this. So, I have a punch, I can find out what is the force that I have applied to pierce this. So, I can also find out, what is the kind of stress happening on this surface. It is actually shearing. Fine? And because the problem is simple, straight forward, we can also determine the magnitude of shear stress as; whatever is the force *F* and whatever is the area of the perimeter, I can also say this as *F*/*A*. And this *F*/*A*, where would I put it in the tensorial notation? In this case, F/A what we have calculated is only representing the shear. Can you write that? We have already looked at how to write a tensorial representation. We have written it in a form, fine? So, I have τ_{xy} and τ_{yx}. So, the tensorial representation of state of stress at any point on this is nothing but what I have as a matrix.

See, shear was difficult. People have not had that idea to start with. So, there is a scientist, Parent; he introduced it in 1713 and Coulomb developed these ideas extensively further. So, right now what we will do is, we will go back to this simple tension. We will understand it little more.

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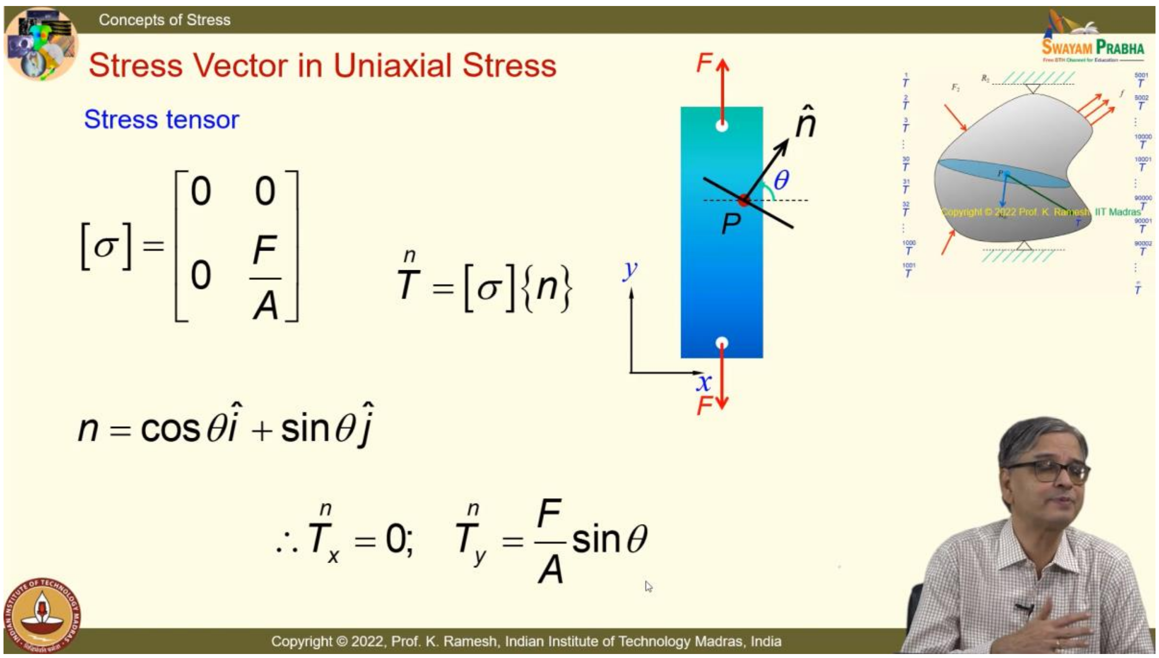

It looks very simple. I have the stress tensor now. You know that you have only just pulled. Now I said by stress tensor, we have the information of totality of stress vectors acting on all the possible infinite planes. Now I have this matrix populated with zero, only one element is non-zero. But for the same problem, if I look at from a different plane of reference, if I look at a different plane passing through the point of interest, I can have a normal stress as well as a shear stress.

Though visibly you are not applying a shear stress on the object, but when I look at a plane of interest; so that is what is needed when you want to understand why a chalk fails when I apply an axial tension, when I apply a torque, in different manner. Fine? Now we have already looked at the expressions. So, we have the direction cosine for an arbitrary plane. When I represent this as θ, I can write it as $n=\cos\theta\hat{i}+\sin\theta\hat{j}$. So, I can find out the stress vector on any plane.

We have expression for finding out the normal stress as well as shear stress. Let us look at the quantities one by one. I have the famous Cauchy's formula.

$$\overset{n}{T} = [\sigma]\{n\}$$

So, I get $\overset{n}{T}_x$ when I have this multiplied by the direction cosine goes to zero and $\overset{n}{T}_y$ equal to $(F/A)\sin\theta$. We will also subsequently see what is the expression for normal stress and what is the expression for shear stress.

And we would like to see a graphical representation. There are multiple graphical representations. We would like to see the stress vector, it's $\sin\theta$ and I suppose you know how a polar plot is done. For any value of θ, put it when it is positive in one direction, negative in another direction, and you have the expression.

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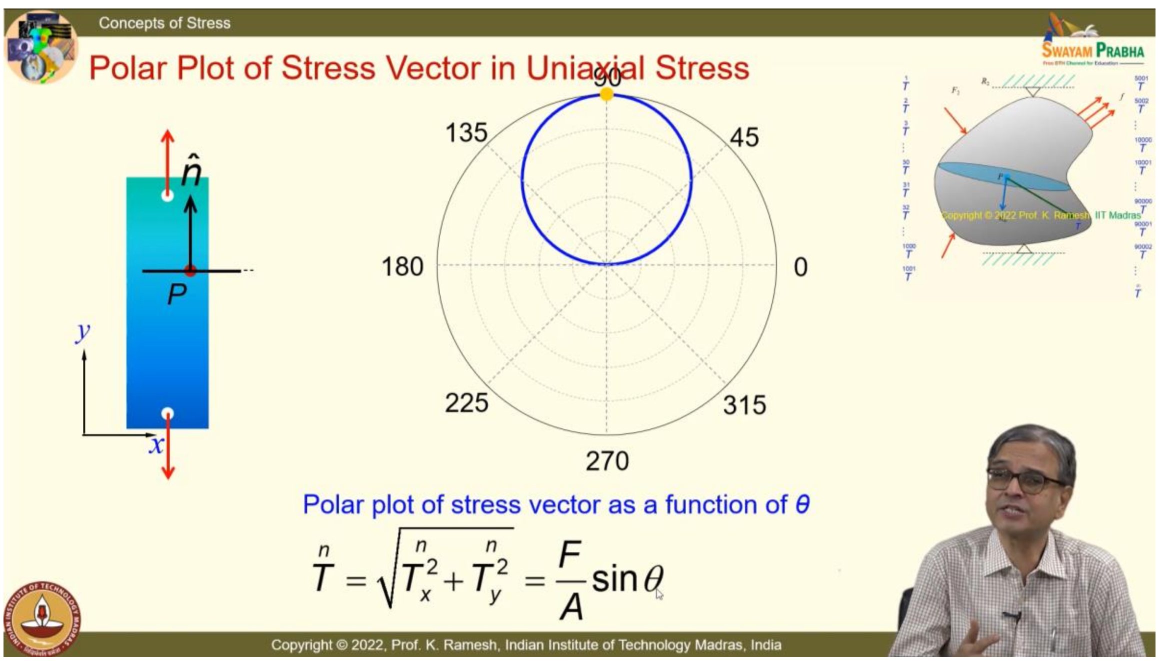

And when I plot this, I have a beautiful plot like this. Is the idea clear? This is one graphical representation. More than the stress vector, from failure point of view, we are interested in the normal and shear stresses. So, we will go and then see, what is the expression for the normal stress. You know! You have $\overset{n}{T}_x$, $\overset{n}{T}_y$ is determined. When you take the dot product, you get the normal stress. Can you calculate what is the normal stress for this? You can check with me later. Please calculate the normal stress acting on any arbitrary plane, straightforward.

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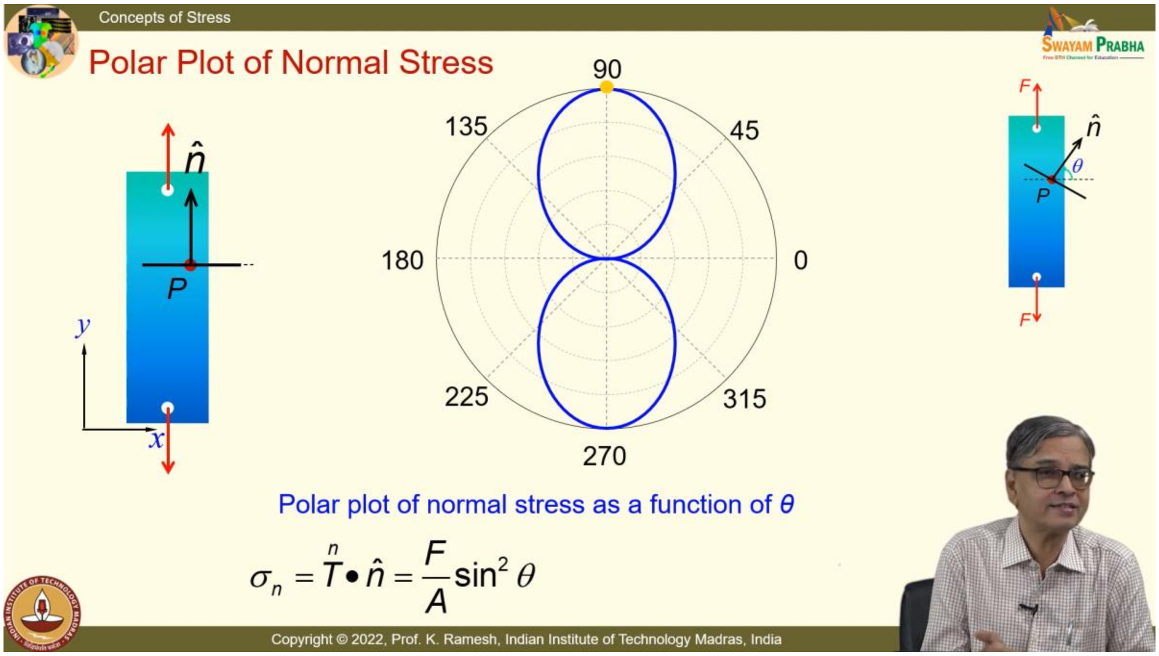

So, when I get the normal stress, the dot product gives me normal stress as $\sigma_n = \frac{F}{A}\sin^2\theta$. And suppose I want to do it as a polar plot, the polar plot will appear like this. See, the problem what we have taken is a very simple one of simple tension, but I have a beautiful plot. The pattern is photogenic, isn't it? And you also get some interesting features of it. What is the maximum value? Suppose I put this as one and this as zero, it goes from, when $\sin\theta$ is zero, you get this as zero; when $\sin\theta$ is 90^{o}, I get this as F/A; so, it goes from zero to F/A maximum. And we can also write the shear stress. Shear stress diagram is much more interesting to see. Ok? Can you get the expression for shear stress? Because you have the expression for stress vector, you have the expression for normal stress. We have seen $\overset{n}{\tau}$ is nothing but $\sqrt{\overset{n}{T}{}^2 - \overset{n}{\sigma}{}^2}$. So, if you do that, I get an expression which is very simple. I think by the time I show the slide, you should have your expressions. These are very simple trigonometric simplifications.

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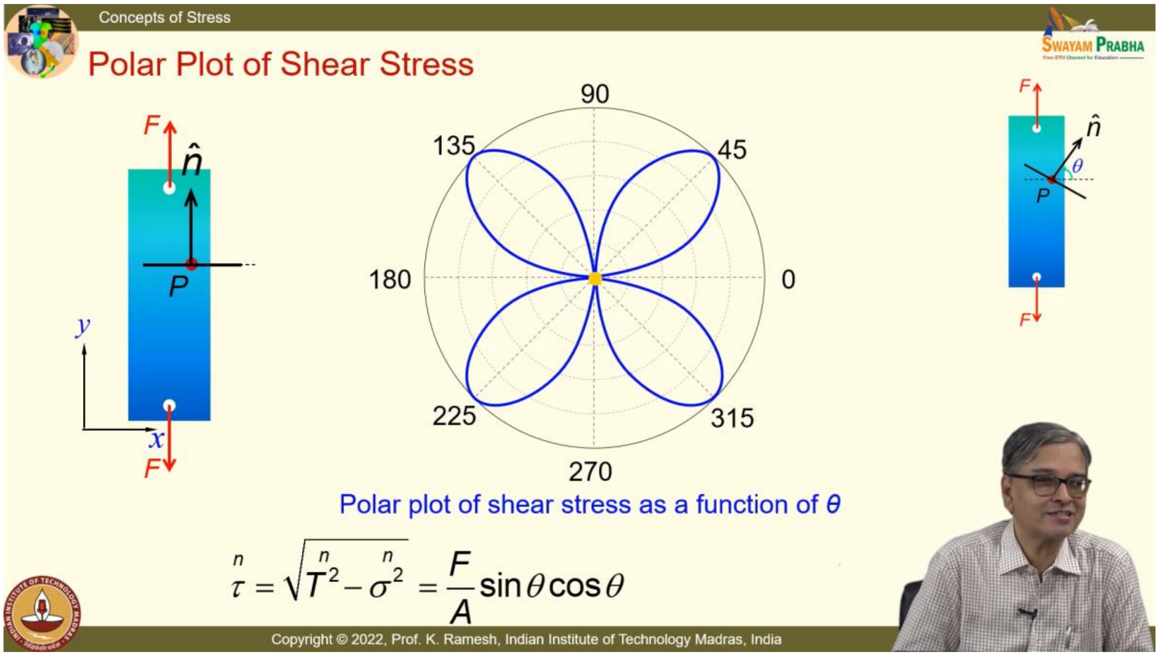

I get that as $\dfrac{F}{A}\sin\theta\cos\theta$. And in order to see this variation, I am not bringing in the scale here. Ok? We will compare these two in the next stage. Let me plot this. When I plot this, this has a beautiful figure like this.

So, what do you think immediately once you see a shape like this? Does it look like a butterfly? They are very nice to see, isn't it? See for environmental sustainability, they say butterflies are needed. Fine? So, what I want to tell you is, when you look at the expression, when you want to do a polar plot, I get beautiful plots. There is definitely beauty attached to this. Nature loves beautiful patterns. But from your point of view, it is very difficult for you to plot and gather the information. Now what we will do is, we will understand this comprehensively in a single diagram.

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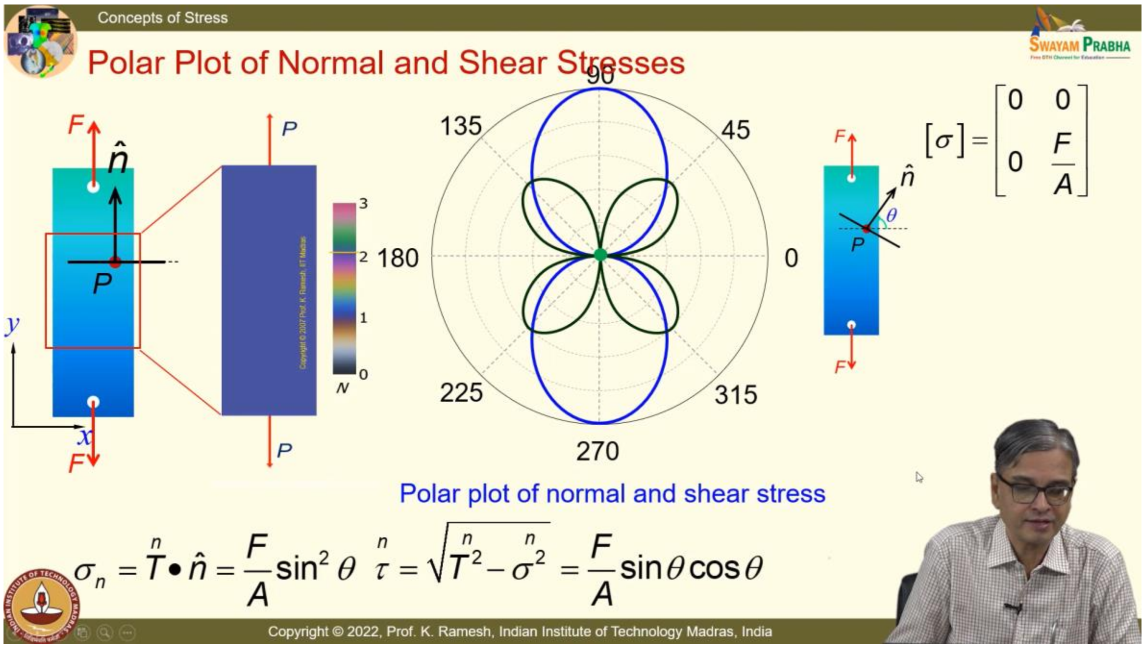

A simple tension; here again I have to caution you one more thing. When I showed you this photoelastic fringe pattern, I have said it is away from the loading points. How many of you noticed it? I do not know. Now I have also shown it pictorially here. So, we have to worry about what happens near the load application points.

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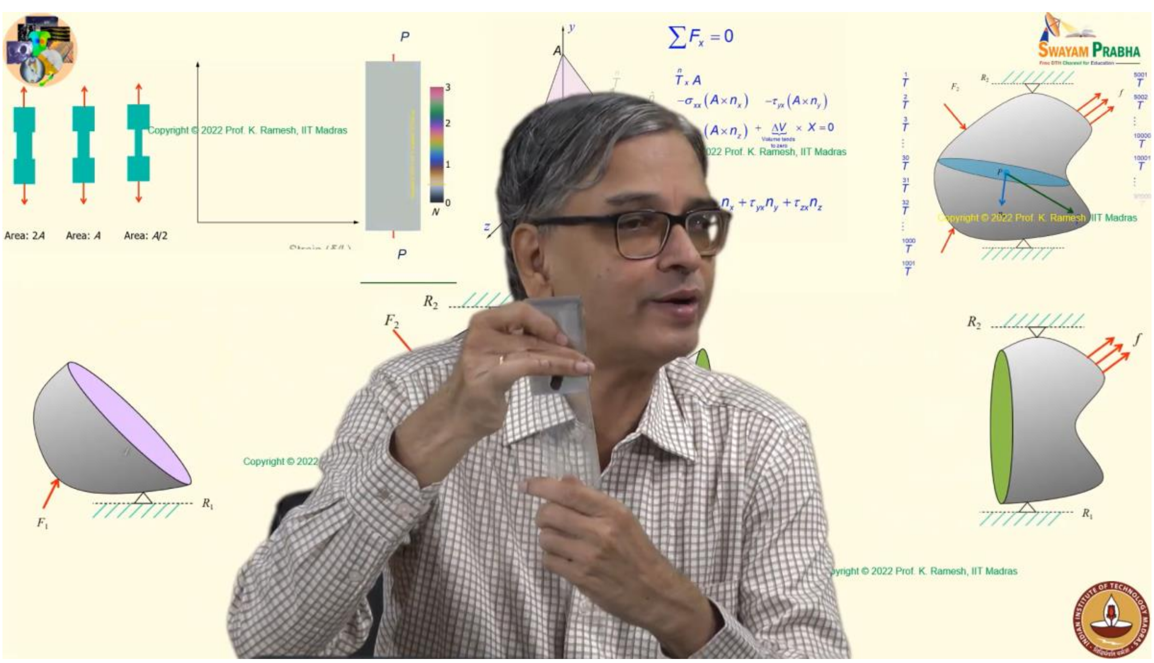

Because when somebody gives me an object to apply tension, I can have only an object like this, put a hole and then I have a provision for me to hold it like this and then put the pin and then pull it. This is how I can do it. Fine? But my intention is, across this cross-section I should have a uniform stretching.

But physically when you go, you can only do like this. Probably you can improve it further by having multiple holes. Instead of one hole, you go with two holes so that you make it little more uniform, or you go with three holes, you make it further uniform, or you also have another system wherein what I have is, I have this. I suppose you are able to see the color, maybe I will put this, it has a different contrast background. What I have is, I have multiple holes, I have about five holes. Fine?

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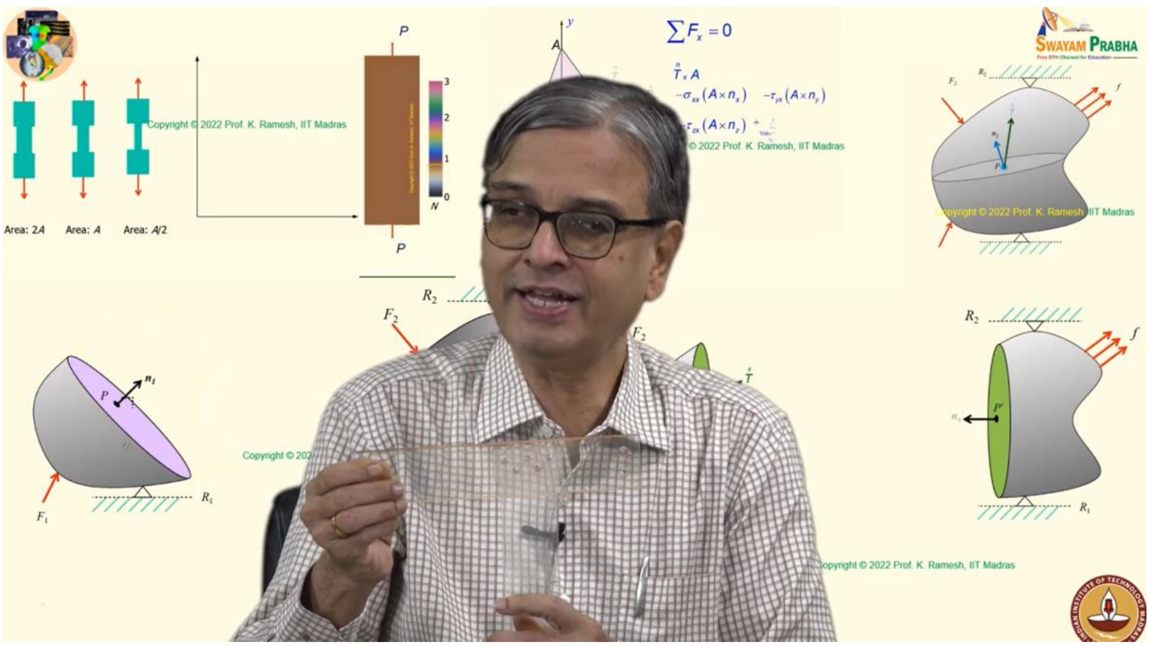

So, you can try to have your fixtures designed in such a manner that I can go closer to uniform pulling of the member. We would also see those details later. Now our focus is, what happens away from the load application points. Only for the region away from the load application points, I show you the photoelastic fringes. This is again a graphical representation for various loads, that is what you are seeing it.

But what you are plotting here is, for a given load what is the variation of normal and shear stress. Now I am going to put it in the same graph. I have the normal stress expression as $\frac{F}{A}\sin^2\theta$. I have the normal stress plotted like this. So, this goes from zero to F/A and you have various points put here and then you also have the associated plane that is also animated.

And if you want to see again, you can have a look at it because now, you can associate this with the particular plane. So, a simple tension can throw you surprises like this. So, in those planes, whatever the representation of stress tensor what you have written will not have zeros in all the places, it can be populated also. Fine? And, what is the expression for shear stress? I have this as $\frac{F}{A}\sin\theta\cos\theta$. And what is the maximum value this it can go? Only sin 45 and cos 45, I get this as (F/A)/2. The maximum will be one half of what is seen in the normal stress. And this is plotted, the graph is reasonably good, but it should be still smaller than this. But it gives an impression that normal stress is of larger magnitude and shear stress maximum is one half of it. And this graph also can give you additional information. One information is difficult to get; see, only when you see these graphs; the beauty is very good but what I extract from this, certain ideas are difficult to see.

Only when you appreciate this, when you go for a simpler representation, you can fully appreciate the beauty of that kind of a graphical representation. Here, I can see what happens on a plane, what is the magnitude of normal as well as shear stress. So, from the plot I can find out, I can have a plane where I will have only normal stress where the shear is zero. That is evident from this, if you are able to recognize, fine. Suppose I go to a maximum shear stress, I have maximum shear stress; that plane also has a normal stress, it is not zero. You do not have a situation where I have only shear stress in any one of the planes. But what we find interesting is, because the way that we have applied the load, that is one thing, but we find you have two such planes where I have only normal stress, there is no shear stress at all. So, these two observations you can make out of this. Ok?

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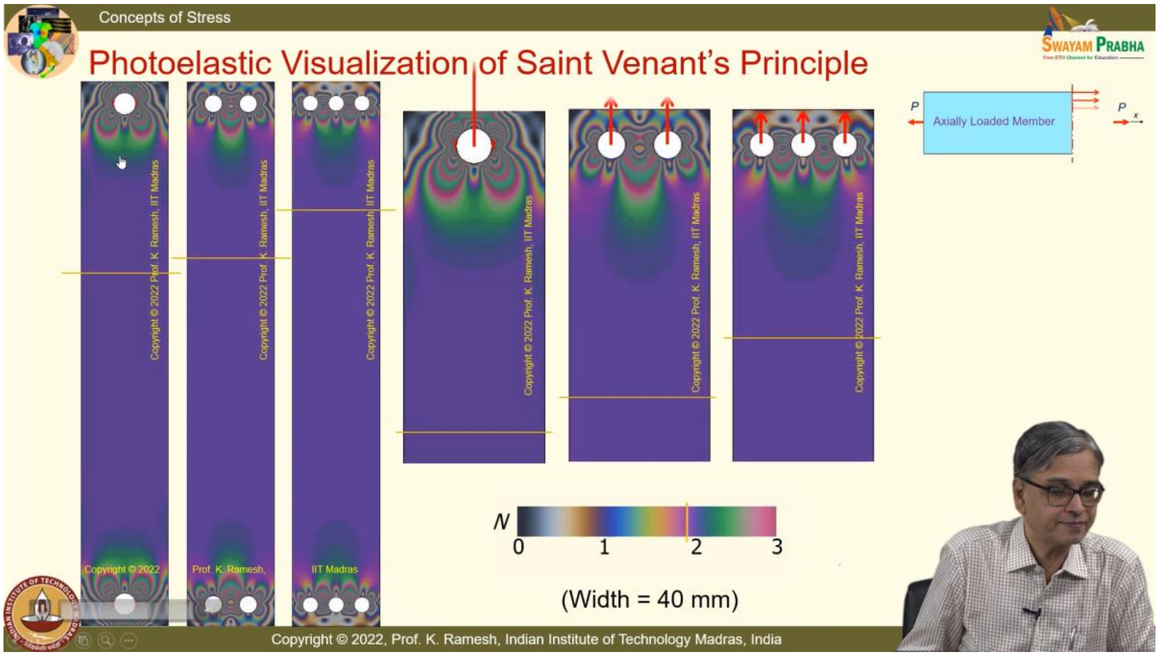

Now what we will do is, we will go back and then reinvestigate what is it we have found out when I have a single hole. You are all now well trained to look at fringe patterns and what you have here is, the full specimen is shown here, you have a single hole, you have two holes and then you have three holes. And a blow-up of this is shown in this and if you look at very carefully, I have shown what way the loads are applied. The figure is reasonably geometrically drawn. Can you find out what is equivalent in these three specimens? I have shown the length of the arrow indicates the magnitude of the load. Can you just look at that statically, is there anything that you get out of it? Suppose I find a resultant of this, you know I have the two-hole specimen and you have these forces shown, I have three forces shown. Can you find out this from the geometry? You are not able to see that?

I am applying the same load, I am just putting it by one force, I have divided the force into half. I apply it at two different points; divided the force into three and I have applied it. The resultant remains same. Fine? And what you see here, I also have these horizontal lines put. And you find near the holes, you have beautiful fringe pattern seen and they all die down here. We have always seen when I apply a simple tension, in the region away from the loading point, uniform color. And what you see in this is, I see the uniform color only after this line when I apply a single load.

When I apply two loads, I see it after this. And as I go nearer and nearer as I apply a uniform loading, fine? What I find is the fringe pattern becomes of uniform color earlier than the previous case. And this is embedded in a nice principle, which goes with the name Saint-Venant's principle. Ok?

In fact, he developed this concept in the context of bending stress. See, bending stress, you have to apply a triangular variation. When you have a model, you can only go and apply a bending moment like this. You cannot apply a triangular variation by any other gadget. So, the principle is - as long as the load applied is statically equivalent, after some distance from the load application point, whatever the distribution that you are conjecturing is achieved. This is a very very useful principle. We will repeatedly use it in solid mechanics. In fact, none of the books give you a pictorial representation in a manner that is represented here. Ok? See, the idea is to have a uniform pull. And what you find is by the way that you apply the load, there are local disturbances and these local disturbances die down after a distance. Fine? And this goes by the principle called Saint-Venant's principle.

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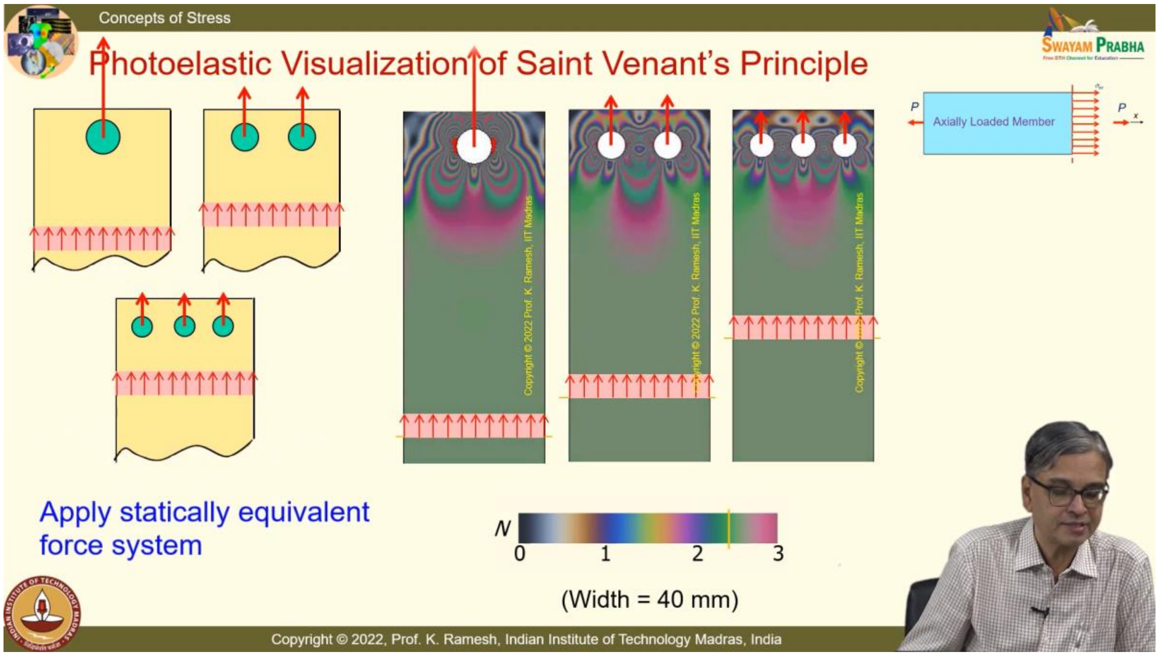

What I have is, I have a single hole, I have a double hole and I apply statically equivalent force system. The requirement is to apply statically equivalent force system. This is a very nice mathematical advantage when you want to solve a problem from a mathematical perspective.

So, as long as you apply a statically equivalent system, you can conjecture the similar distribution is achieved at distances away. And this distance can reduce if I make my application of the force system closer to what it is intended for. It is a very powerful tool. None of the books really give you any practical demonstration like what you have from photoelasticity here at this stage of development of the course. So, you have the advantage. I suppose you appreciate the utility of Saint-Venant's principle. Many times, we will simply say Saint-Venant's principle because with the development of mathematics at this stage, it is too difficult to comprehend what is happening near the load application points.

Unless you go to a numerical or experimental analysis, there is no way you can find out the forces. Ok? One thing you can notice in all these problems; these corners always remained black. That is a very, very important clue. Notice this. I will discuss about it in a later class.

So, in this class, we have looked at further development on the concept of state of stress. And we have particularly focused on the simple problem of axial tension. Even a simple axial tension, if I look at different planes, some of the planes can have both normal and shear stress. And we had done one type of graphical plotting; polar plot is what we have looked at. The polar plot, the representation was beautiful. Nature really loves beautiful patterns. But to cull out information that we want, it is little difficult for you to do that. And what we noticed was, you can identify two planes in which the normal stress alone can exist. But when I have maximum shear, you always have some amount of normal stress. Thank you.