Strength of Materials
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Lecture - 37
Stability 1 Governing Equations, Fixed-free and Pinned-Pinned

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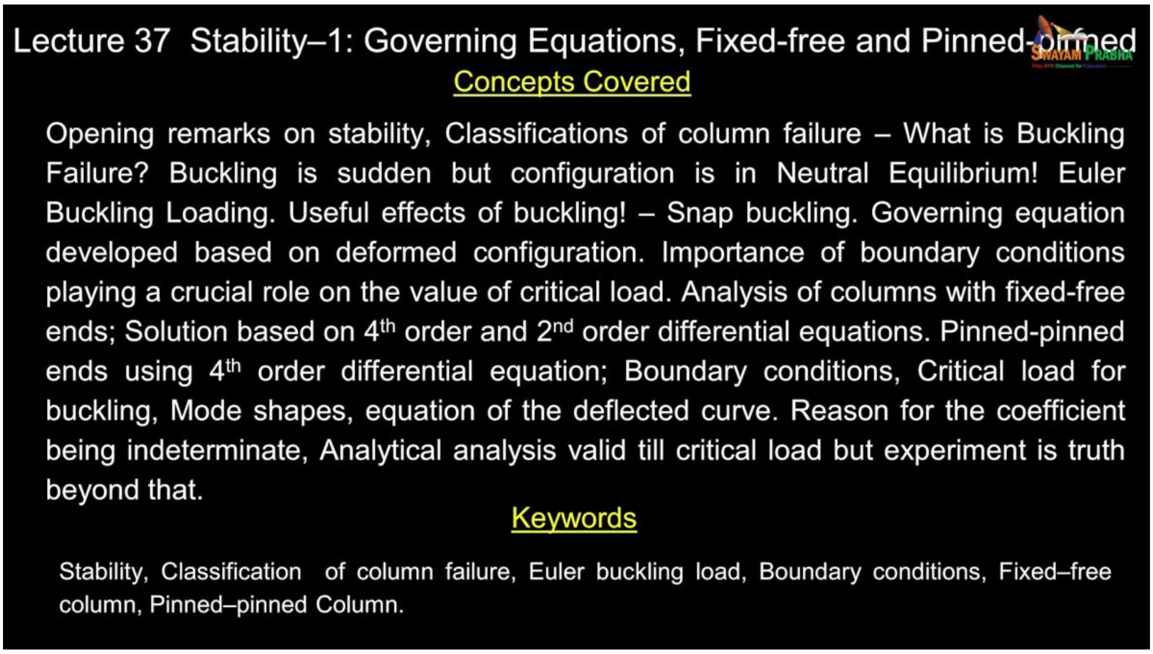

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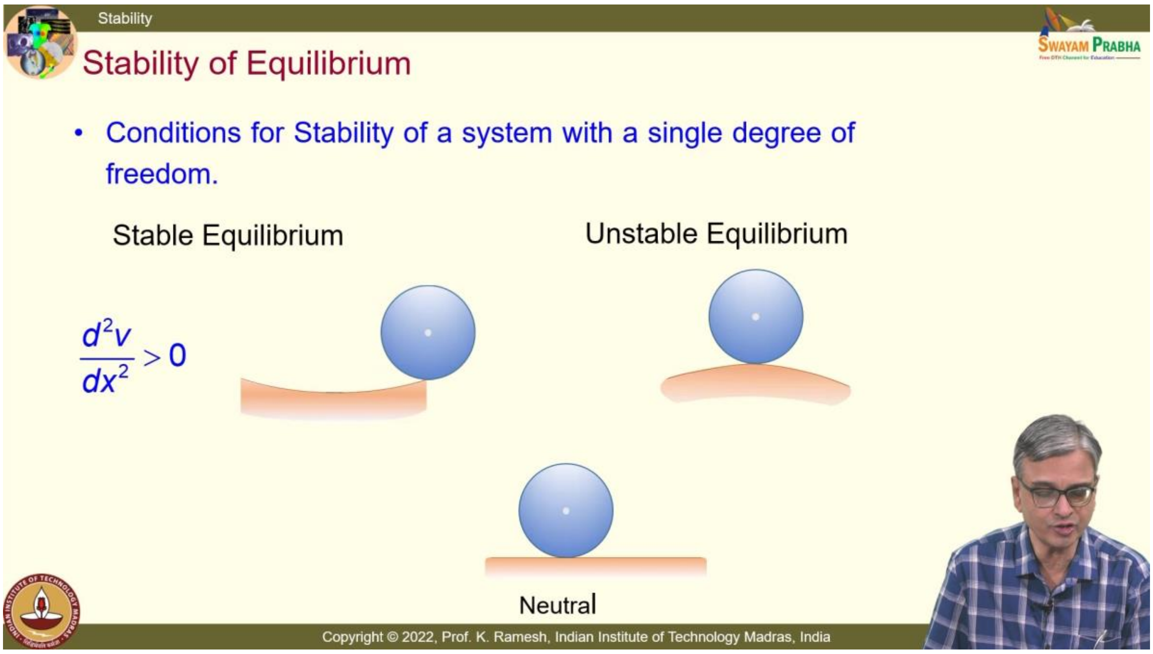

Let us move on to our next topic, stability. See, in this short series of two lectures, we will confine our attention to find out the stability of columns and we will do the mathematical analysis. Wherever you come across compressive loads, whether it is in bending or torsion or even in shear, you will have to keep at the back of your mind; if the compressive stresses exceed the limit and your optimized structure has minimal thickness in those areas, it can lead to buckling. But mathematically, for this purpose of this course, we will confine our attention to columns. And normally, one of the, you know, fallout of this discussion is students tend to think only columns buckle. I want to caution you; we do a mathematical analysis for columns. We do not have the mathematical background to analyze for other situations. Wherever you have compressive load, you have the problem of instability.

And you know, you have studied in an elementary manner, even in your physics courses and also in your rigid body mechanics. There you know, we have investigated what is the potential energy. And potential energy, we had used the symbol capital *V*. See, mind you, when we discuss advanced mechanics, there is paucity of symbols. You cannot have one symbol reserved for the entire course. In this course, we have used capital *V* for shear force. And while we developed rigid body mechanics, we had used capital *V* for potential energy. So, understand the relevance of the symbol with respect to the concept that we are discussing, ok.

So, when you have a stable equilibrium, if I disturb it, you have a restoring force; it brings it back to the position. It is a simulation which is very closely representing it. You know, all these motions, one has to do it with free hand. There is no geometric curve I can fit in. At least the PowerPoint people should learn from this lecture that they should have such facility incorporated in the future versions. It is all free drawn. So, it is difficult to do beyond a point.

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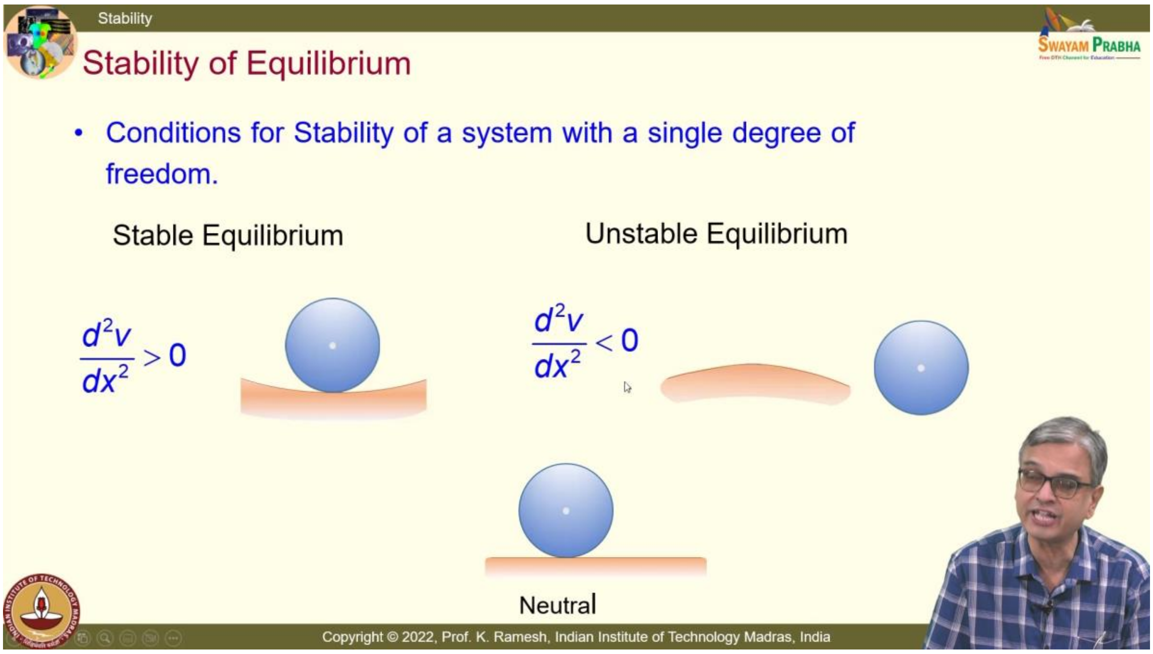

And I have a situation; when I have a situation like this, when I disturb, it does not come back to its original position. You have investigated what is the sign of $\frac{d^2v}{dx^2}$, then you say it is unstable equilibrium.

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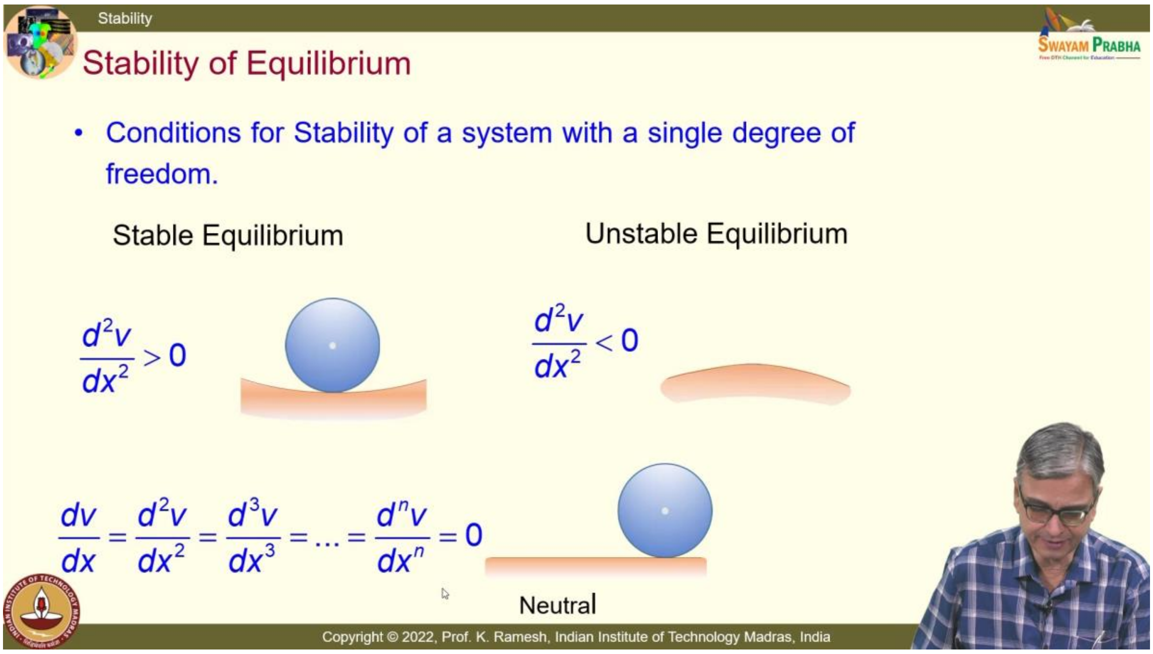

And you also have a situation, when all these differentials are zero,

$$\frac{dv}{dx} = \frac{d^2v}{dx^2} = \frac{d^3v}{dx^3} = \ldots = \frac{d^nv}{dx^n} = 0$$

When all of these go to zero; when you disturb, it goes to another equilibrium position. See, it is only an illustration. The other equilibrium position can be of different shape. That is what we are going to look at in stability of columns.

So, you have an idea, it does not become like what you have in unstable equilibrium; it occupies another equilibrium position. How the transition is? Here the transition is smooth. The illustration what I have given is, the transition is smooth, which is not smooth in stability of columns.

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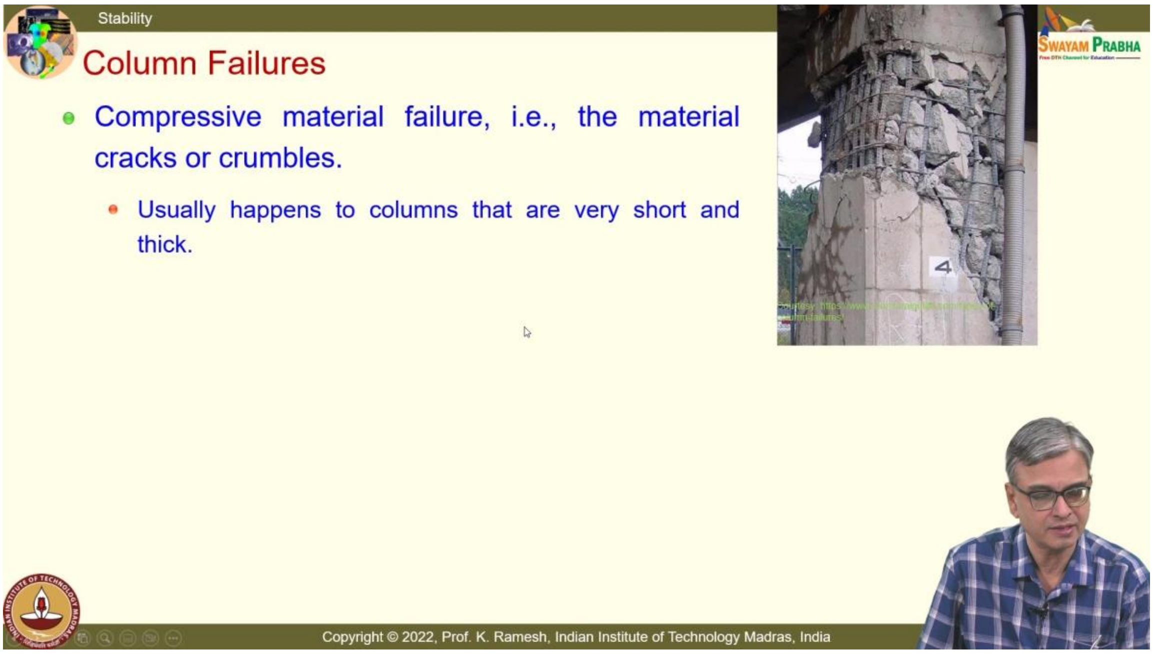

And once you say column failures, one of the things that you can think of is the material failure. You have a compressive material failure, when you have a column which has a big cross section. When you see the next set of slides, you will see that those columns are thinner compared to this. And when I have a short column, you can have a compressive material failure. So, it usually happens for columns that are very short and thick, it just crumbled, fine.

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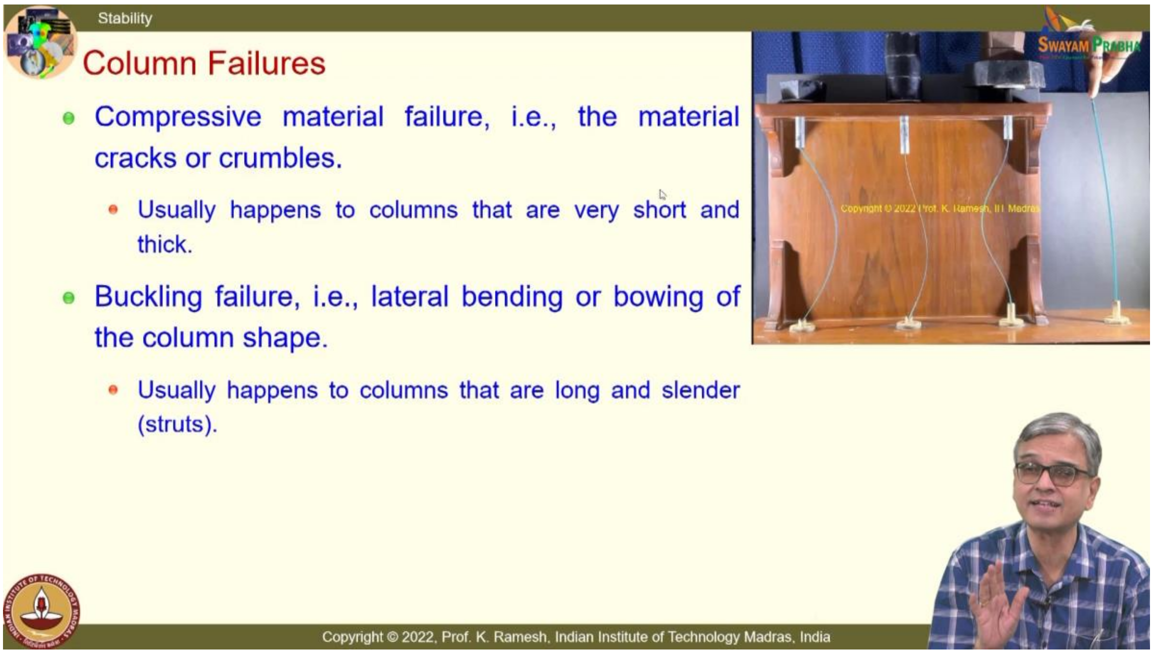

You also have what is known as a buckling failure. It is called lateral bending or bowing of the column shape. And which is very beautifully illustrated in this animation, in this experiment. And you should also recognize, you know, I am using discrete weights. If I have to do instability analysis, I should have a provision to increase the load gradually. I do not have that provision in the experiment. So, in that sense, this experiment is little crude. Nevertheless, it illustrates a very important basic aspect of buckling beautifully. And what you have here is, you have the boundary conditions. I have this as pinned end and this also pinned end. And you can identify, because you have already looked at bending deflection. By looking at the deflected picture near the load application point; near the support, you can always figure out what is the boundary condition that is being simulated. Can you tell me what is the boundary condition simulated here? Close to fixed end, fine. And this is also fixed end, and this is also fixed end.

So, when you have columns that are long and slender, you have the problem of buckling. And one of the aspects is, when we have a structure which is supported by columns, you do not want buckling to happen, ok. So, what people have done is, they have done the mathematical analysis and also based on experience, they find there is a load which is labeled as a critical load, beyond which if there is any lateral disturbance; even a small disturbance, it will go into buckled shape. The load by itself does not introduce buckling, if there is a disturbance, ok.

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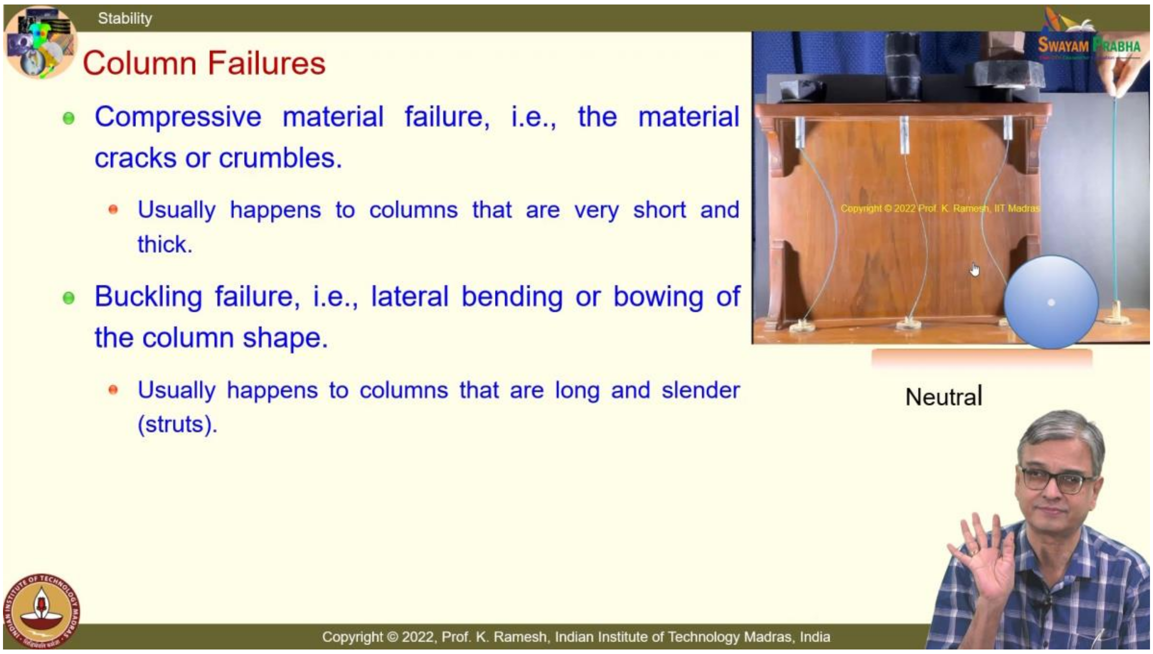

Now, the question is, from where the disturbance comes? That also has to be looked at. See, I have just now said that you have what is known as neutral equilibrium. When I disturb, it goes to another equilibrium condition. I am not sure how many of you looked at this experiment closely. Please watch the experiment closely. You have the idea of this neutral equilibrium beautifully illustrated in this. See, in the first case, I have applied a load which is much beyond the basic critical load. So, it has assumed this shape. The second one you look at it. Can you see that it has started deflecting? See, what you will have to look at is, when you increase the load, it cannot remain straight. You want the column to remain straight. If it does not remain straight, then it is a problem. If you look at this, this satisfies the equilibrium. And suppose I increase the load further; it goes to another equilibrium position. You get the idea! It goes from one equilibrium position to another equilibrium position. And look at the fixed-fixed end. You could see that it has taken another shape beautifully. I have this load; I have the deflection. So, the system is in equilibrium, not with the column being straight, but in a deflected shape. Suppose I increase the load further, you could see that it has taken another shape. And I increase it further, it takes another shape. So, it beautifully illustrates the buckling phenomena; what you see is actually neutral equilibrium. It cannot remain straight. The moment it cannot remain straight, our functionality of the column is lost. And we do not want to reach that in our structures. That is the idea behind it.

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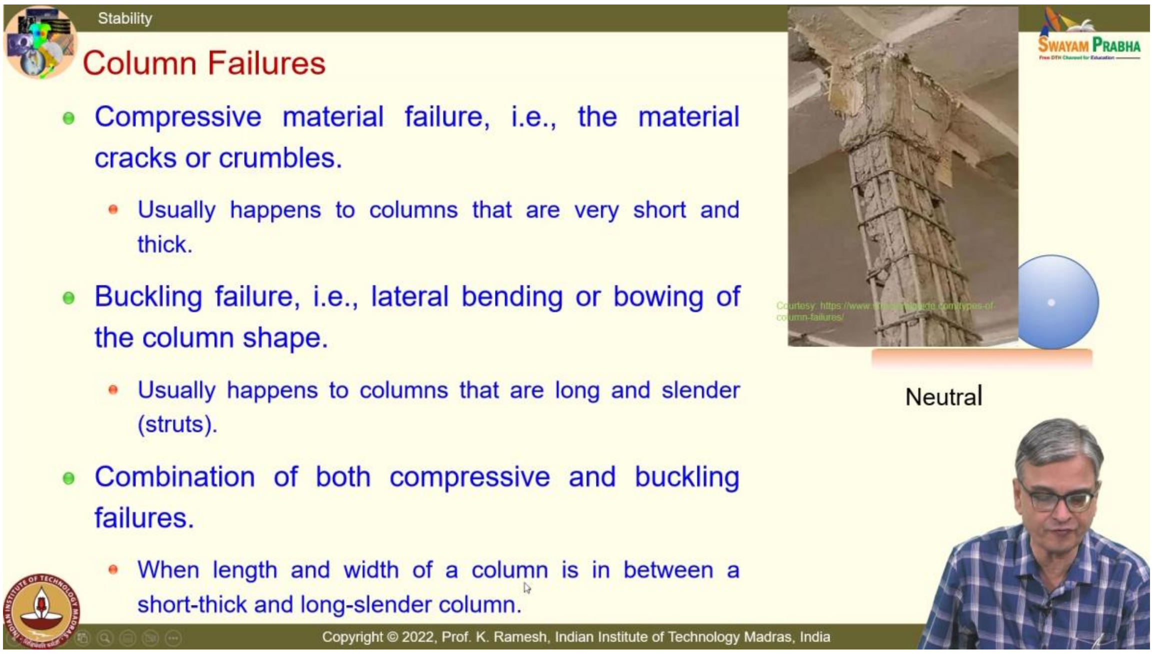

And you will also see, I have a concrete beam. So, you could see visibly the cross-sectional dimensions are much smaller and see it has buckled, fine. It has beautifully buckled. Then in practice, you know, you could have a combination of both compressive and buckling failures. I do not have an image which illustrates that. This happens when length and width of a column is in between a short, thick and long slender column.

You know, we have already looked at, in this course, we are going to work on slender members, fine. All our discussions are confined to slender members. At times, we may borrow that and use it in a, even in a spur gear teeth. They analyze the gear teeth as beam; cantilever beam. You may make some corrections of it. You know, engineers try to solve a given complex problem as best as possible. You should appreciate from that perspective, ok.

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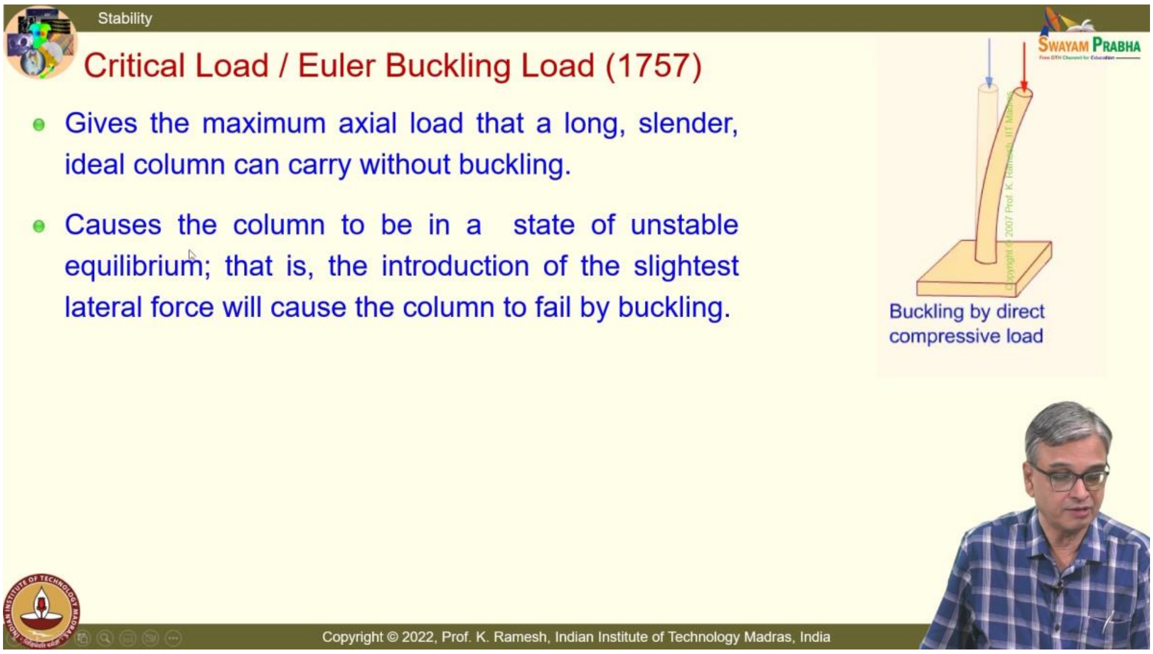

And I want you to look at this animation very, very carefully. See, this illustrates what is instability beautifully. See, when I have the load as blue, it is going on increasing, fine. At some point, without a warning; see, that is what you have to look at, if there is an alert, if there is a warning. See, you have your gas cylinder at home, if there is a leak, you get a smell. Naturally, the gas has no smell. From safety consideration, the manufacturers and the courts dictate that you should induce that smell, so that you detect that there is a leak, before a catastrophic blast takes place, you have avenues to avert it. Either you go and close the gas cylinder or at least open your windows or run out of the place. Any of those options which is possible within the time, that you can take. If there is a failure which happens without a warning, you know, nothing can be done.

In fact, yesterday, there was a; two days back, there was a bridge collapse; 100 years old bridge collapse in Gujarat. It failed in seconds! That is mainly because one of the possible reasons from stress analysis point of view is in a suspension bridge, you are not supposed to make any periodic oscillations. It seems that revelers did periodic oscillations, that would have created a resonance that would also added apart from the weight, apart from poor maintenance. It would have also added from a structural point of view for a failure, all of a sudden.

So, if you look at here, when the load becomes red; it buckles! We do not want that to happen, ok. I have taken a column that is circular, fine. I have shown buckling in one way. Was there any visible external interference to this? I said, when you apply the load, if there is a perturbation from outside, you can have buckling. There is no restoring force to bring it back to original shape, so it buckles. See, in this course, to make our life simple, we have said, we have an elastic continuum. Our mathematics, we could develop it so beautifully. When $\Delta x \to 0$, there is was material available. But real material, how they are? They all have defects. So, one of the perturbations that can precipitate buckling can come from inherent defects on that column.

See, I have a circular cross-section. I have already told you; how does the mathematics understand the cross-section. How does the mathematics understand the cross-section? You have the moment of inertia, fine. Suppose I take any axis when the geometry is circular, what is the moment of inertia on any of these diagonals? The moment of inertia; not polar moment of inertia, I am talking about inertia about an axis. Because it is circular cross-section, the moment of inertia is same. So, the probability for buckling to happen can be any direction. It can buckle like this; it can buckle like this; it can buckle like this; it can buckle like this. Please understand that. You know, some of these concepts do not come out very clearly when you read a book. Because the book publisher wants to minimize the pages, so the author can show only buckling in one direction. If you have seen buckling to a right or left, whichever way you look at it. I see it as left; you see it as right. You will think always buckling happen in one direction. No, it is dictated by the moment of inertia. Because I have taken a circular cross-section, the probability exists in all the 360 degrees. It could be precipitated by inherent defects. It could also be precipitated when I apply the load. I may not apply it exactly along the axis as intended it to be. There could be an eccentricity. So, there is a perturbation introduced in the process itself.

Suppose I take a rectangular cross-section. Here what happens? You should be able to say, I have a rectangular cross-section. Which way the probability of buckling can happen? Why it is so? Very good! So, what happens is, you have the moment of; it will buckle in the direction where it has the least moment of inertia, fine. So, I have also modulated the expression in that fashion. So, we will see.

So, you have to appreciate what causes the column to be in a state of unstable equilibrium. Introduction of the slightest lateral force will cause. That can be inherent in the material itself, ok. There could be material defects or there could be problem because of the loading. And column buckling, the credit goes to Euler, and he was a great mathematician. See, if you look at history, you know, people have not had a very healthy life. It seems Euler lost his, one of his eyes in 1735 and in those days, cataract is also a big issue. And he started losing his second eye; he was almost blind, and the queen of Russia provided him all the support with all assistance. And the buckling was determined in 1757. He had assistance, he would go and explain what he wants. They will translate his work. He was so prolific. He was patronized by the king in Prussia, later in, the queen of Russia. They provided him all the facilities so that he can peacefully work on scientific development. He published his paper at the age of 20. At the age of 16, he has completed his masters. And most of his prolific work he did when he was almost blind, very sad to hear! So, you know, with able-bodied people, you should at least read your subject properly, ok. And 1757 is the time; he was almost blind at that time, that is what the history says, ok.

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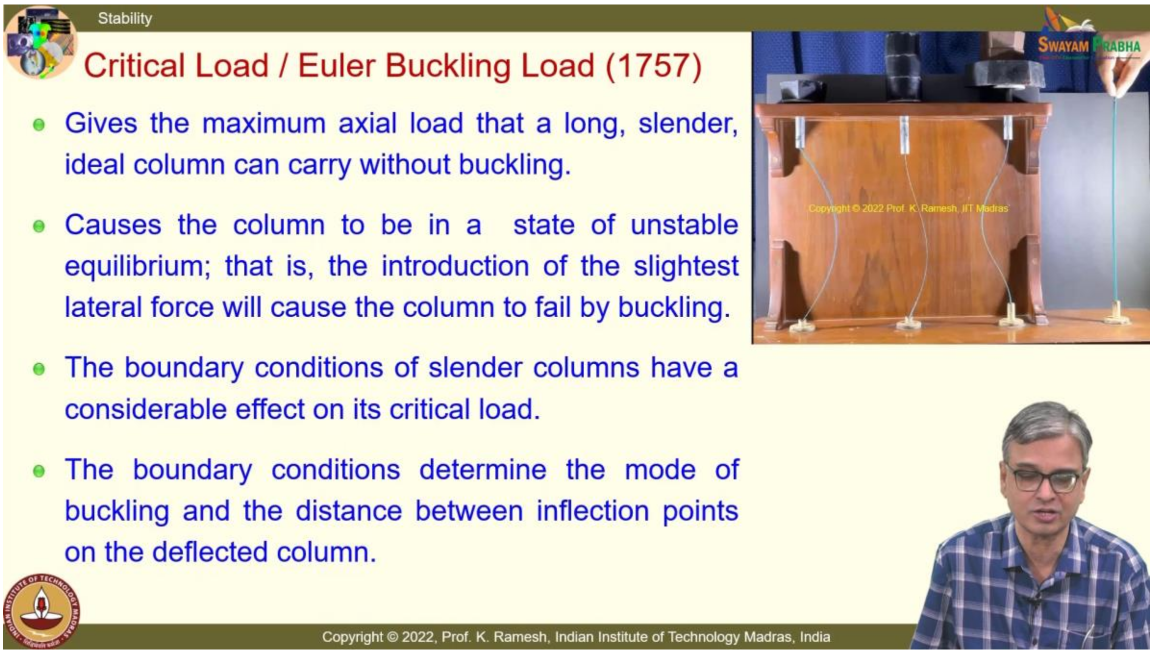

And buckling is one phenomena that is greatly influenced by the support conditions. So, that is very clearly seen. See, the critical load drastically changes from one stage to another stage. So, that is what you have looked at it here. And you know, if you ask a question, is buckling is useful or not useful? The question is very similar to whether the friction is useful or not useful. Friction is useful for you to walk. Friction is not needed when you want to have your IC engines.

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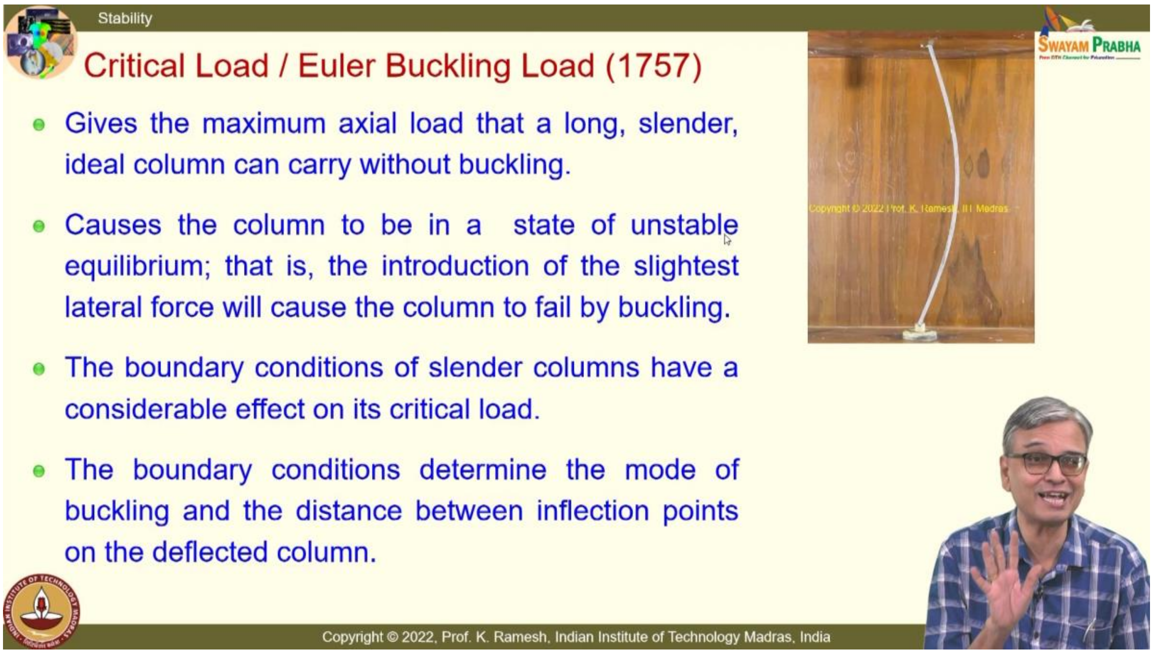

And what you have here is, people have used this deflected shape, it is called snap buckling. If I put a force here, it will go to the other shape. This is used in electrical appliances as a switch. So, buckling also has found; and I do not know, when we were young kids, we used to have a toy which is very, very cheap. You allow a simple plate, you press it, it will go to another buckled shape. In the process, it will create a sound; tick, tick, tick, tick like that. It will be irritating, but toy industry have utilized buckling for entertainment, ok. In fact, I wanted to get the toy and show it to you, I could not get it.

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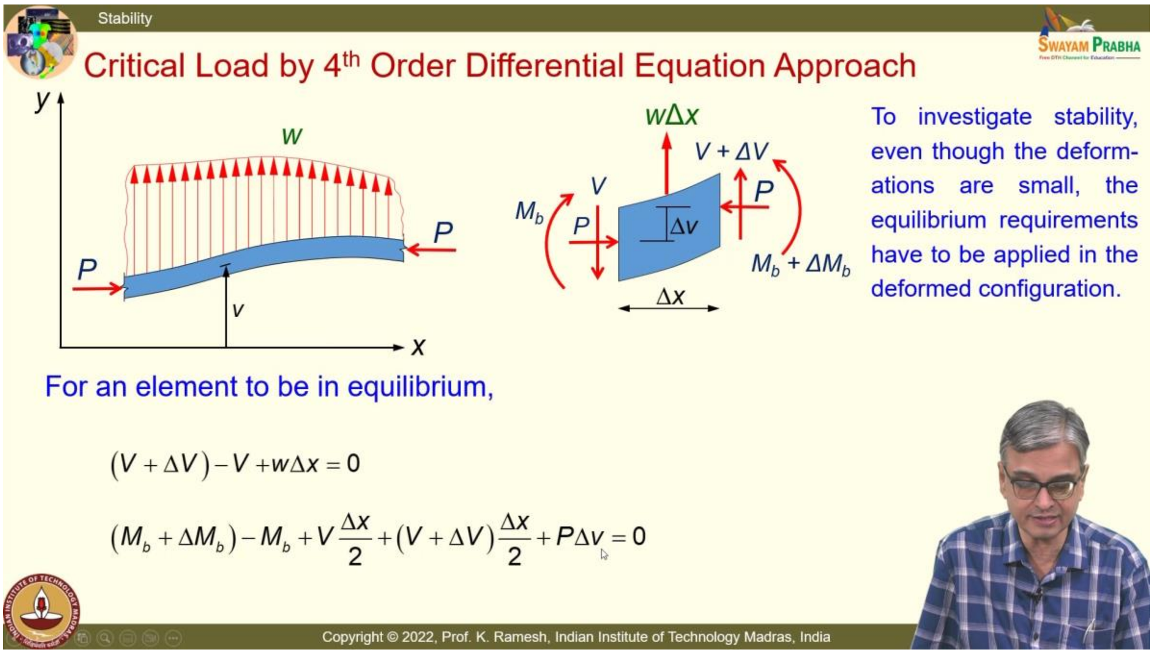

And in this chapter, you know, we also make a conscious decision to investigate stability even though the deformations are small; the equilibrium requirements have to be applied in the deformed configuration. This I alerted you even long time back. In this course, because the deformations are small, we will work with undeformed configuration for all our mathematics. We make a departure when we want to investigate stability. So, I have shown the beam which is subjected to a transverse load w, and I have this compressive load applied. I have shown this as deflected. You know, we have also analyzed interrelationship between shear and bending moment where we have taken a load w. There we would have taken this as straight; go back to your notes and then see. So, the difference what we make is, we consciously make, our goal is to investigate stability. Since I have to investigate stability, I do this, and I take a small element out of it.

So, we consciously make a decision to analyze it in deformed configuration and I have the deformed shape taken. You can put the shear force, you can put the bending moment, all that follows the usual convention that we have followed. What is extra here? I have a compressive load. Because I have considered this as a deflected shape, you must read this as small v, which is showing the deflection, ok. So, these loads P are separated by a distance Δv That is the only difference between what we have done for finding interrelationship between shear force and bending moment. We had taken this beam as straight. Here we have considered this as deformed so that I can investigate, and I get an extra term in this. Otherwise, the mathematics is simple. Please write down the mathematics.

I am going to write the equilibrium equations. I am going to write the force equilibrium. So, that gives me

$$(V+\Delta V)-V+w\Delta x=0$$

So, force equilibrium in the *y* direction. And I can also write the moment equilibrium. I have this as

$$(M_b+\Delta M_b)-M_b+V\frac{\Delta x}{2}+(V+\Delta V)\frac{\Delta x}{2}+P\Delta v=0$$

So, the additional term what you have is $P\Delta v$ and understand this v is the displacement v, ok.

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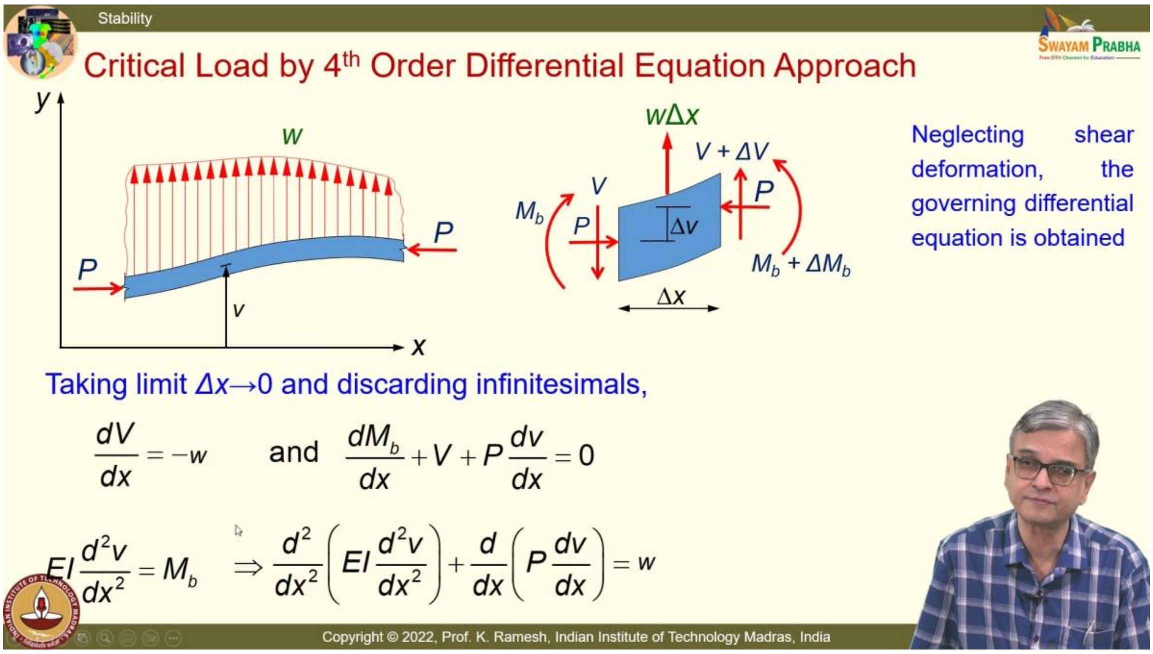

Taking the limit $\Delta x \to 0$ and discarding infinitesimal; that is the usual approximation we do. And we also know the interrelationship $dV/dx=-w$. And using all that, we have

$$\frac{dM_b}{dx}+V+P\frac{dv}{dx}=0$$

That is what I am got from the earlier expression. And you will also have to appreciate this does not take account the shear deformation. Even in our bending deflection, the governing equation we never worried about shear deformation. We were worried only about the contribution from bending. So, the same limitation applies to this expression also. Neglecting the shear deformation, the governing differential equation is obtained. And you know, we know from our earlier knowledge that bending moment is related to the curvature.

$$EI\frac{d^2v}{dx^2}=M_b$$

The very famous relation we use for deflection. So, this gives me

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)+\frac{d}{dx}\left(P\frac{dv}{dx}\right)=w$$

And you know, we have put this in a generic fashion, and this is known as bending rigidity. Suppose I consider a cross-section, which is of constant cross-section and the material is same material, then I can take *EI* also out.

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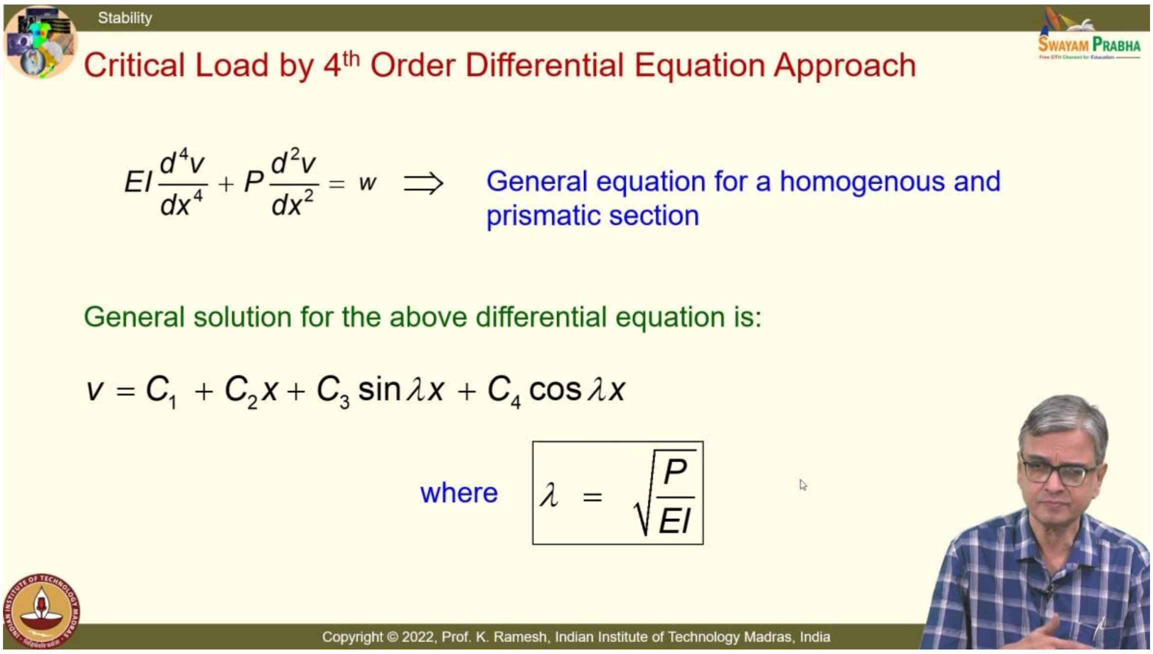

I can take this out and get the final expression as

$$EI\frac{d^4v}{dx^4}+P\frac{d^2v}{dx^2}=w$$

So, we have got the governing equation for buckling of columns. When you get an expression and you do a systematic analysis, you all feel that you have got the right expression and mathematics you have done. Let me ask the question, is it an exact expression or an approximate expression? What way it is an approximate expression? Very good, very good! What they are saying is, see moment-curvature relation, we have linearized it, fine. We have not taken that as non-linear. In fact, it is very surprising when Lagrange and Euler analyzed, they had taken non-linear expression and then did all the derivation, fine. They have also indicated that. But here, we have used only linearized moment-curvature relation. So, in that sense, it is an approximate expression. Where does this hit in our solution? That also we will see.

So, it is a general equation for a homogeneous and prismatic section. That means, I have one material and then the cross section does not change. And general solution for this is in this form, v equal to the deflection,

$$v = C_1 + C_2x + C_3\sin\lambda x + C_4\cos\lambda x$$

And you have multiple boundary conditions. So, when you supply the boundary condition, you are in a position to evaluate the constant C_1, C_2, C_3 and C_4. And you are in a position to solve this as an eigenvalue, eigenvector problem, ok. And we also have simplified λ and

$$\lambda = \sqrt{\frac{P}{EI}}$$

So, what we will have to do is, we will have to go and analyze the fixed-free, that is, a cantilever beam; hinged-hinged. Likewise, we can analyze, and we can evaluate the critical load.

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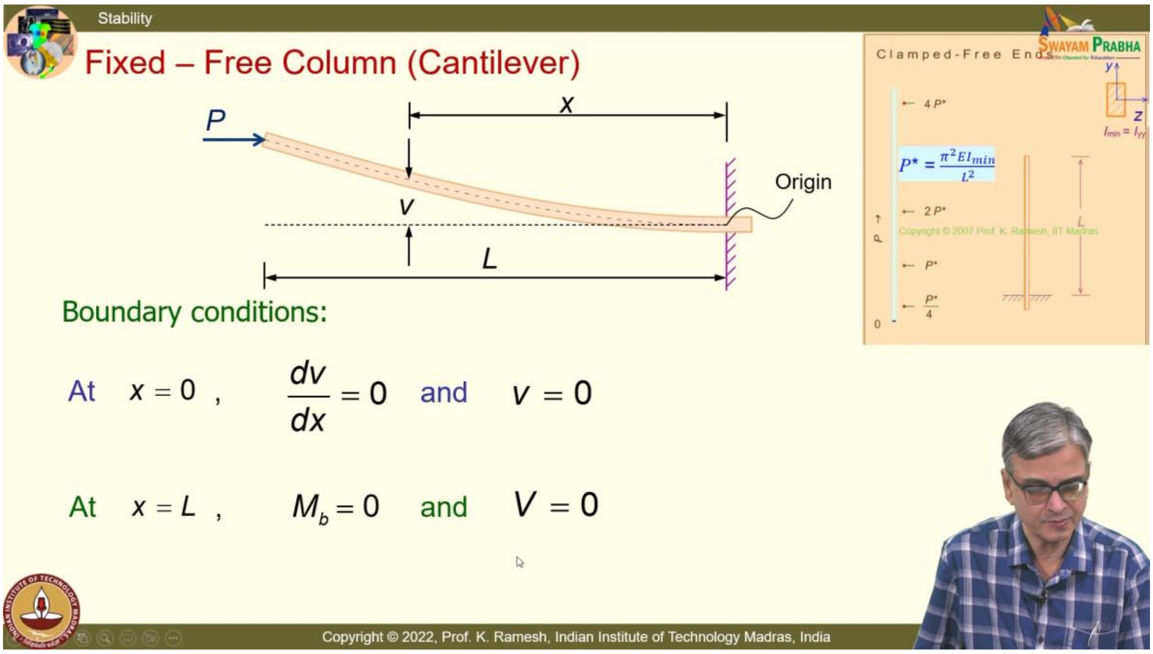

I have a fixed-free column. So, the column is deflected under the action of the load P. And this is the length L and you measure X from this, ok. And you take a deflection as v and you know the boundary conditions. See, I have four constants. I need to have four boundary conditions minimum for me to solve. Here you can write the boundary conditions very easily based on the deflection v. What happens to v here? v is zero here and then your slope is also zero. So, you have at

$$\text{At} \quad x = 0 \,, \qquad \frac{dv}{dx} = 0 \quad \text{and} \quad v = 0$$

So, these go to the deformation picture. We also have boundary conditions based on the equilibrium requirements. That you can write in the end B. Can I have a shear force there? At $x = L$, can I have bending moment there? Bending moment is zero and your shear force is also zero. So, please understand, you know, when you write, make it a small v, make it as big V. While writing it in computer, I can do this easily. You must make the distinction. And now you apply the boundary condition and solve for it.

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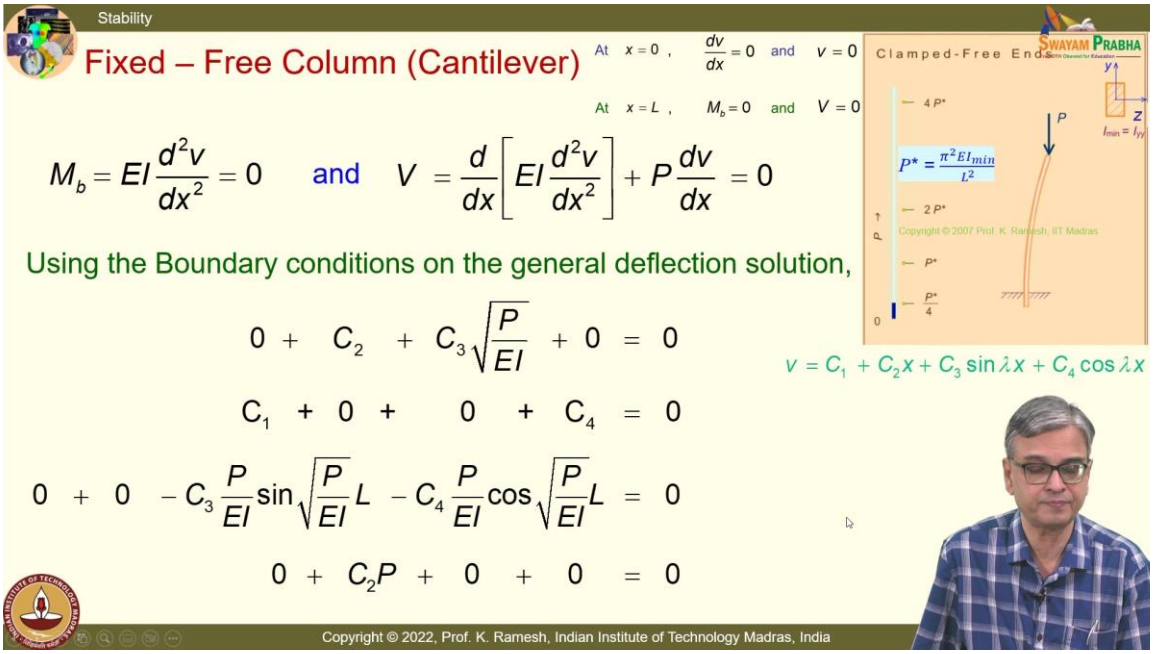

I have the boundary conditions and I have, you know,

$$M_b = EI\frac{d^2v}{dx^2} = 0$$

I should construct that expression. Similarly, I should construct

$$V = \frac{d}{dx}\left[EI\frac{d^2v}{dx^2}\right] + P\frac{dv}{dx} = 0$$

These two are straightforward. So, I can write the expressions by substituting those values. This is the governing solution. So, when I substitute $\frac{dv}{dx} = 0$,

$$0 + C_2 + C_3\sqrt{\frac{P}{EI}} + 0 = 0$$

Then I have $v = 0$. So, I have

$$C_1 + 0 + 0 + C_4 = 0$$

Then I have $M_b = 0$. So, when I do this, I get this as

$$0 + 0 - C_3\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L - C_4\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L = 0$$

L is outside. It is not inside the square root, equal to zero. So, the fourth condition is $V = 0$ That gives me

$$0 + C_2P + 0 + 0 = 0$$

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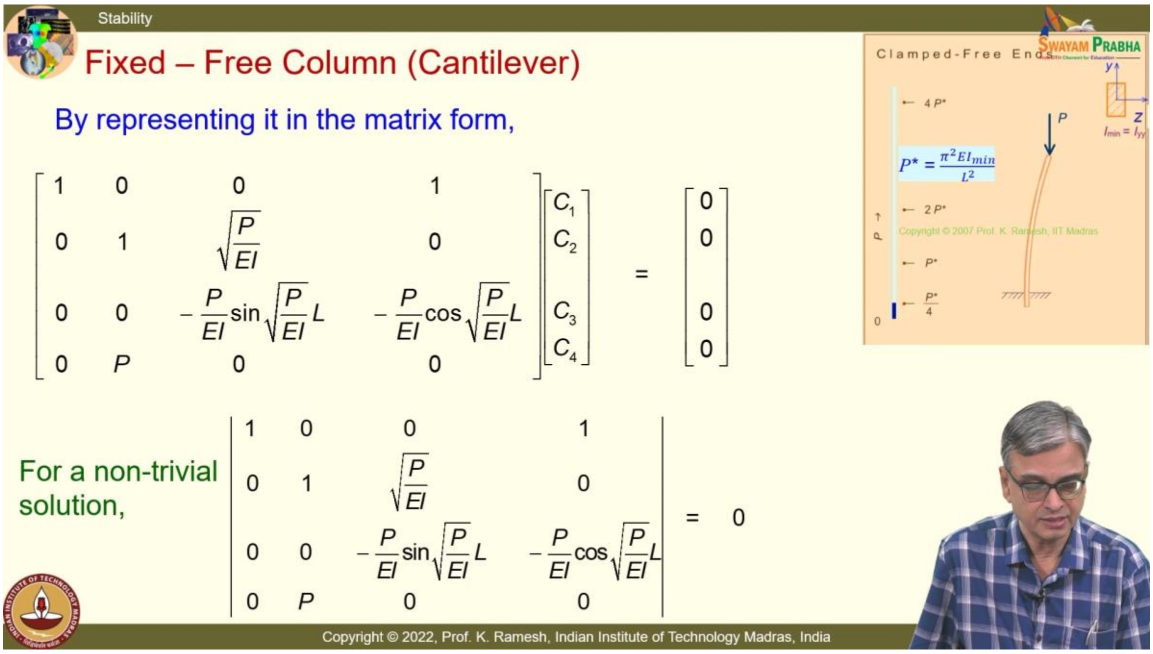

And this I can put it in a matrix form in a convenient fashion. And you have this equal to $[0 \quad 0 \quad 0 \quad 0]^T$. So, this is nothing but an eigenvalue and eigenvector type of problem. And for non-trivial solution, this determinant should go to zero because when the determinant is not zero, the trivial solution is C_1, C_2, C_3, C_4 are zero. That is a straight column. We want a non-trivial solution. So, the determinant should go to zero.

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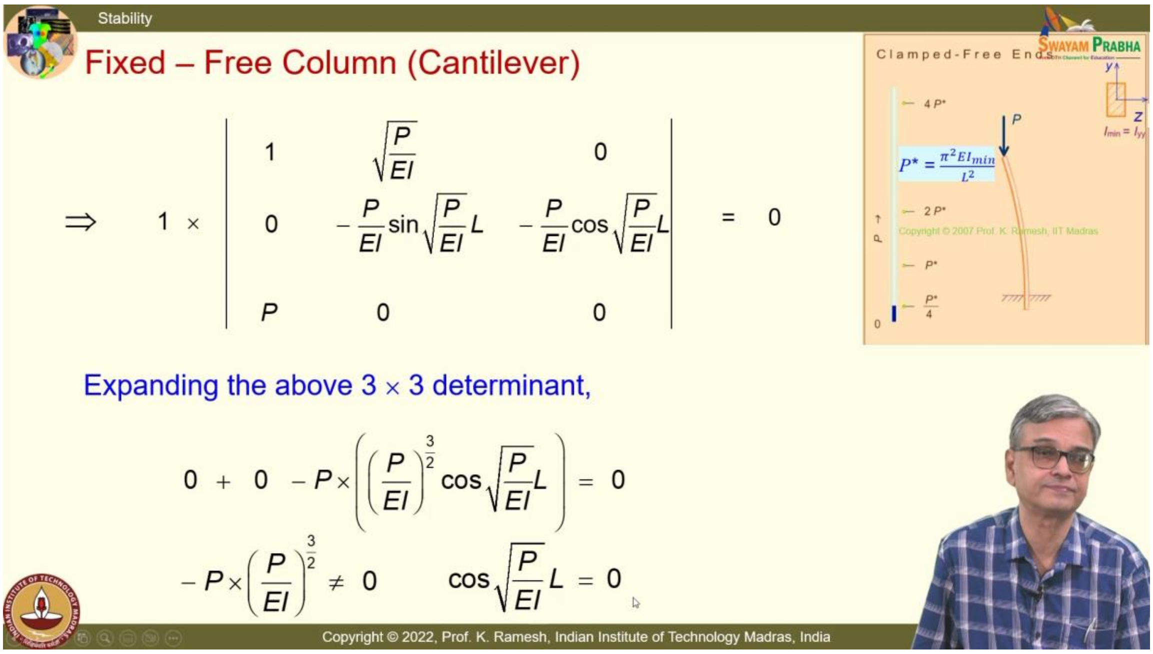

And you know, we can evaluate this determinant in stages. I think it is shown, ok. So, you can expand in the above 3×3 determinant. I get this as

$$0 + 0 - P \times \left(\left(\frac{P}{EI} \right)^{\frac{3}{2}} \cos \sqrt{\frac{P}{EI}} L \right) = 0$$

So, this is the final expression. See, P is the applied load and EI is the bending rigidity. So,

$$P \left(\frac{P}{EI} \right)^{\frac{3}{2}} \neq 0,$$

because it is a non-zero quantity. So, this tells us that

$$\cos \sqrt{\frac{P}{EI}} L = 0$$

So, that helps us to construct the solution, fine.

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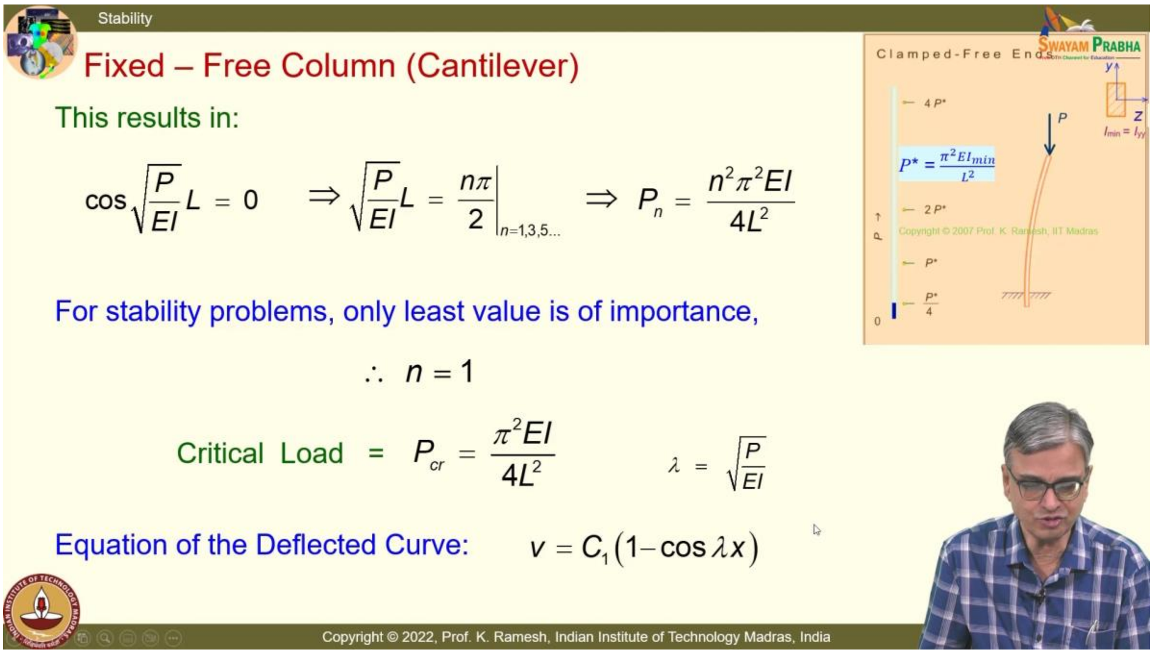

So, you have this and this is again a multi-valued problem, ok. Any inverse, you will have a multiple values. So, this says

$$\sqrt{\frac{P}{EI}} L = \left. \frac{n\pi}{2} \right|_{n=1,3,5...}$$

And you have

$$P_n = \frac{n^2 \pi^2 EI}{4L^2}$$

And you know, in this what we have done is, we have taken the solution for a hinged-hinged connection as P^*. Compared to P^*, all the values for fixed-free, fixed-hinged, fixed-fixed values are indicated. And you also have a rectangular cross-section, and your minimum moment of inertia is I_{yy}. So, explicitly in the expression, it is desirable to put I_{min}. If I have a circular cross-section, I_{min} is same as I at any one of the diagonals. But if you take a non-circular cross-section, you will have to understand that buckling or the critical load, you should look at for the least value. If you take another I, you will get another value, but that may be higher than the least value. You do not want the column to deflect, you do not want the column to buckle in a given application. So, $I_{\min}$ is a better representation, only least value is of importance. So, we will have n =1. That gives me critical load as

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

See, this is for the case of fixed-free column.

And equation of the deflection curve, when you go back and substitute these values, I get an expression

$$v = C_1\left(1 - \cos\lambda x\right)$$

and we know what is λ.

$$\lambda = \sqrt{\frac{P}{EI}}$$

How many of you are convinced with this solution? There is one catch still there. We have used four boundary conditions, fine. Have you been able to find out what is the value of C_1? C_1 is still indeterminate. What has caused this? See, when I apply the boundary condition, I should get one unique solution or not.

See, if you, we have done the linearization of the moment-curvature relation. If you do not make that as a linearized solution, if you take that as non-linear, then you will also be in a position to find out what is C_1. But nevertheless, whatever the solution that we have got, good enough for investigating instability, fine. But you should understand that C_1 we have not been able to catch, ok.

See, what you will not appreciate is experiment is truth. The moment the column has buckled, it is large deformation. We have said our hypothesis is small deformation and we will investigate. Even though it is small deformation, because we want to investigate stability, we go for a deformed configuration to write the governing equation. Until it has buckled, our analysis is all right. Once it has buckled, you have to go for improved mathematics to analyze post-buckling behavior. Suppose I apply a load higher than the P_{cr}, what happens? You cannot do analysis, but experiments can reveal it. Experiment does not get limited by your poor ability to solve your differential equation, fine. That is what we had seen in the experiment. We were able to see, we have applied load beyond the critical load. We saw deflected shapes. They are all neutral equilibrium positions. So, do not mix up the experimental result with your analytical analysis. Analytical analysis stops the moment you find the critical load. Beyond critical load, experiment gives you more information for you to ruminate over it, ok.

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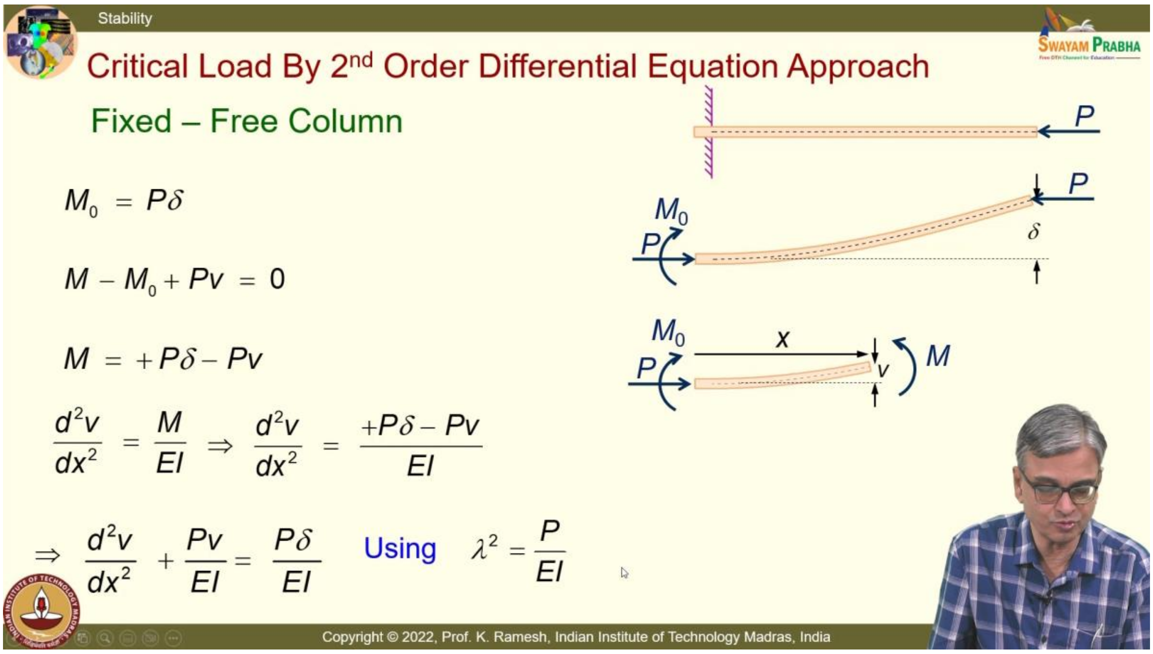

See, there is also another approach which is usually adopted by civil engineers. They use a second-order differential equation. It is discussed for a fixed-free column. You have enough background to do that, and this is a statically determinate problem. Whereas, if you have hinged-hinged, even hinged-hinged is statically indeterminate. Hinged roller support is statically determinate. So, all the other problems are statically indeterminate. While we developed deflection equations, we also developed $\frac{d^4v}{dx^4}$. We have developed that. We said that equation is useful even for statically indeterminate problems. So, that is why you find that equation is used for developing the critical load because except the fixed-free, all other situations are statically indeterminate. But there is also school of thought that people directly use the second-order differential equation approach. So, you have to write the equation when you have this deflected. Because of the deflection, this compressive load causes the bending moment $P\delta$. So, if you take an arbitrary section at a distance X which is measured from the fixed end, you have this as v and this as bending moment M. And $M_0 = P\delta$ that is obtained from this. So, if you write for this free-body diagram,

$$M - M_0 + Pv = 0$$

So, this reduces to

$$M = +P\delta - Pv$$

And then you have

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

which we have learnt in deflection. So,

$$\frac{d^2v}{dx^2} = \frac{+P\delta - Pv}{EI}$$

So, I can get

$$\frac{d^2v}{dx^2} + \frac{Pv}{EI} = \frac{P\delta}{EI}$$

And you know $\lambda^2 = P\delta/EI$, which we have already used.

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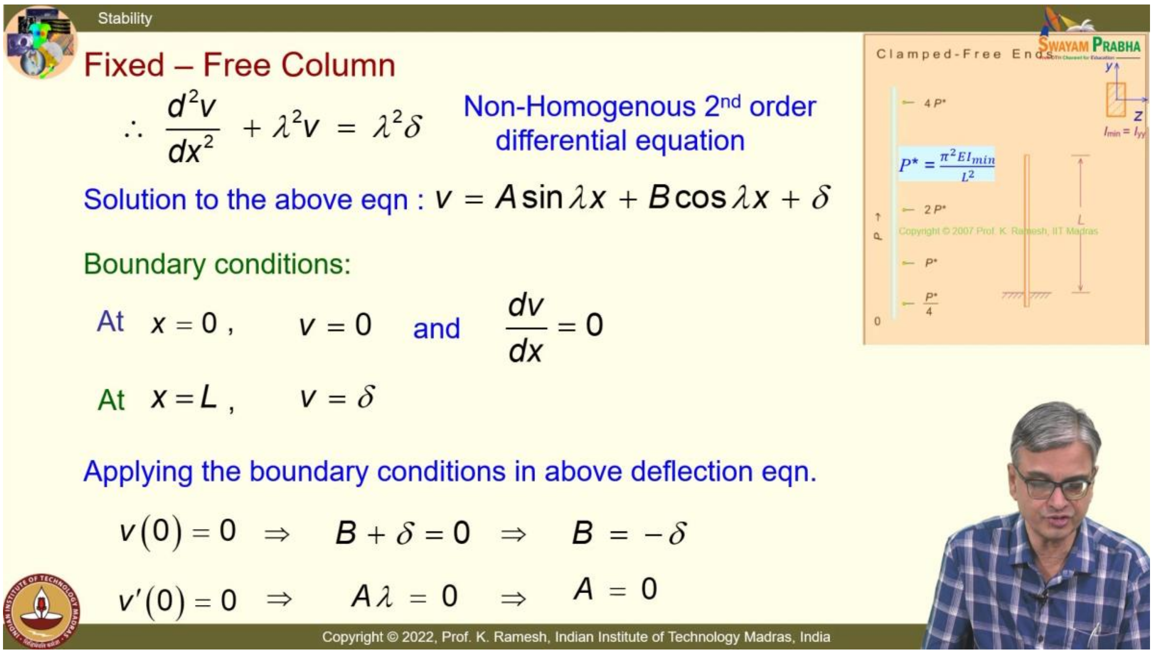

You can also find, rewrite this as

$$\frac{d^2v}{dx^2} + \lambda^2 v = \lambda^2\delta$$

It is a non-homogeneous second-order differential equation. Solution to the above equation is of the form,

$$v = A\sin\lambda x + B\cos\lambda x + \delta$$

And you have the boundary conditions

At $x = 0$, $v = 0$ and $\frac{dv}{dx} = 0$

At $x = L$, $v = \delta$

So, applying the boundary conditions in the above deflection equation, we get $v(0) = 0$ because this is straight forward mathematics. There is no difficulty, you have enough background you can in fact, do it much faster than what I am showing.

$$v(0) = 0 \Rightarrow B + \delta = 0 \Rightarrow B = -\delta$$

$$v'(0) = 0 \Rightarrow A\lambda = 0 \Rightarrow A = 0$$

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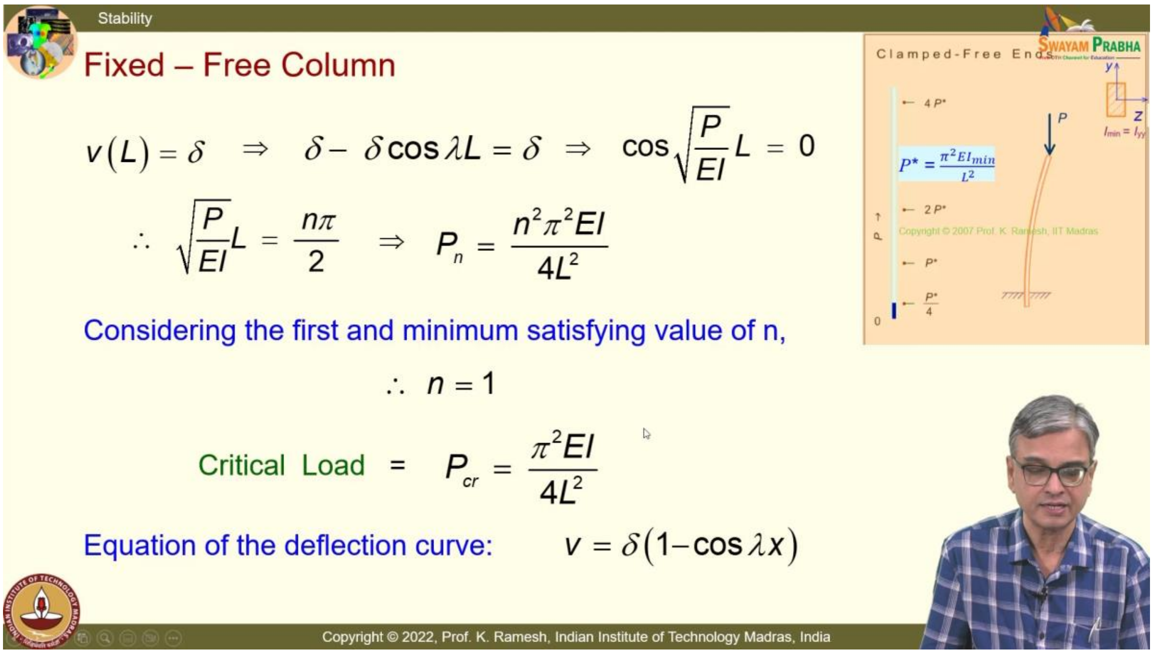

So, when I substitute all of this,

$$v(L) = \delta \quad \Rightarrow \quad \delta - \delta\cos\lambda L = \delta \quad \Rightarrow \quad \cos\sqrt{\frac{P}{EI}}L = 0$$

Identical same solution like what we have got. And you get the critical load as

$$\sqrt{\frac{P}{EI}}L = \frac{n\pi}{2} \quad \Rightarrow \quad P_n = \frac{n^2\pi^2 EI}{4L^2}$$

Considering the first and minimum satisfying value of n, $n = 1$, I get the critical load as

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

Equation of the deflection curve, so

$$v = \delta(1 - \cos\lambda x)$$

So, this is another form of analysis.

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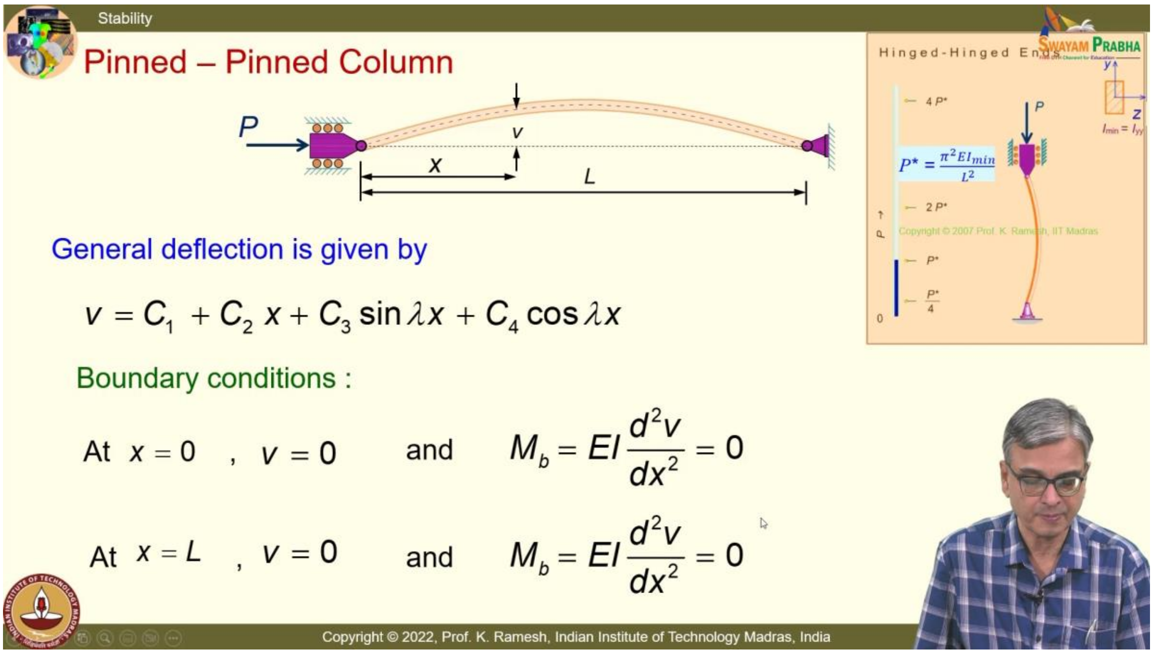

We can do the similar analysis for a pinned-pinned column. What is the difference? The difference is only the boundary conditions. You have the governing equation; you have the basic solution, and you have the expressions. So, it is just substitution of the boundary condition and get this.

$$v = C_1 + C_2 x + C_3 \sin \lambda x + C_4 \cos \lambda x$$

And you have the boundary condition

$$\text{At } x = 0 \;,\; v = 0 \quad \text{and} \quad M_b = EI \frac{d^2v}{dx^2} = 0$$

$$\text{At } x = L \;,\; v = 0 \quad \text{and} \quad M_b = EI \frac{d^2v}{dx^2} = 0$$

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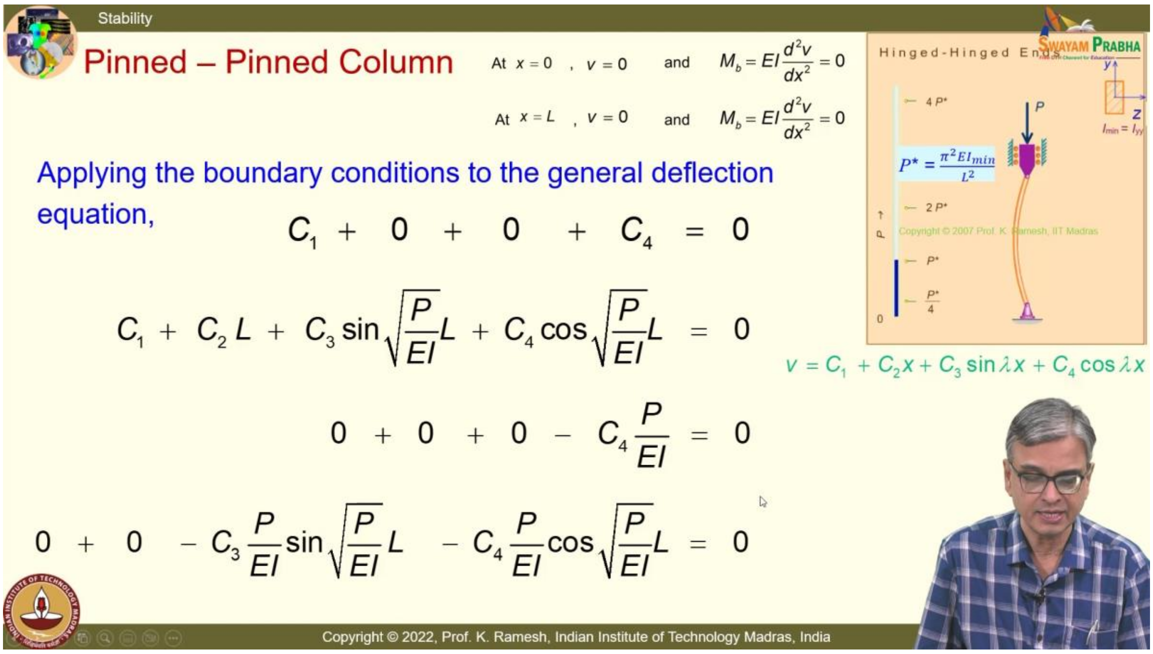

So, once you substitute this, I get this set of equations. It is very easy for you to do that. Please work it out in your rooms. That is also a practice for you.

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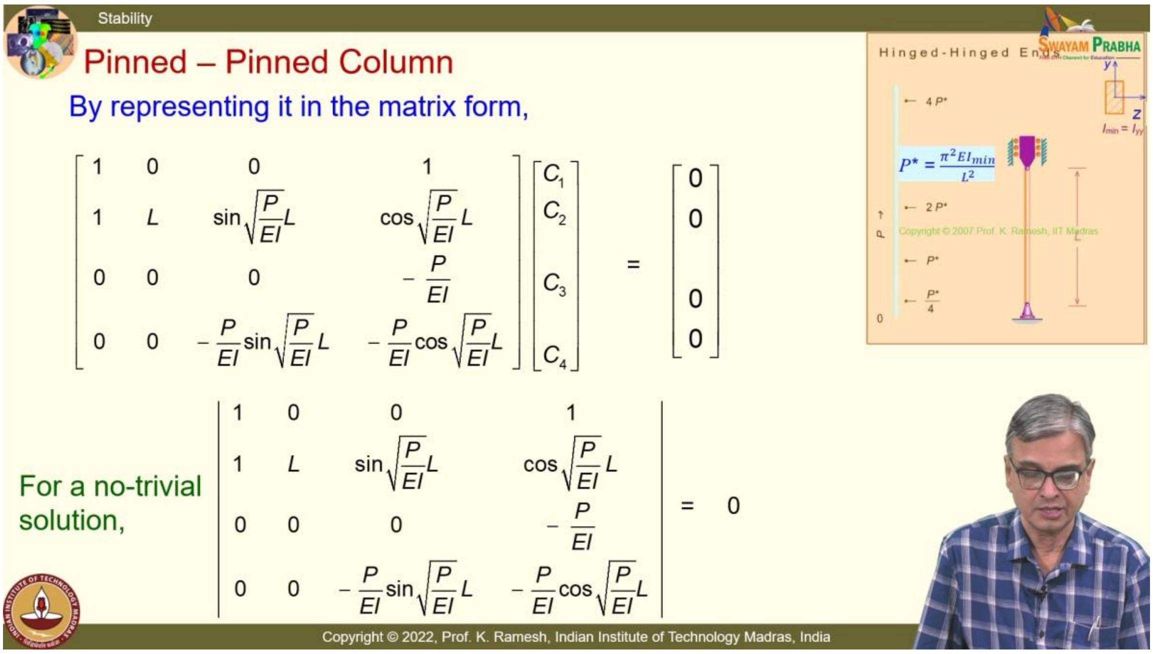

And I, for non-trivial solution, I want the determinant to go to zero. And that will give me the basic expression for me to find out and investigate what are the eigenvalues and eigenvectors. For non-trivial solution, this determinant should go to zero, ok.

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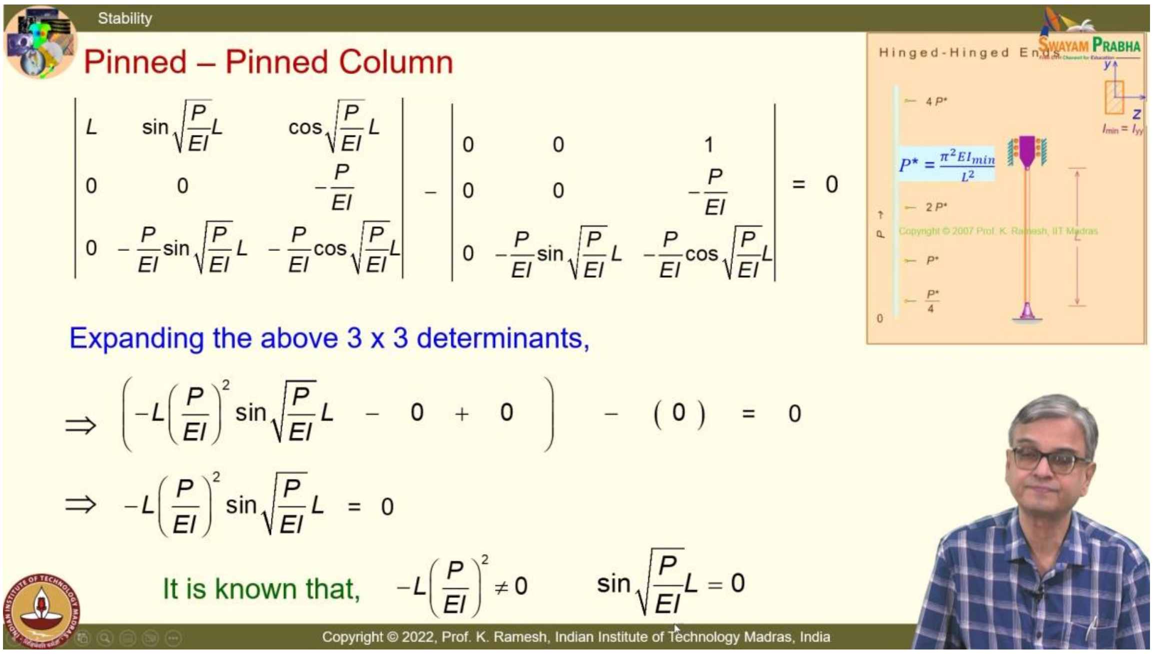

And I have this, this gives me finally, I have an expression like this, and this gives me

$$\sin\sqrt{\frac{P}{EI}}L = 0$$

And once you investigate this, it will also be, all inverse functions are multi-valued.

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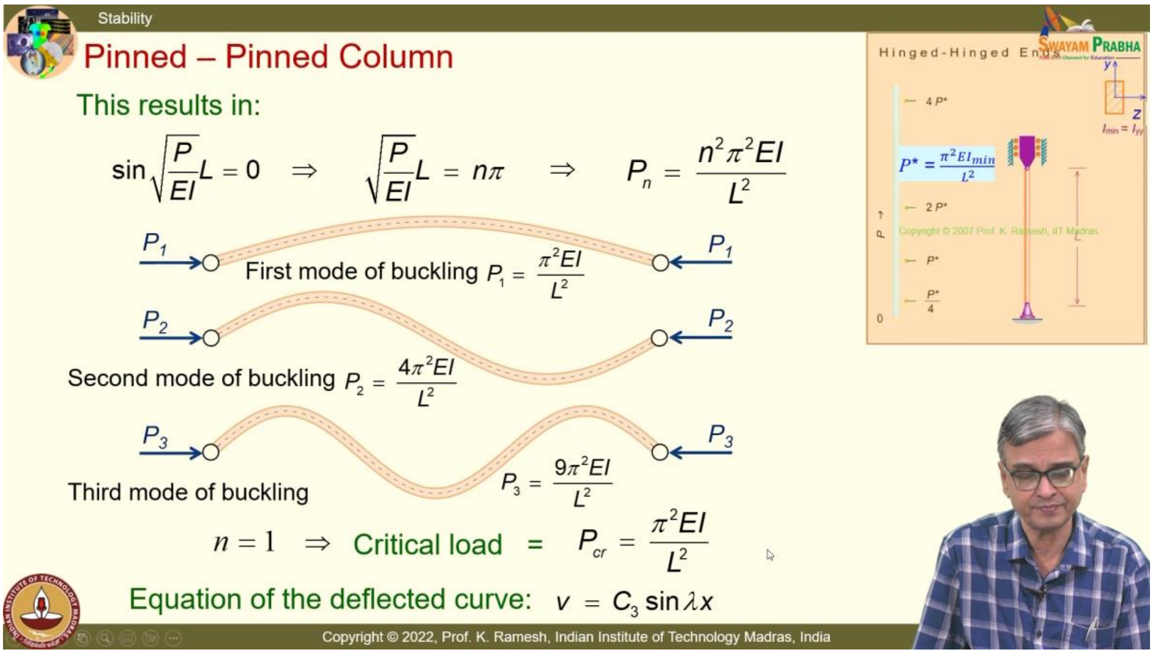

So, I have

$$\sqrt{\frac{P}{EI}}L = n\pi \quad \Rightarrow \quad P_n = \frac{n^2\pi^2 EI}{L^2}$$

And this is a very basic expression, which is used for comparing what happens in fixed-free, that is like a cantilever beam. So, if you look at, this shows what is the critical load for a cantilever. It is one-fourth of the critical load in pin-pin condition. When I have a fixed and pinned, it is twice the load. If I have the fixed-fixed, it is four times the load of the pin-pin condition. So, that is how the graph is plotted. And this simulation is also good. It shows you the buckled shape alternately. To remove the mental block, that buckling happens only towards your left or right. It is precipitated by inherent defects or even external disturbances. If there is a breeze going from one side to another side, that can also precipitate buckling. If the wind direction changes, if the wind direction changes, so the buckling is precipitated by the disturbance and also dictated by the $I_{\min}$. You should have both in your mind.

And you know, when I have $n > 1$; when $n = 1$, you have this simple say first mode of buckling. I have this as

$$P_1 = \frac{\pi^2 EI}{L^2}$$

And when I have $n = 2$, I get the second mode.

$$P_2 = \frac{4\pi^2 EI}{L^2}$$

And when I have this $n = 3$, I have the third mode. See in vibration problem, all of these modes are important. In the case of buckling, higher vibration modes are difficult to get unless you do prompting of the nodes experimentally, ok. And my TAs, I have done a very good experiment. I will show you that also. And you have the

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

I would urge you to write this as

$$P_{cr} = \frac{\pi^2 EI_{\min}}{L^2}$$

That is what I have shown it in my animations here. Now, let me look at this deflection curve.

$$v = C_3 \sin \lambda x$$

So, you are not in a position to find out what is the magnitude of C_3. That is because you have linearized the moment-curvature relation.

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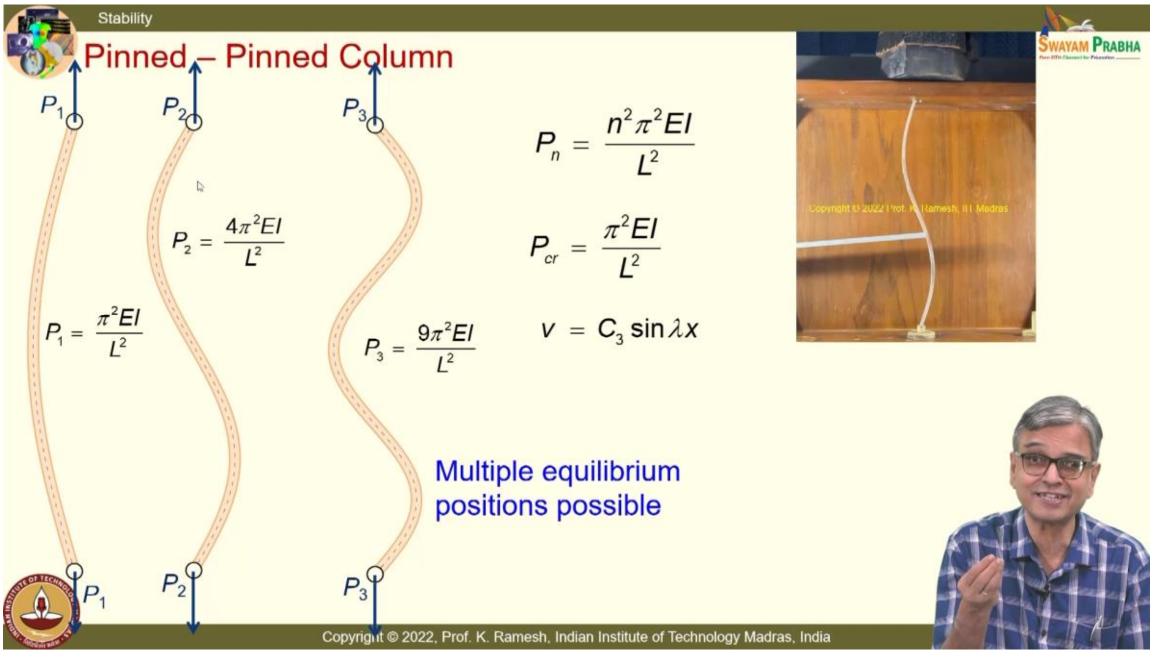

And you have multiple equilibrium positions possible. And let us see the experiment. You have to put your restraint at the node. For the given load, these higher-order positions are not realized in practice unless you prompt the structure to have that. So, this is a very nice experiment my TAs have done. And it shows you, you can have this shape. You understand? You can have this shape. And so, there were mathematical analysis not for fun. What you see? It is too difficult to realize that you can have the column bent like this. And long time back, we had also seen torsional mode of vibration of the Takoma bridge. You will not realize that concrete bridge can have such motions, fine. So, experimentally it is possible provided the experiments are done carefully.

So, in this class, we have looked at stability. And I said the stability is an issue wherever you have compressive load. From a mathematical analysis point of view, we will confine our discussion to column buckling. And because we are investigating stability, we have to write the governing equation only in a deformed configuration. We do that to find out the critical load. The moment I get the critical load and the column has buckled, though I can see deformation very clearly in my experiment, you do not have the mathematical analysis to get the deflected shape. That requires higher order theories for you to do it. Thank you.