Strength of Materials

Prof. K. Ramesh
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture - 35
Theories of Failure 1 Overview

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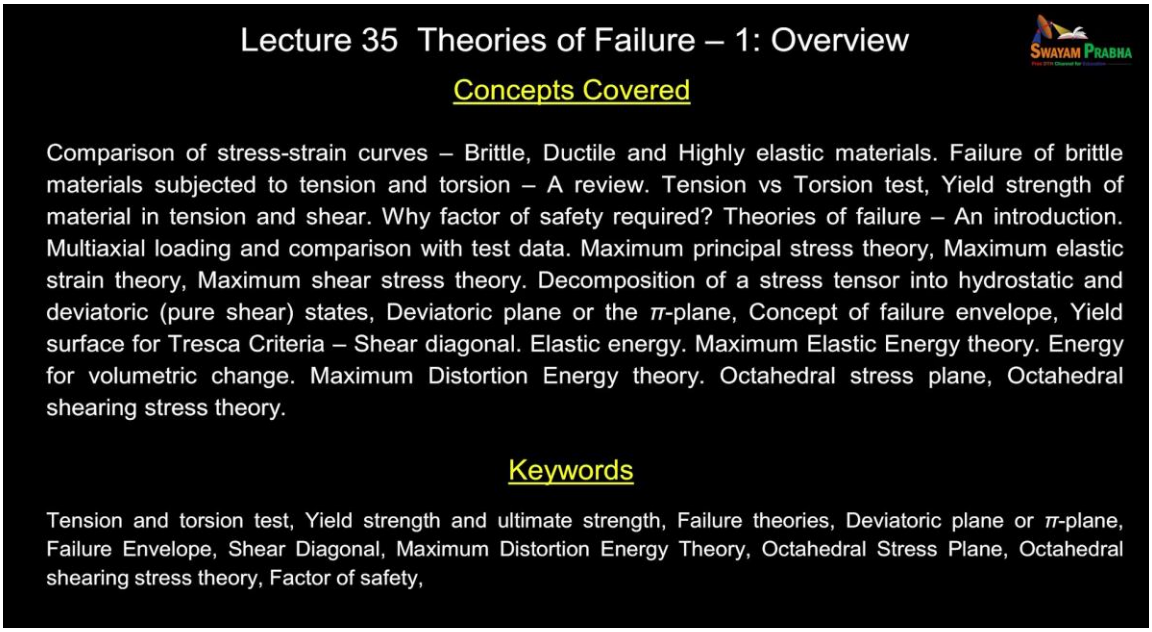

Let us move on to the next topic on theories of failure. In fact, this would provide a very good recapitulation of many of the concepts that we have developed in the early part of the course. So, it will also help you to be well prepared for the final examination.

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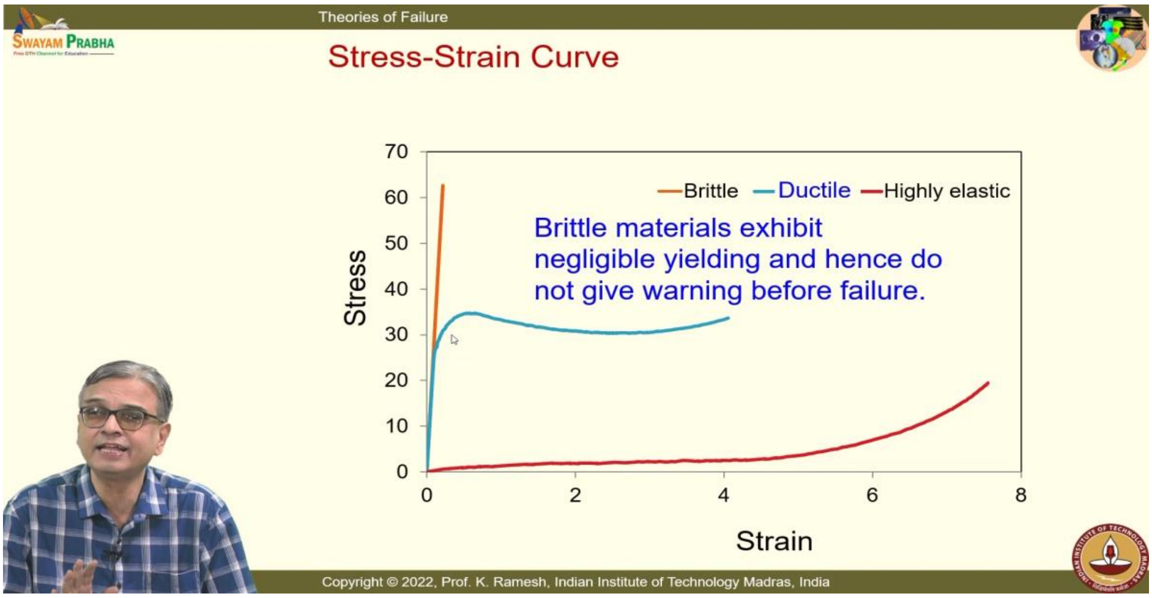

See, we are going to look at what is that we look for in the case of brittle materials, ductile materials. And though highly elastic material; stress strain curve is drawn, we are not going to pay attention on that in this course. So, if you look at a brittle material, you are interested because it does not give you any warning before the failure.

It exhibits negligible yielding and you will only try to get the ultimate tensile or compressive strengths. On the other hand, when you go to ductile materials, you know, yielding is considered the undesirable feature because when you have components that are working together with a particular tolerance, if you have large deflection-deformation, it is going to affect the performance. So, when you handle the ductile material, yielding is considered as to be avoided.

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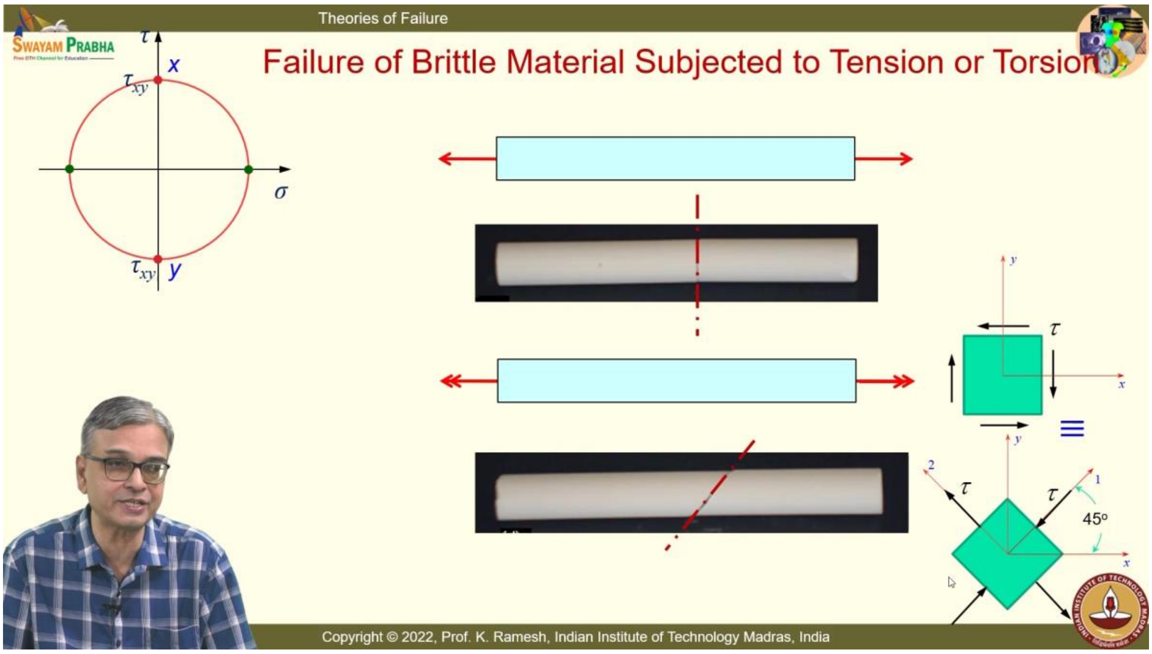

Then you know, we have also looked at what is the need for stress vector and how stress vector is evolved into stress tensor. Ultimately, you have to interpret when I apply a tension, the chalk breaks in one particular fashion. Why it fractures in this plane? On the other hand, when I apply torque, it fractures at approximately 45 degrees. It depends on because when you do the application of the load with hand, you may not apply a pure torque, fine. If you put it in a machine, it will always be exactly 45 degrees. And you all know that when I apply torsion, you know very well that you have a pure shear stress state introduced and you know how to draw the Mohr's circle for this.

So, it is all review. You know torsion produces pure shear stress state. Then you mark the *x*-plane and *y*-plane, following the convention that we use. We need that convention so that any rotation in the physical plane, I can correlate it with the Mohr's circle. And I could visualize this pure shear stress state as a combination of tension and compression.

And when you say a brittle material, it fails by maximum normal stress. So, you have normal stress here. So, that introduces a fracture at 45 degrees. Chalk piece does not have an intelligence, but the stress phenomena what you see introduces a normal stress at 45 degrees when you apply torsion. If I change the sign of torsion, then you should know how to find out the complementary plane. You should find out what is the plane. You have to find out what is the state of shear. This you convert into tension and compression. If you look at here, you know these two arrows means so it reduces to tension. And this also goes to tension and the other two becomes compression or you draw the Mohr's circle and find out how to read it. Either way you can do it.

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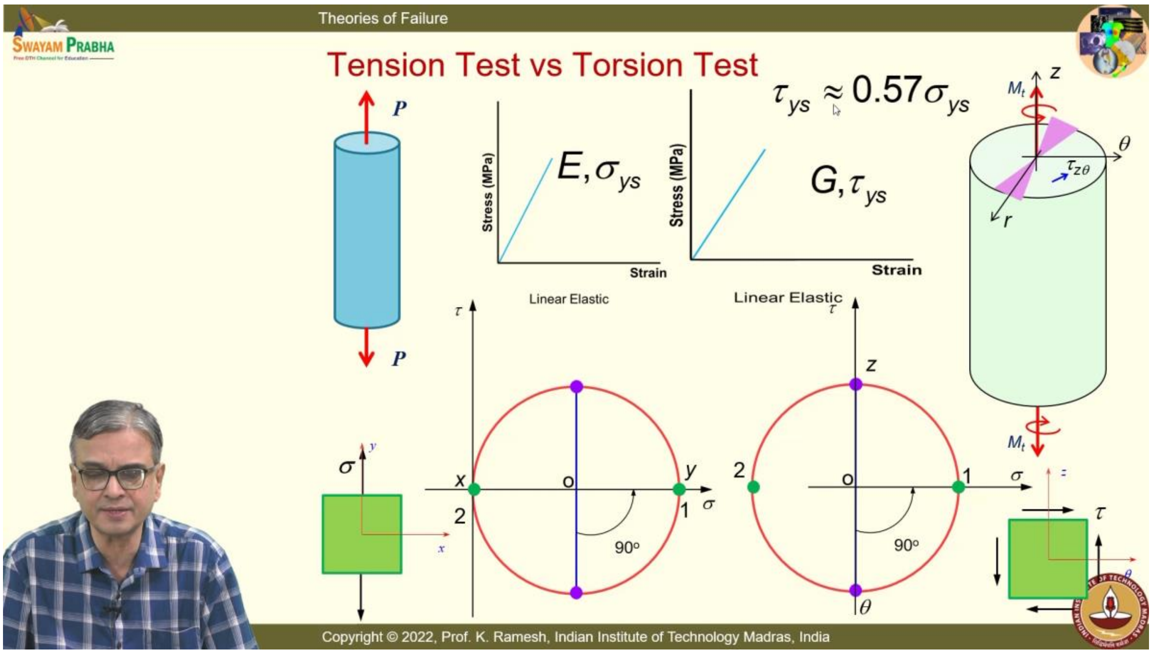

Then next question is, you know in this course we have focused on tension test and when you have a tension test, you know how to draw the Mohr's circle. So, you have the principal stresses and you also have planes of maximum shear stress. And you are in a position to evaluate from the tension test, basically Young's modulus and yield strength. If you want, you can also find out how to get the Poisson's ratio.

And you can also do a torsion test. See, the accuracy achievable in a tension test is much more than in a torsion test. Torsion test is also difficult to do and you have a state of pure shear and you know how to draw the Mohr's circle for pure shear stress state. And from a torsion test, you essentially get the shear modulus and the yield strength in torsion, fine; shear stress. Instead of σ_{ys} , you call this as τ_{ys} . And you know the slopes will be different because you know $G = E/2(1+\nu)$. So, the slope will be smaller than what you have seen in this, ok. And you also have interrelationship; it is approximate, this about 0.57 of σ_{ys} .

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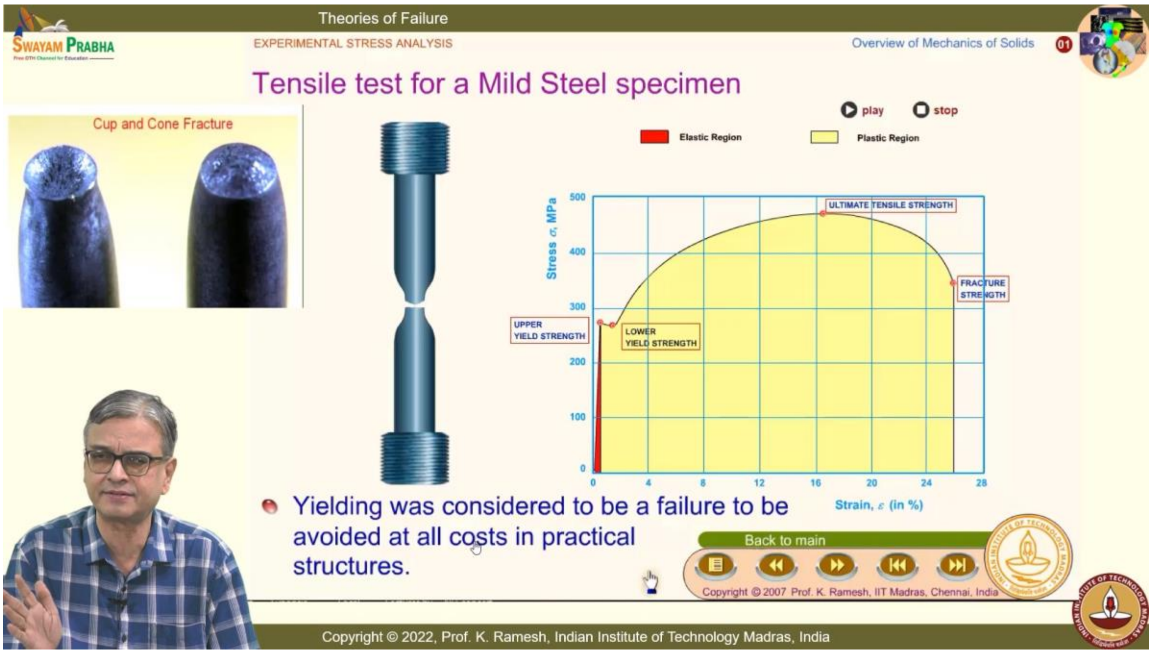

And you know we have looked at, if I take a specimen and then pull it, this is taken from a bulk material. So, I have a linear region which is very small, then I have a yield region, then I have a strain hardening region, then reaches the peak, then you have necking starts. When necking starts, fine, eventually the specimen fractures. And you know in material modeling, we never attempt to model the complete stress strain curve, because many of our applications are confined to linear region; only the elastic region which is very, very small. We have seen the maximum strain that you allow for it to find out yield, when yield is not very clear, is 0.2 per cent.

So, you have visible change in cross-sectional area. And related to this, we have also looked at what is a true stress and true strain graph, because near necking you find that this, there is a drastic reduction in the cross-section. So, you have to take that reality, and you also have a very characteristic cup and cone fracture. If you look at these angles, it will be 45 degrees. That you can explain from your Mohr's circle, because you have shear reaches the maximum, ok.

So, yielding was considered to be a failure, and it should be avoided at all cause in practical structures, was the initial philosophy about design. Now, we never go anywhere close to yielding stress, you will have a factor of safety. You will have a design stress and design stress is determined from factor of safety, which has multiple ways; multiple considerations go into how to decide the factor of safety.

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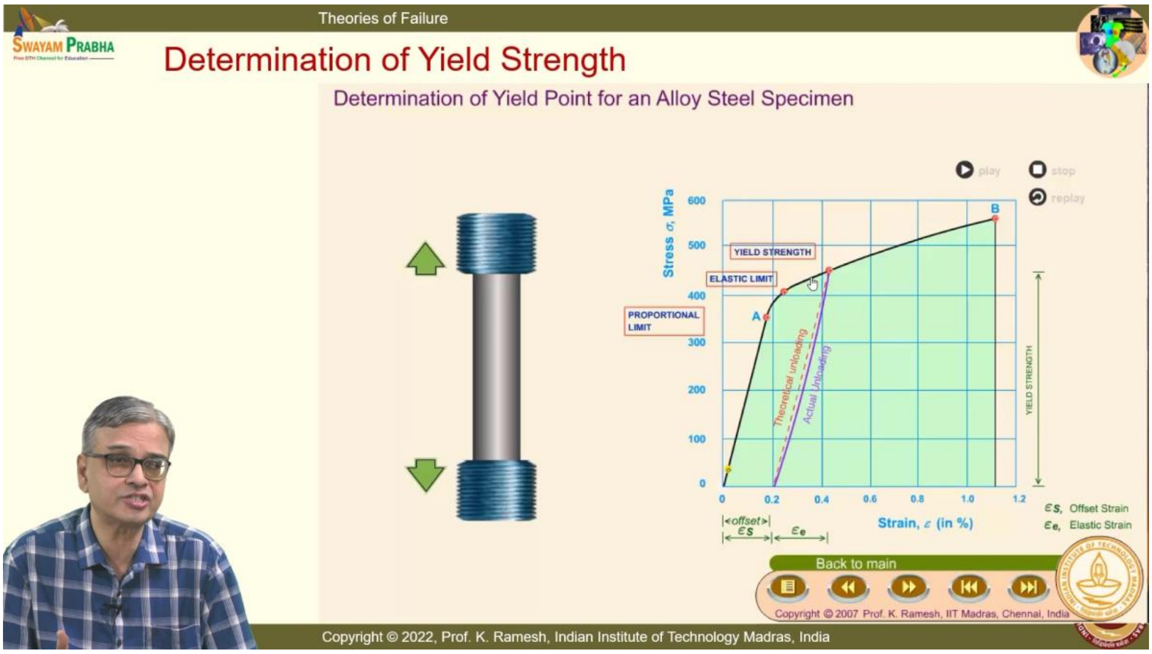

And as I said, suppose I have an alloy steel, which does not show any sharp change to identify what is the yield strength, you have a non-linear curve like this. The recommended practice from codes is - you have an offset strain of 0.2 percent, you draw a line parallel to the initial slope, and whatever that it meets, this value is taken as a yield strength.

And you also have a very marked point *A*, and another point which is the elastic limit. You have relationship is linear up to this, from *A* to elastic limit, it can be non-linear, fine. So, the offset strain is very important. So, you allow a small amount of plastic deformation, when you are not in a position to find out the yield strength by a sharp change. You take that as a yield strength for all practical purposes.

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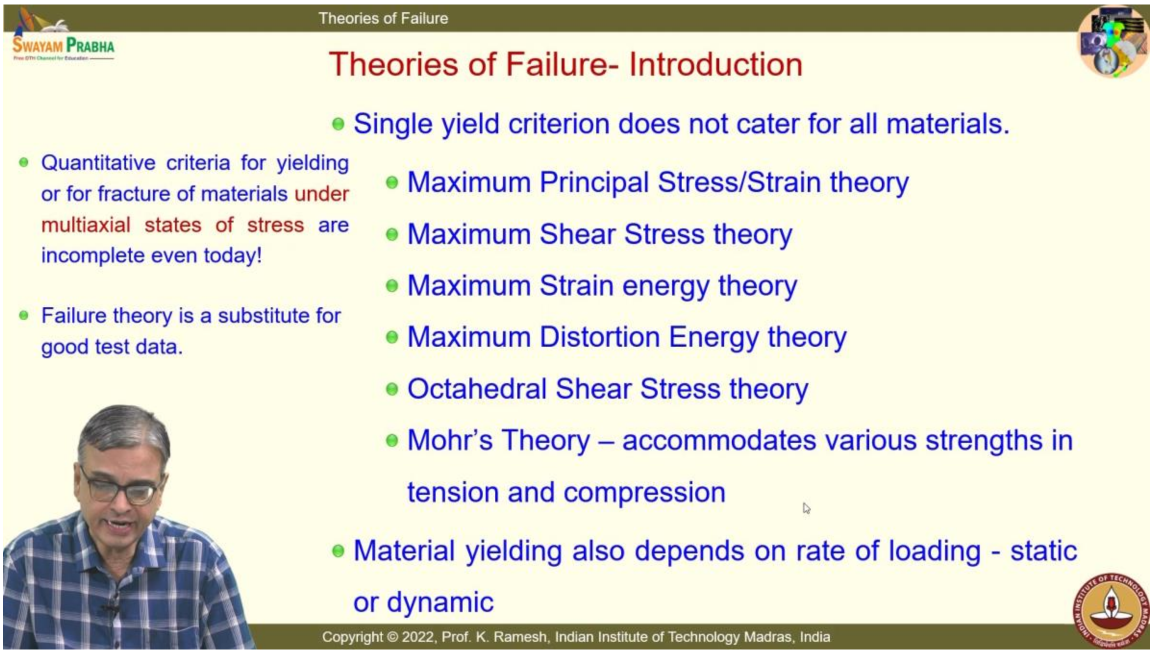

And you know, this is a very, very important statement that you should write it. Quantitative criteria for yielding or fracture of materials under multi-axial states of stress are incomplete even today. See, if you go to a practical structure, it has a very complex type of loading. Even in your bench, the way you come and sit, the load changes, you may sit smoothly, you may jump on the chair, all that. So, the loading is a very, very complex aspect to model, and many of the practical structures like you take an aircraft, if you have a gust of wind; how the air frame is exposed to variable loads. One of the biggest challenge in engineering modeling is how to idealize the load acting on the structure.

And you know, you cannot reproduce all that happens in nature, which happens in multi-axial fashion. You cannot do it in a laboratory. And what we have cleverly done is, we want to make our life simple and cost reasonable. Just do a simple tension test, extract maximum out of it. And you know, you have seen brittle materials and ductile materials, you cannot have one criteria for all of these materials. Even among the brittle materials, you need multiple criteria; even among the ductile materials, you need multiple criteria, because each material behaves in a different fashion. And from an academic perspective, we may see all of these theories, and you should also recognize failure theory is a substitute for good test data. If I have a good test data, I would put more emphasis on the test data. See, you must have heard some of the cars that developed in India, they go for a crash test. And then you hear these cars have survived the crash test, and some of the models have not survived the crash test. So, ultimately for very critical situations, the manufacturers have dedicated, highly sophisticated expensive test facilities; you should understand all of this. If you want to do that for every component you want to manufacture, then only kings can buy anything, you and I cannot go and buy. So, you have to look at intelligently that I take a simple tension test, evolve a failure theory, which substitutes test data for a variety of routine situations.

Any special situations, you should do exhaustive test, design new test rigs, all that people do, that is how engineering has developed. And once you look at this, you know we have studied principal stresses and strain. So, people have looked at what happens when principal stress or strain reaches a maximum value. And we have also looked at shear stress, what happens to the shear stress reaches a critical value. Then we also said when we graduate from rigid body mechanics to deformable solids; they can all store energy.

So, one way of looking at this, what happens to maximum strain energy. Then people said no, no, no strain energy is not the right choice, you should look at only the distortion energy. And somebody said you look at a very special plane called octahedral plane on which, what happens to the shear stress. You have really looked at what was an octahedral plane when we studied Cauchy stress formula to find out stress factor on any plane of interest. And you also have Mohr's theory, which we have discussed; brittle materials have different tensile and compressive strengths. So, Mohr's theory accommodates various strengths in tension and compression.

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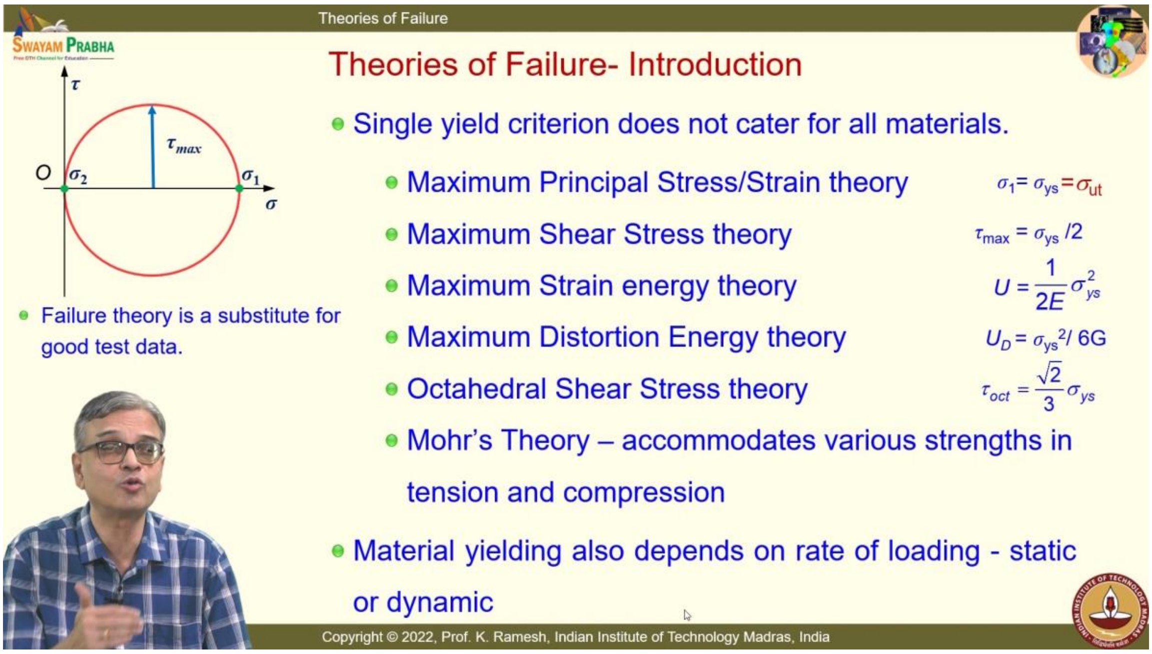

So, what you do is, you do a simple tension test. And in a simple tension test, note down all of these extremum values. You have σ_1 reaches σ_{ys} and usually the maximum principal stress theory is applied only for a brittle material. So, it is more appropriate to talk this as ultimate tensile strength.

So, one can also find out what is the maximum shear stress in a tension test, what is the maximum elastic energy, what is the distortion energy and what is the octahedral shear stress. Ultimately, we will zero down into only three theories, one for brittle material and two for ductile materials out of these. But from an academic perspective, you must know what the theories are, what has been the story. And you know, you should also appreciate that material yielding also depends on rate of loading, whether it is static or dynamic. I have said that in this course, we are going to look at loads that are gradually applied. Even the test is done, you take a specimen of as per the standards and apply the loads gradually at room temperature. If you want to work at extreme temperatures, whether it is high temperature or low temperature; you have to go back and see what are the current understanding of materials at those temperatures, fine.

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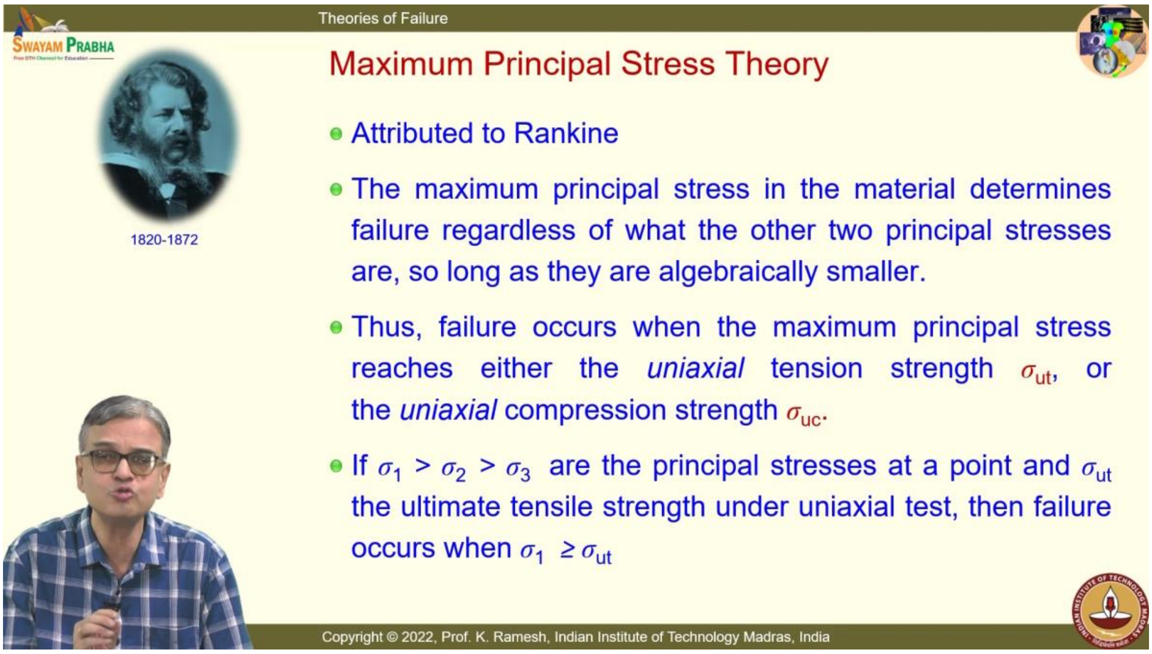

And the maximum principal stress theory says, you know, when you have the maximum value it reaches regardless of the other two principal stresses, as long as they are algebraically smaller, failure happens. And you know, I told you in those days, people looked at whole of science as physics and mathematics. There is nothing like the divisions that you have right now. And you know, Rankine was very famously known for Rankine cycle in thermodynamics. So, this theory and you should also look at the time period in which it ought to have been developed, is one of the earliest theories. When people had started using materials, people also were intelligent; there is something happens, failure occurs. Something reaches the critical value, failure occurs. So, it was baby steps that they were doing, ok.

So, the failure occurs when the maximum principal stress reaches either the uniaxial tension strength σ_{ut} or uniaxial compression strength σ_{uc}. And ultimately, all of these failure theories should explain what is observed in the experiments. If it is observed in the experiment that this is what is happening from the theory, you accept the theory or use it for that material, ok.

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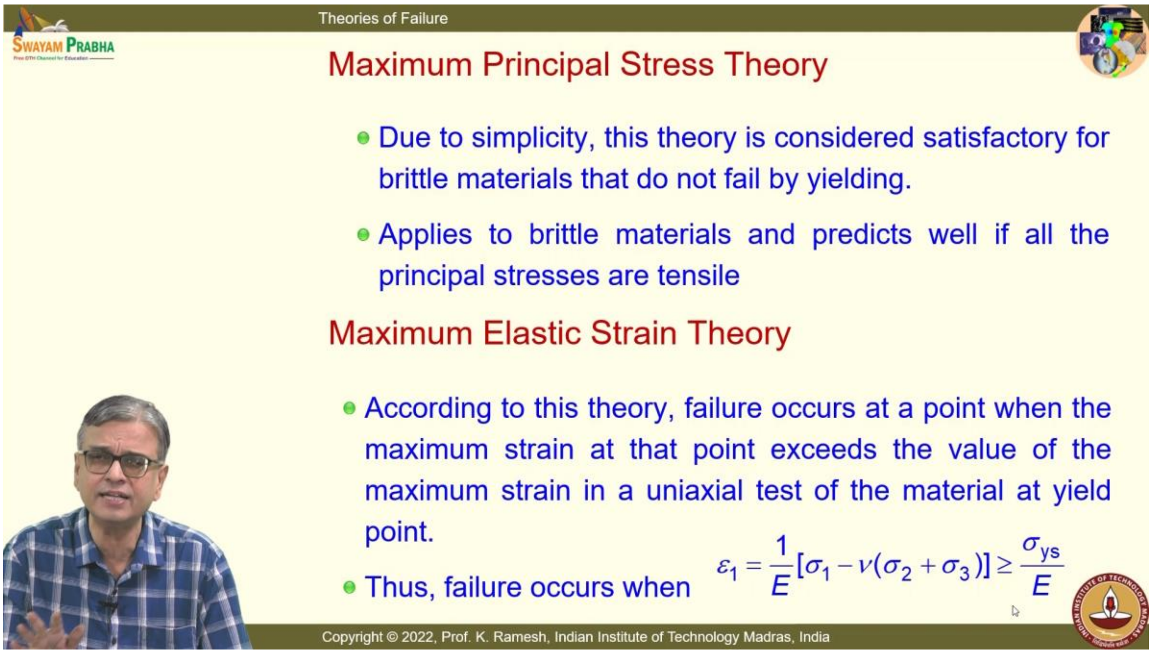

And you know, it is a very simple theory satisfactory for brittle materials and if all the principal stresses are tensile. This is the thumb rule that you have for the principal stress theory. And people thought, why not I also look at elastic strain. And all of these theories are done in principal stresses.

See, when we have looked at combined loading, we were able to combine the loads by looking at the coordinate system appropriately and get the final stress tensor. So, the combined effect you can get it as a final stress tensor. From the stress tensor, find out the principal stresses. And the principal stresses have to be organized in the way algebraically largest is σ_1, intermediate is σ_2 and algebraically the least is σ_3, ok. So, here the name itself says elastic strain. So, you take,

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \geq \frac{\sigma_{ys}}{E}$$

because you are talking about failure. You should not exceed this value is the proposal by the theory, fine. And you know, maximum elastic strain theory is not well supported by experiments.

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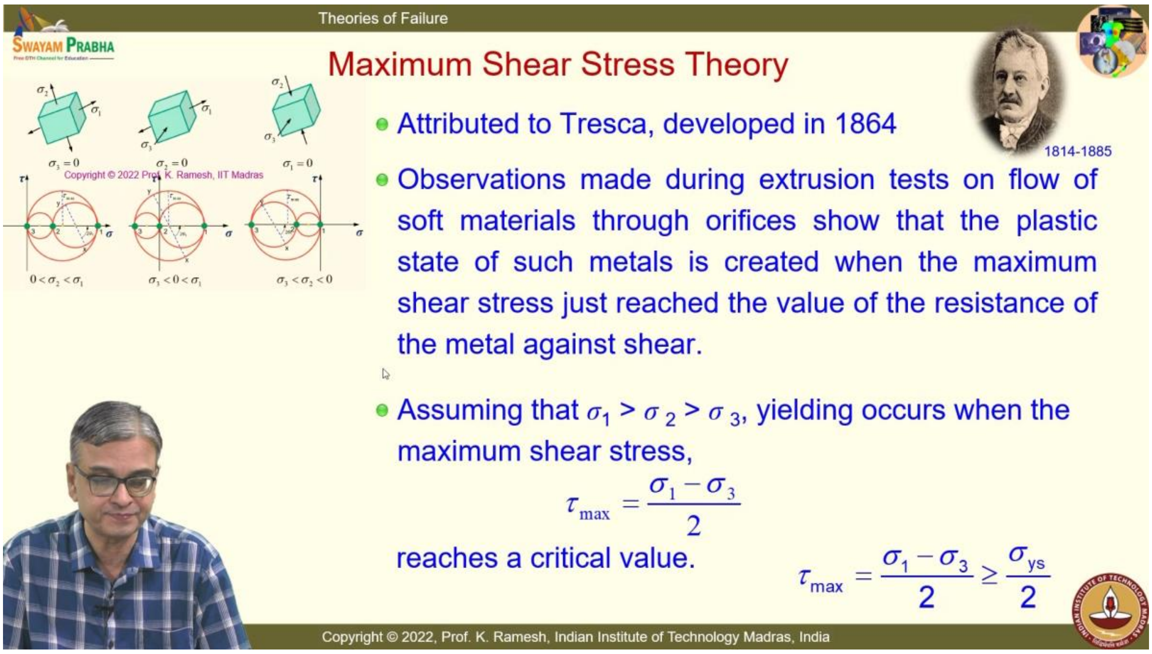

And you also have a theory developed for yield material, I mean ductile materials. And this is attributed to Tresca, which was developed in 1864. This is one of the earliest theories for ductile materials. And it is very simple to use, because the mathematics is very simple. Observations made during extrusion tests on flow of soft materials through orifices show that the plastic state of such metals is created when the maximum shear stress just reach the value of the resistance of the metal against shear.

So, it is a practical observation. And I said extrusion is a common metal forming operation that people use; your I sections are extruded. They are available at standard dimensions. And if you translate into mathematics, it essentially says,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

If you understand it like this, you will never make a mistake. What is shown here is, we have discussed, suppose I have both the principal stresses positive in a two-dimensional scenario and both the principal stresses negative in a two-dimensional scenario, the third principal stress being zero, plays a nuisance value. I have cautioned even your simple thin cylinder — the coke can, experiences biaxial state of stress, where both the principal stresses are positive. So, you should not simply say, use this as $\sigma_1 - \sigma_2$ and then say that this is τ_{max}, whereas it is $\sigma_1 - \sigma_3$ with this is a τ_{max}. So, you can make a drastic error in your calculation, if you do not arrange the principal stresses in this fashion and interpret it in this manner. Do not have it as $\sigma_1 - \sigma_2$. You can also say $\sigma_{max} - \sigma_{min}$, but it is always better that you write it in this fashion, $\frac{\sigma_1 - \sigma_3}{2}$. You will never make a mistake.

Then in those days, people had only access to slide rules, they did not have even calculator, all computers came much later. Now, calculators are very smart, they can also do some symbolic computation these days. So, in those days, if I can predict failure by calculating just a difference or the principal stresses, it was a great boon for all of them. And you should also look at it was developed way back in 1864.

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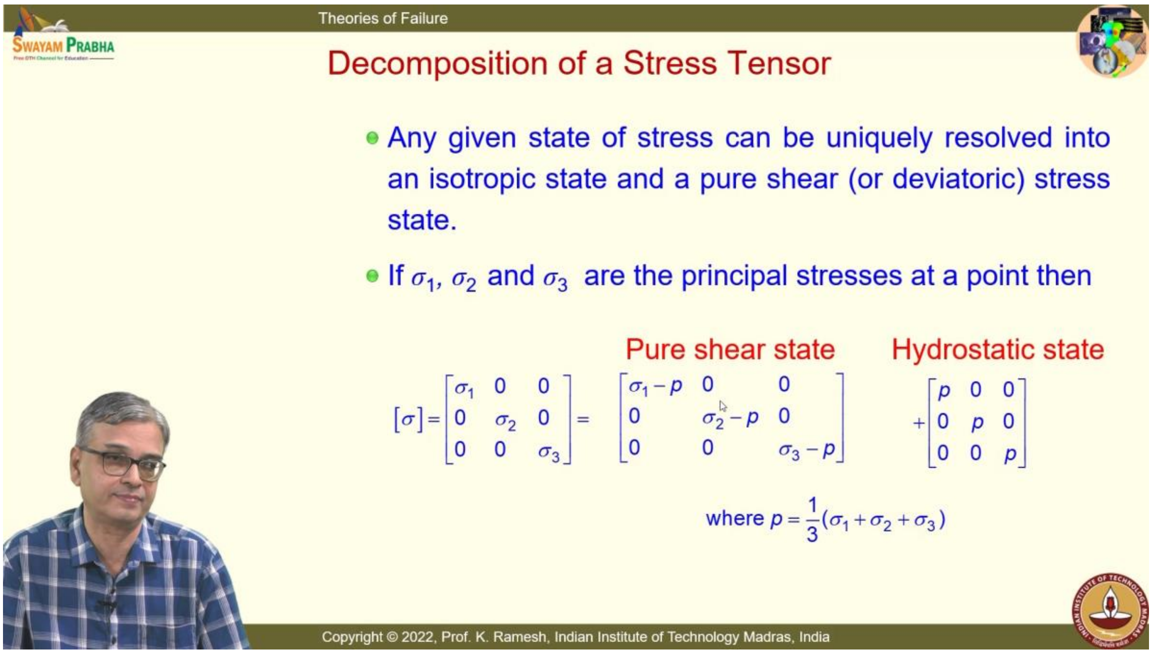

And you know, when you want to understand what is the distortion energy theory, you will not appreciate that a given stress tensor can always be split into sum of two quantities that we have seen. I am going to redo that, I have put the stress state in terms of principal stresses σ_1, σ_2, σ_3. I can always write it as addition of two matrix quantities, when you add them, you get this value. Can you tell me what this represents? I have

$$p=\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$$

I have the stress tensor given to you and I have told you by looking at the invariants, you can qualify whether it is the uniaxial state of stress or a pure shear state of stress. Louder! How? On what basis you said it is pure shear? $I_1 = 0$. You should know. So, it is the review of your course, fine.

So, I have this as hydrostatic state that is like immersing it in a fluid and this is a pure shear stress state. What we will have to look at is, the argument is, if you want to calculate the total energy using this, the energy can be split as one corresponding to this, one corresponding to this. We will do all these calculations today and this you call it as deviatoric stress. That is the way the literature recommends this. So, one theory says when this energy reaches the maximum, failure occurs, ok.

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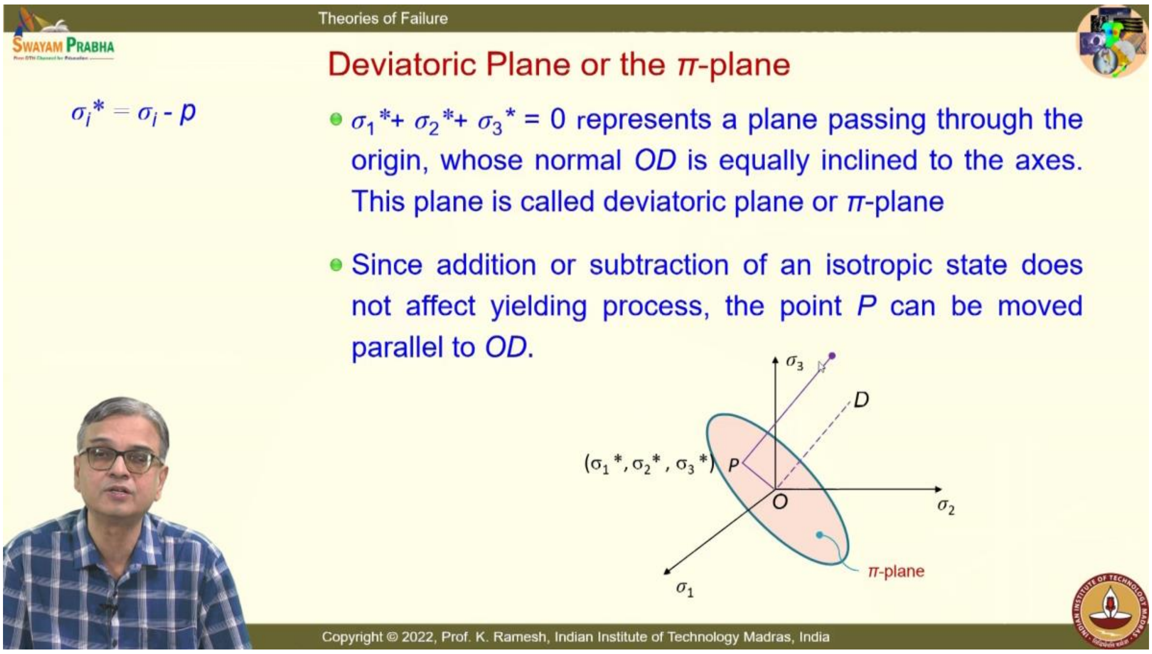

And you also have what is known as a deviatoric plane or the π-plane and which is listed here. Can you see that you have also seen that in a different context earlier? You are referring this plane with respect to σ_1, σ_2 and σ_3, and this is equally inclined. Can you recall what is the other plane that you have read earlier? Octahedral shear, octahedral plane. Octahedral plane is also like this. They are both are you call it one context as octahedral, in another context you call that as a π-plane, ok.

Since addition or subtraction of an isotropic state does not affect yielding process. See, there was lot of development on plasticity theories. And people have simplified the mathematical modeling. And one of the important aspects there is isotropic plastic theory is what they would say and the simplest one, you have a hydrostatic state of stress. If I add any of that, it does not affect the yielding behavior. So, the point P can be moved parallel to OD. So, if I do that, I can also construct a three-dimensional shape of a failure locus. See, the idea is I am performing only a simple tension test, but I will interpret it in terms of σ_1, σ_2 and σ_3, and try to plot a three-dimensional envelope. And if my real stress state in a given problem lies within it, no failure will occur. If it is outside or on the boundary, yielding will happen. That is how people have proposed the failure theories.

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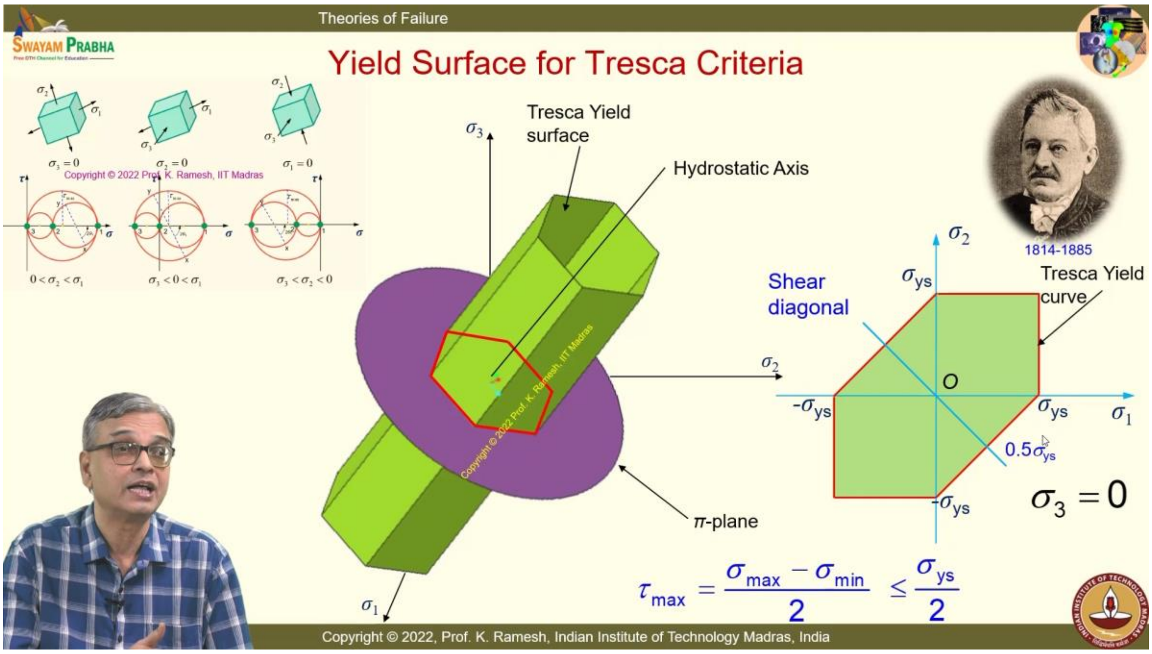

So, you have this, we will see the what is the yield surface for Tresca. And I have the π-plane. And if you look at the yield surface for Tresca, this is the hexagon; very nice appearance of the failure envelope. So, this is called the hydrostatic axis. And if you have the stress state within this envelope, you will not have yielding to take place, which can be easily understood if I do it on a two-dimensional representation. Suppose, I take a projection of it on σ_1 and σ_2 axis and when I look at here, I plot it only on σ_1 and σ_2. The projection of this hexagon will be elongated like this, because it is on an equally inclined plane—the hexagon. When I project it, I will have it like this and this you can easily interpret. See, I do a tension test applying σ_1, when σ_1 reaches σ_{ys}, you also have τ reaching the maximum shear stress, ok.

So, I can locate this point. Similarly, I can also locate this point, whether I apply σ_1 or σ_2, it does not matter. If I have both σ_1 and σ_2 is equal, then also I can draw the Mohr's circle between these two principal stresses and the σ_3 as zero. So, I can locate this point. And when I have one of the stresses is compressive, then the curve becomes like this. The green region shows in any given practical problem, if the σ_1, σ_2, σ_3 values; and σ_3 value is zero, that is how it is plotted.

And σ_1, σ_2 lies within this region, yielding will not take place. It is on the boundary or outside the boundary, yielding would have taken place. You make a sketch of this; you need to have these diagrams. You need to have these diagrams; you should also know how to interpret. And I am going to present it in a manner that is useful to you for your understanding.

Because you know, people also plot the failure envelope for maximum normal stress theory, maximum principal stress theory in this. That becomes irrelevant, because when we have brittle materials, they always have tension and compression strengths are different. So, we will have a different diagram for that. We will not mix it up with ductile materials. For a ductile material, one of the simplest and oldest criteria was by Tresca. It was developed in 1864. It is easy to employ, and you have a failure envelope in two dimensions and failure envelope in three dimensions.

And you also have what is known as a shear diagonal. Shear diagonal is nothing, but I have a pure shear stress state and your Tresca yield criteria says, this value is $0.5\sigma_{ys}$. Why do you have multiple criteria? The material is so complex, you are not able to grasp what the material is doing by one simple criteria. It is a modeling aspect, fine. Each material behaves in a different fashion. When you do a complicated test, when you do any testing other than tension test, it is going to be expensive. Tension test is the simplest test that you can do. Even that is costly, that is not cheap because you have to prepare the specimen as per ASTM standards. You should have a calibrated testing machine and follow all the procedure and evaluate the parameters. And what we do is, we exploit what we have got in tension test carefully done in terms of hypothesis or proposals. That is how failure can occur. Later on, we will also see experimental verification of these theories.

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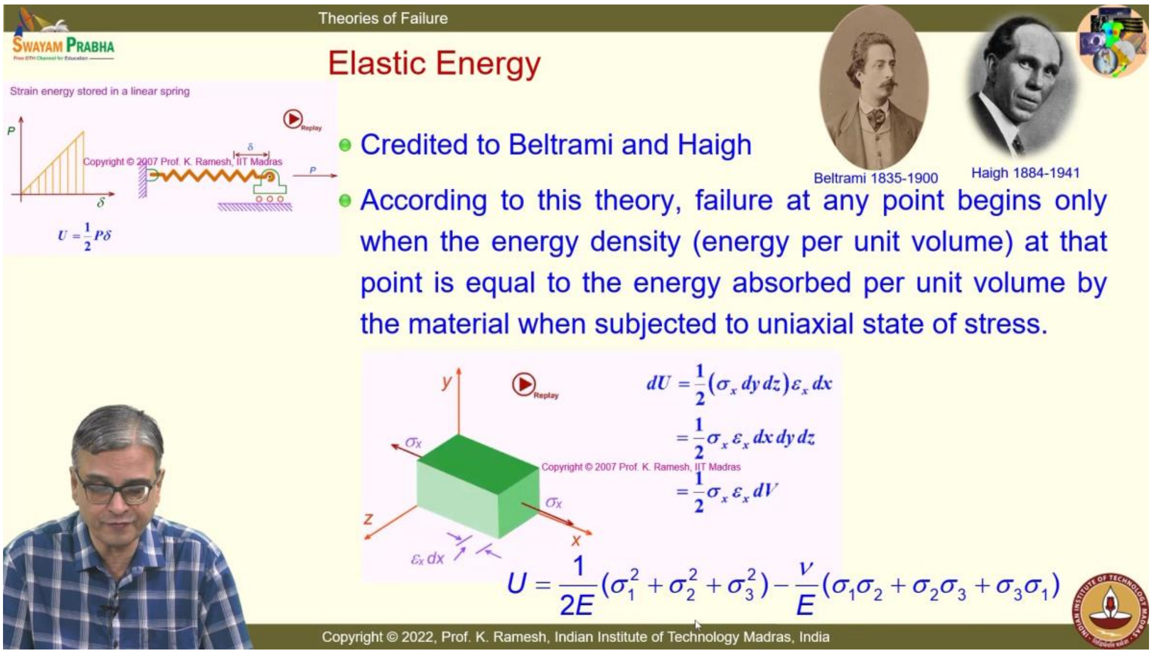

And now, you know, we will also have to look at how to calculate the elastic energy. We have already discussed when we were talking about Castigliano's theorem, how to find out the energy stored in a spring. And this is credited to Beltrami and Haigh. See, you look at the time period, you do not think somebody proposes a failure theory and it stays for years together. Somebody proposes certain elements of it, somebody else comes in the later, generalizes it and makes it more perfect. So, you can imagine it has taken two lifetimes, fine.

And according to this theory, failure at any point begins only when the energy density at that point is equal to the energy absorbed per unit volume by the material when subjected to uniaxial state of stress. These are all like, you know, when you are proposing a theory, you feel instead of concentrating on other quantities, let me concentrate on the energy.

That is how they proposed it. In all the theories, you are going to say what happens in a tension test. That is going to be common. But the quantity that what you look at is different. Whichever quantity that has worked in real life, that is the final theory that people will use it for design, fine.

But from an academic perspective, you should look at all the theories. And you know how to calculate the energy when I have it in terms of stresses,

$$dU = \frac{1}{2}\sigma_x \varepsilon_x dV$$

Now, I want you to work it out. Suppose, I express the stress state in terms of $\sigma_1, \sigma_2, \sigma_3$, you can also write $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Can you substitute and then tell me what would be the expression for strain energy? Please do the computation. It is not very difficult for you. It is also like brushing up your understanding of stress strain relations because when you write the strain, you should write the strain completely. ε_1 is influenced by σ_1, σ_2 as well as σ_3. Only in a uniaxial tension test, if I am having σ_{yy} only applied, then ε_{yy} is, you have σ_{yy}/E. Otherwise, in a generic situation, strain is influenced by all the three stress components.

So, if you do that, you will also get a very simple expression. It is not difficult for you to visualize once I show. I have the strain energy labeled as capital U. It can be organized in this manner,

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

That is what you get it as total elastic strain energy. Now, we will see what is the value, if I have σ_2 and σ_3 is zero, what is the value at uniaxial tension test? So, that tells you what is the criteria. So, that criteria getting is very simple.

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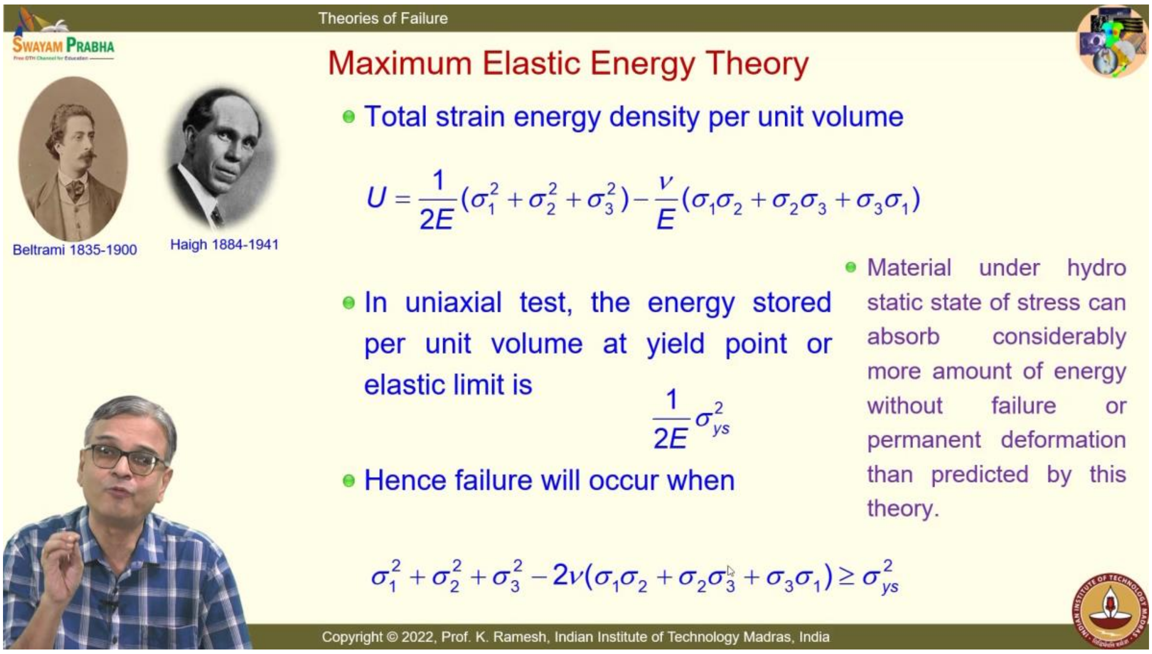

Right now, what we are saying is what happens in a tension test? You say that as the limiting criteria. Whether it is valid or not, only an experiment can say, fine. So, you are only looking at different entities. So, when I have a simple tension, it is reduced to

$$\frac{1}{2E}\sigma_{ys}^2$$

Hence, failure will occur when you have this is greater than or equal to σ_{ys}^2. But if you go to the test, what happens?
See, people have observed materials can withstand very high levels of hydrostatic stress. Nothing happens to them. Even if you take a brittle material like glass, you take a glass bulb and immerse in a liquid. If you do a careful experiment, see experiments have to be done carefully. What people have found is, it is able to withstand a very large pressures. Nothing happens to it. And if you, when you release the pressure, if you are not careful, failures people have observed. And we have also looked at and made an idealization that we have an elastic continuum.

In reality, nothing is an elastic continuum. You have micro cracks. So, what we have people have found is, if there are micro cracks and a fluid gets into that micro cracks, then failure has happened much earlier. So, one of the why I made the people suggested is, you cover this material with a thin metallic cover, metallic sheet, so that the liquid does not get into the crevices. Even glass has withstood very high hydrostatic stress. So, this is the reason people thought your elastic strain energy is not the right index for the failure analysis. You have to look at the distortion energy. Is the idea clear? You have to move and see distortion energy, because the experiments do not support the predictions. The theory simply says, instead of looking at 10 different things, look at the elastic strain energy, see what happens in the attention test. That is how people have coined different theories. What theories have stood the test of time? If you look at whichever is supported by experiments, people have adopted it.

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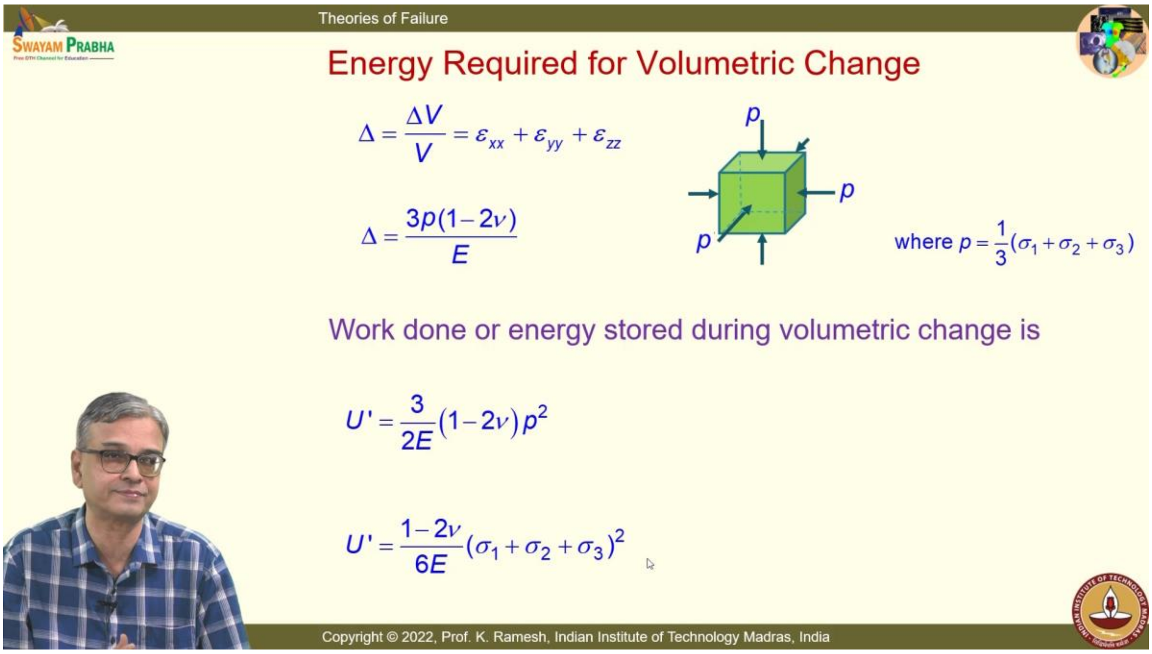

And when you want to find out distortion energy, we have already looked at when we develop stress strain relations, when we are talking about bulk modulus. We have said, you can find out the volumetric strain as

$$\Delta = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

in the context of principal stress reference. It is the invariant I_1. It can also be written in terms of $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. It is one and the same and invariant does not change. So, I have this pressure applied and

$$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

And if you write ε_{xx}, ε_{yy}, ε_{zz}, they will be identical in terms of p. So, we have evaluated this. If you remember those derivations, it is nothing but

$$\Delta = \frac{3p(1-2\nu)}{E}$$

So, now, I have the strain, I have the load, you know how to calculate the energy stored. So, what I am really doing is, what is the energy stored for volumetric change? So, it is nothing but

$$U' = \frac{3}{2E}(1-2\nu)p^2$$

and I substitute p. See, p has 1/3, ok. So, when I put p^2, I will have 1/9. So, this simplifies to

$$U' = \frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

So, what you have here is, what is the energy because of volumetric change? If I have to find out the distortion, what is that I have to do? I have the total energy; I have the volumetric change. So, if I do the subtraction, I can find out the distortion energy. Because our goal is to get the distortion energy, which is more realistic in capturing the material behavior.

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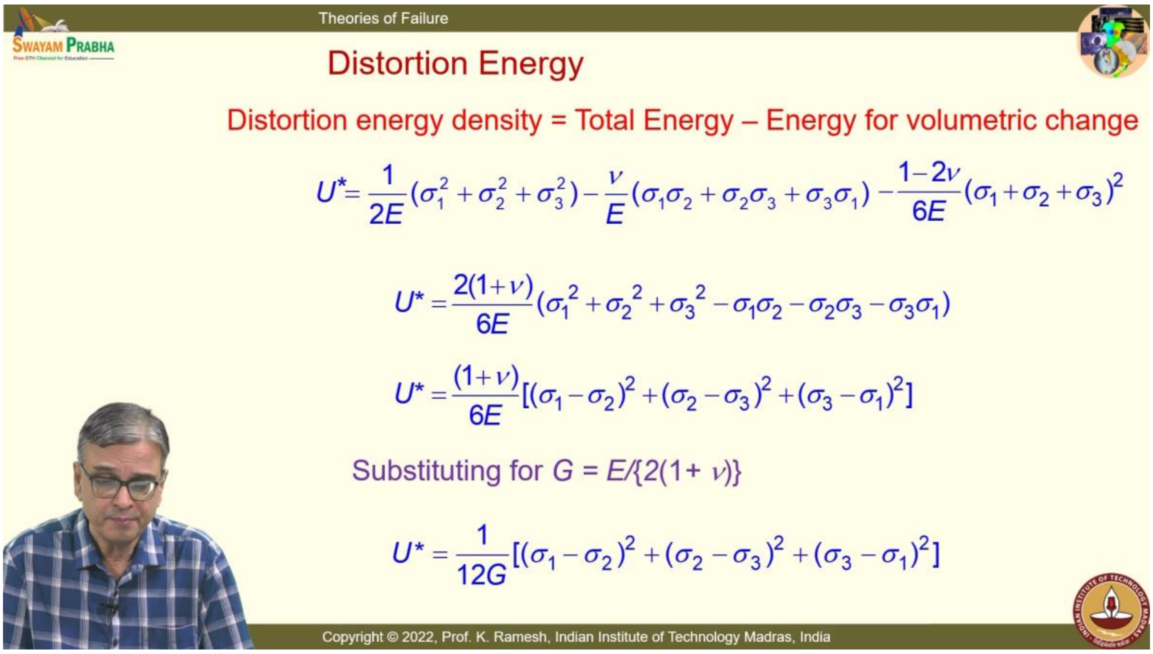

So, the distortion energy density is nothing but total energy minus energy for volumetric change, and we have already seen distortion energy is given in this context U^*. We saw U^* in a different context. We used that as a complementary strain energy. So, some of these symbols due to paucity of it, you must understand their interpretation in the local context. So, I have the total energy, which we have derived, which is very simple to do. Just now, we have derived what happens because of the volumetric change. So, if I do the arithmetic, this simplifies in a very nice fashion. This is the first simplification, but the second simplification is much interesting for us to do, which could be rewritten in this fashion

$$U^* = \frac{(1+\nu)}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

Please remember this. This we will be using it for all our failure theories, that form of representation, which is much more convenient. And now, what you will have to see? You have to see what happens in a simple tension test and then see what is the limiting value. Before that, you know, I am still replacing it. I do not want to have $(1+\nu)$ here. I can write it in terms of one elastic constant, which is G. So, I can write

$$U^* = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

It is also very easy to remember. It is not difficult to remember. Mnemonically, you can understand.

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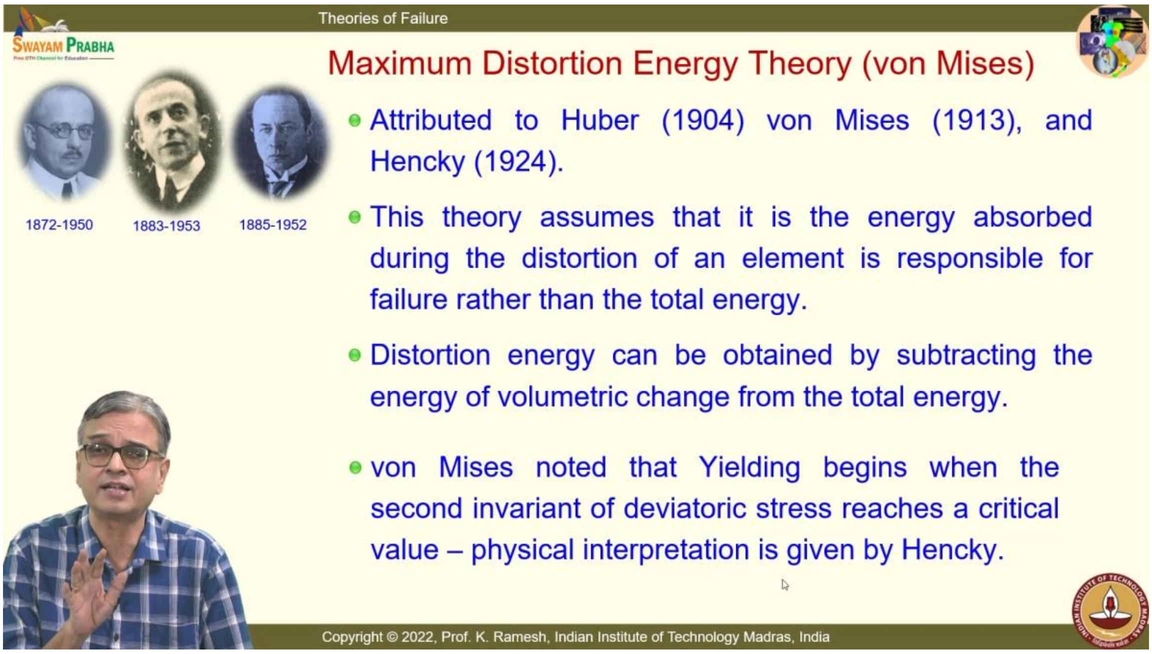

And if you go to the history of development, many people have contributed to this understanding. It is not just one. You have Huber, who proposed it in 1904, improved by von Mises in 1913. A better explanation was given by Hencky in 1924. Imagine, we have also seen bending theory took 400 years. Even the failure theories, it is not that somebody gets a dream. The next day morning, he says this is the failure theory and everybody follows. A dictator can do that. Science does not operate like that, fine. So, people propose the theory, somebody perfects it, somebody provides a rational explanation, fine. So, that is how the theory has got developed.

And so, the theory is very simple to express also, because in all the theories, we will look at one physical quantity. And when it reaches the value that is happening in a tension test, we say failure occurs. So, the energy absorbed during the distortion of an element is responsible for failure rather than the total energy. So, it is a fundamental difference. It is a fundamental shift in thinking.

So, we have already got the distortion energy. And von Mises noted that yielding begins when the second invariant of deviatoric stress reaches a critical value. You know, we have looked at utility of invariants. When you have I_1 goes to zero, what happens to state of stress? I_2 goes to zero, what happens to state of stress? You have found out the invariants in a given stress tensor. Suppose, we split the stress tensor in terms of deviatoric component and hydrostatic component. Now, for the deviatoric component, you again write I_1, I_2, I_3. Instead of labeling them as I_1, I_2, I_3 to distinguish, you label it as J_1, J_2, J_3, fine. von Mises noted when J_2 reaches a critical value, failure occurs. And J_2 is nothing but what we have seen as that $(\sigma_1 - \sigma_2)^2$, some form of that expression. So, people have also looked at it from the point of view of invariants. And this is also called as J_2 material, ok. If the material follows J_2, distortion energy, you call that as J_2 material, ok. And physical interpretation, further development based on physical argument was given by Hencky. So, people have got convinced that this is the way that failure is precipitated.

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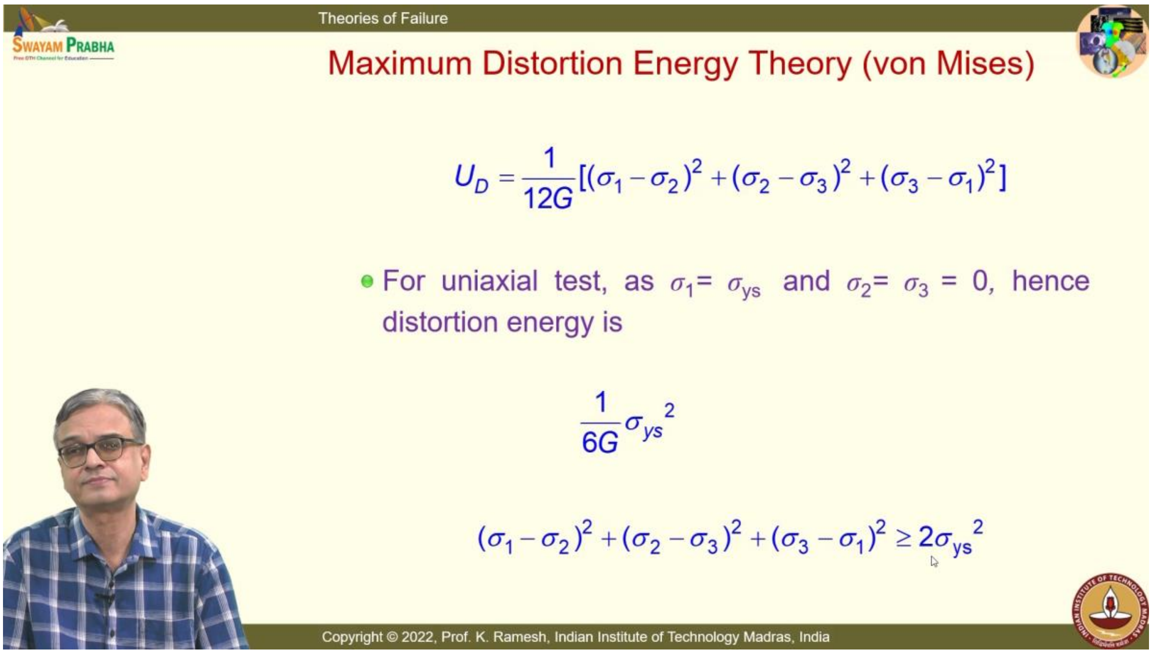

So, you have distortion energy is now given a symbol, sensible symbol U_D.

$$U_D = \frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]$$

For uniaxial test, $\sigma_1 = \sigma_{ys}$ and σ_2 and σ_3 are zero. When you substitute this, the critical value becomes

$$\frac{1}{6G}\sigma_{ys}^2$$

So, I have this as

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 \geq 2\sigma_{ys}^2$$

See, if you look at any theoretical development, people might start from different perspectives. And if you converge to one quantity, it also has an inbuilt verification of our mathematical approach, fine.

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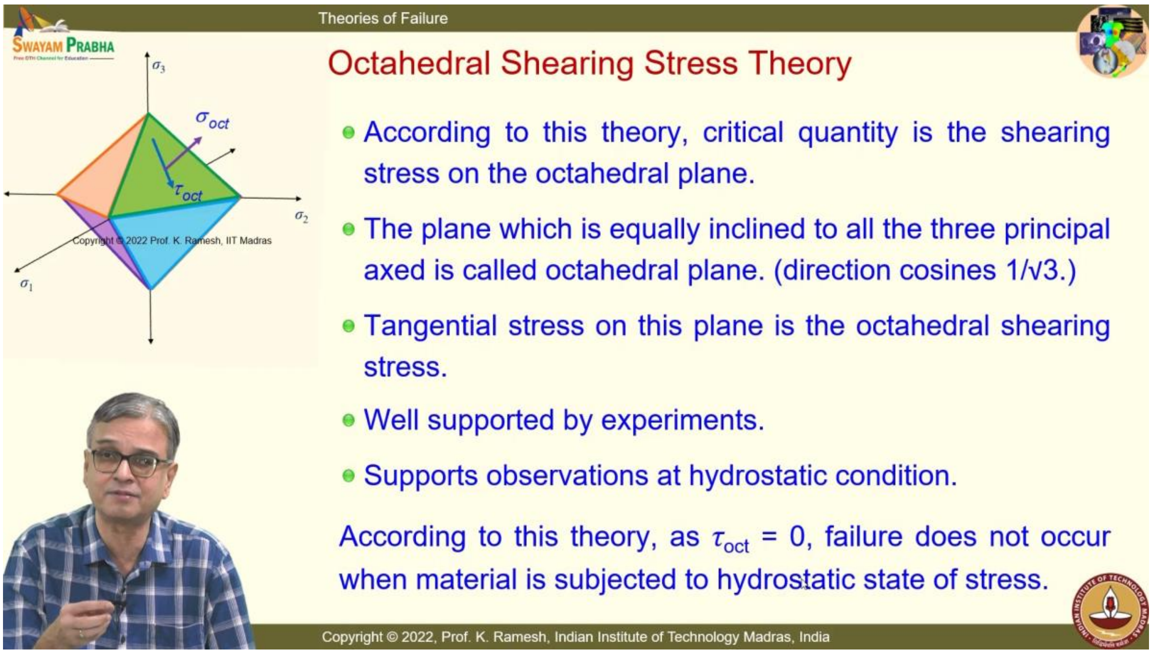

You find there is a development in terms of octahedral shear stress. You know what is the octahedral plane. And it is nicely shown since you have 8 such planes possible, I have shown it with different colors. So, it is referred in the principal stress reference axis σ_1, σ_2, σ_3 and this plane is equally inclined. That means, the direction cosine is known $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and that is also labeled as π-plane. They are one and the same, ok. This theory states, if the shear stress; not the normal maximum shear stress that the Tresca conjectured. They said monitor what happens to the shear stress on the octahedral shear stress plane. When this shear stress reaches a critical value, failure will occur. Look at what happens in the tension test and then find out the expression. And we will also see the expression how they are, ok. So, the plane which is equally inclined to all the three principal axis is called octahedral plane. Direction cosine is $1/\sqrt{3}$ and well supported by experiments. It is very well supported by experiments. Supports observations are hydrostatic condition and once you see the expression, I have written the result here, but once we see expression, you will see octahedral shear stress is zero when I have hydrostatic stress. So, it does not predict like we have said that the point P can be moved along the axis that you have justification from this basic theory, if you analytically examine what is its behavior, ok. So, now I have to get the octahedral shear stress. In fact, you have developed it as part of your assignment, but anyway we will go back and then do this.

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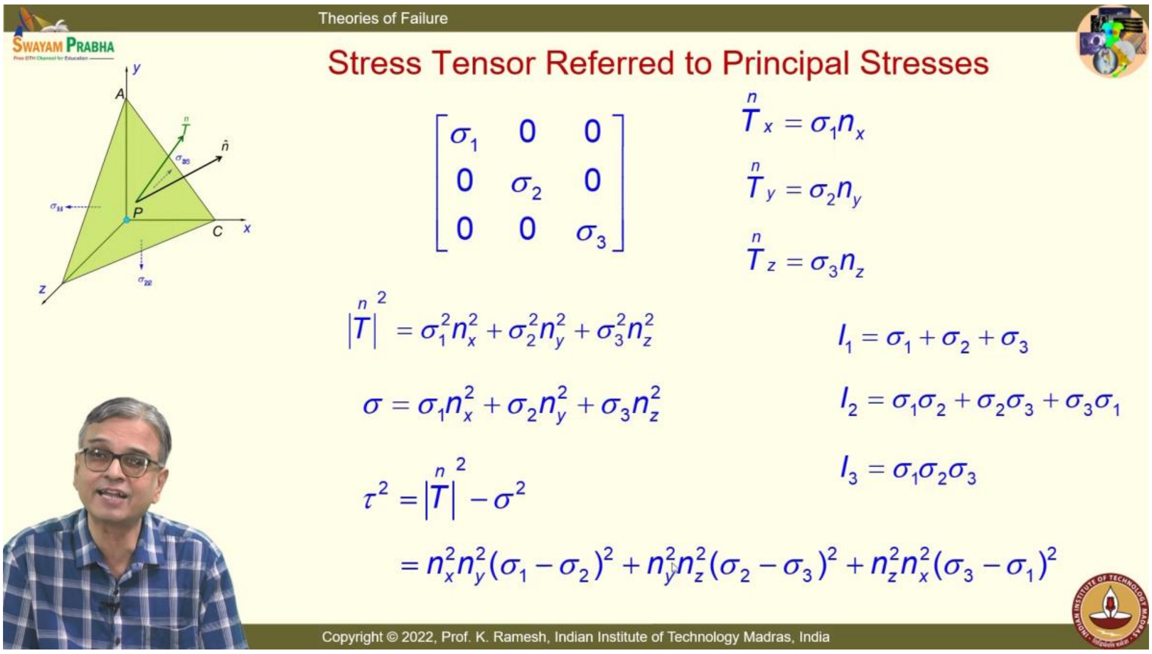

So, I refer the stress state in terms of

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

and your I_1, I_2, I_3 are simply

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

So, all your mathematics becomes much simpler to handle once you refer it in terms of principal stresses and all of the failure theories uses principal stresses because you can combine the load easily, get the final stress tensor and get the principal stresses, then apply the theories of failure.

And if I want to find out stress vector on any plane, I need to have the direction cosines. If it is n_x, n_y and n_z, I can have $\overset{n}{T}_x$, $\overset{n}{T}_y$ and $\overset{n}{T}_z$ in a very simple fashion and you can also calculate what is the stress vector. Stress vector is nothing but

$$\left|\overset{n}{T}\right|^2 = \sigma_1^2 n_x^2 + \sigma_2^2 n_y^2 + \sigma_3^2 n_z^2$$

and this is the squared of this. So, I can take the square root to get the stress vector magnitude. And once you get that your normal stress is nothing but the dot product of this vector with direction cosines. I get this as

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2$$

And your shear stress can be evaluated as

$$\tau^2 = \left|\overset{n}{T}\right|^2 - \sigma^2$$

that reduces to this form

$$= n_x^2 n_y^2(\sigma_1 - \sigma_2)^2 + n_y^2 n_z^2(\sigma_2 - \sigma_3)^2 + n_z^2 n_x^2(\sigma_3 - \sigma_1)^2$$

Do you see where I am going to? We have looked at distortion energy. We looked at some similar form of the expressions. And when you come to octahedral shear stress, I know what is n_x and n_y and n_z; they are $1/\sqrt{3}$, fine.

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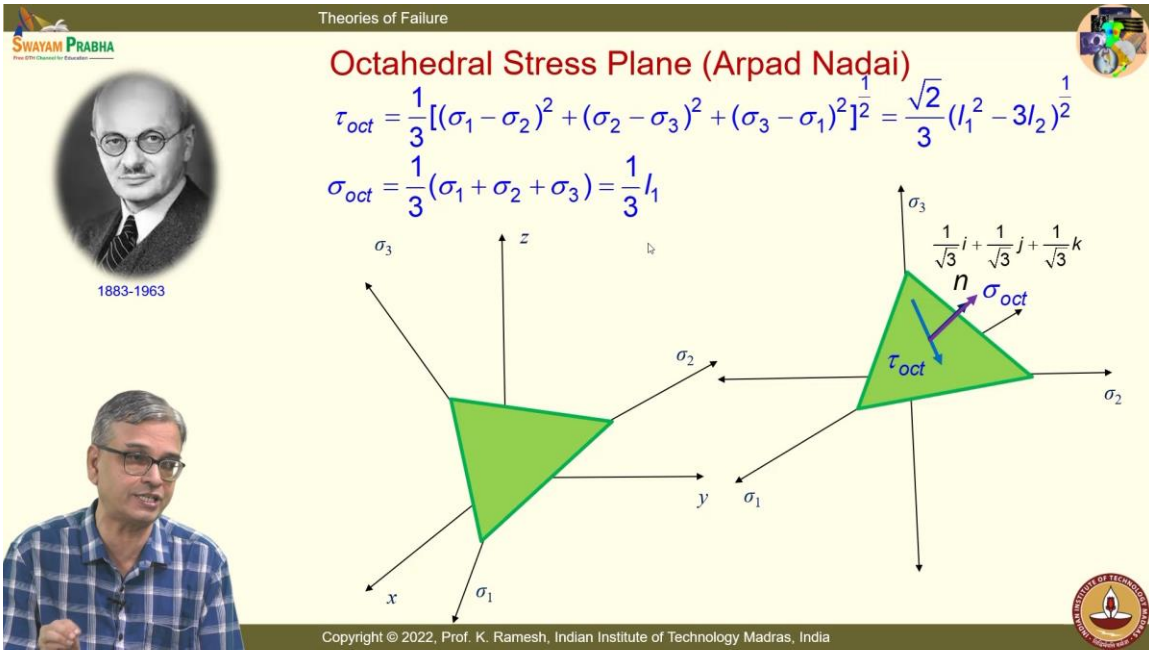

So, you have the octahedral shear stress; it is also very nicely illustrated here because when I have a reference axis as *x*, *y* and *z*, you can have different orientation of σ_1, σ_2, σ_3. From there it starts and it goes to the reference axis as σ_1, σ_2 and σ_3. And you have normal stress, you have the direction cosine $\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$, and you have the normal stress and you have the shear stress acting on it. Normal stress is

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}I_1$$

And you have octahedral shear stress. We have just now got the expression. So, octahedral shear stress reduces to

$$\tau_{oct} = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{\frac{1}{2}} = \frac{\sqrt{2}}{3}(I_1^2 - 3I_2)^{\frac{1}{2}}$$

It can also be expressed in terms of invariant of the deviatoric stress component. It is related to J_2, if you say that J_1, J_2, J_3 ,it is related to J_2.

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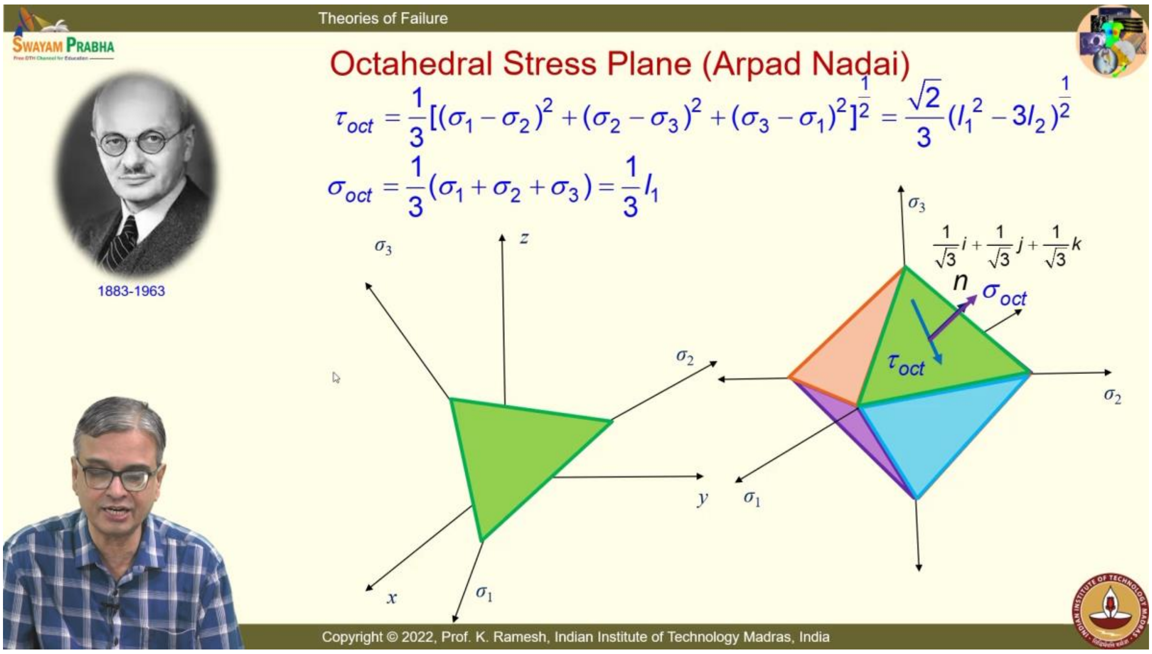

There are multiple ways. You know people, once they develop, they do not stop. They want to see it from multiple perspectives, so that they want to get convinced that we are proceeding in the right direction. That is the crux behind it. And if you look at the time period of Arpad Nadai, it is in the last century, ok. If you look at Tresca, it is very, very old and you saw Huber developed this in 1904 and this is somewhere around middle of 19th century or 20th century, ok. If you put it in the right fashion. So, I have this shear and this one, ok.

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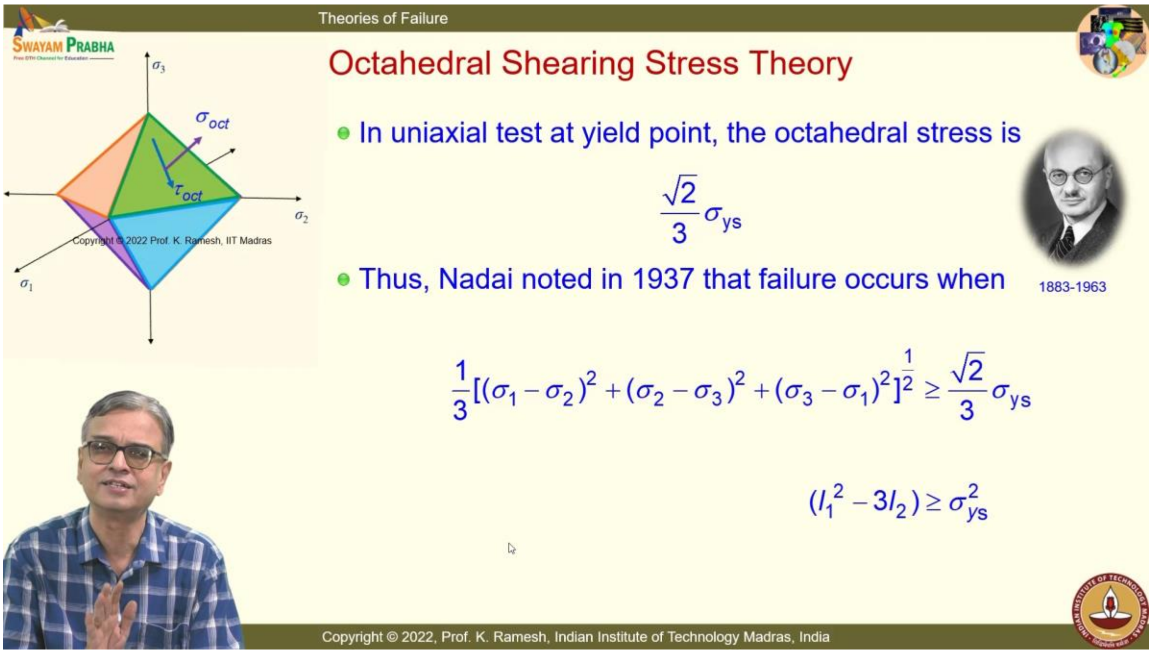

So, this theory says what happens in a uniaxial test when you substitute the values of σ_1, σ_2, σ_3. σ_2 and σ_3 are zero and σ_1 is σ_{ys}. It becomes √2/3. So, he noted in 1937. So, we saw Henkcy gave an explanation in 1924. 1937, he noted that failure occurs when this reaches a critical value, ok. And this can also be put in terms of invariants. So, there are different ways of putting it. You know researchers want to have nice publications, they want to coin new names and then make the mathematics look different, you get a paper out of it. That is how the research goes. There is interesting development in that sense, ok. Can you see that they are one and the same? We will see.

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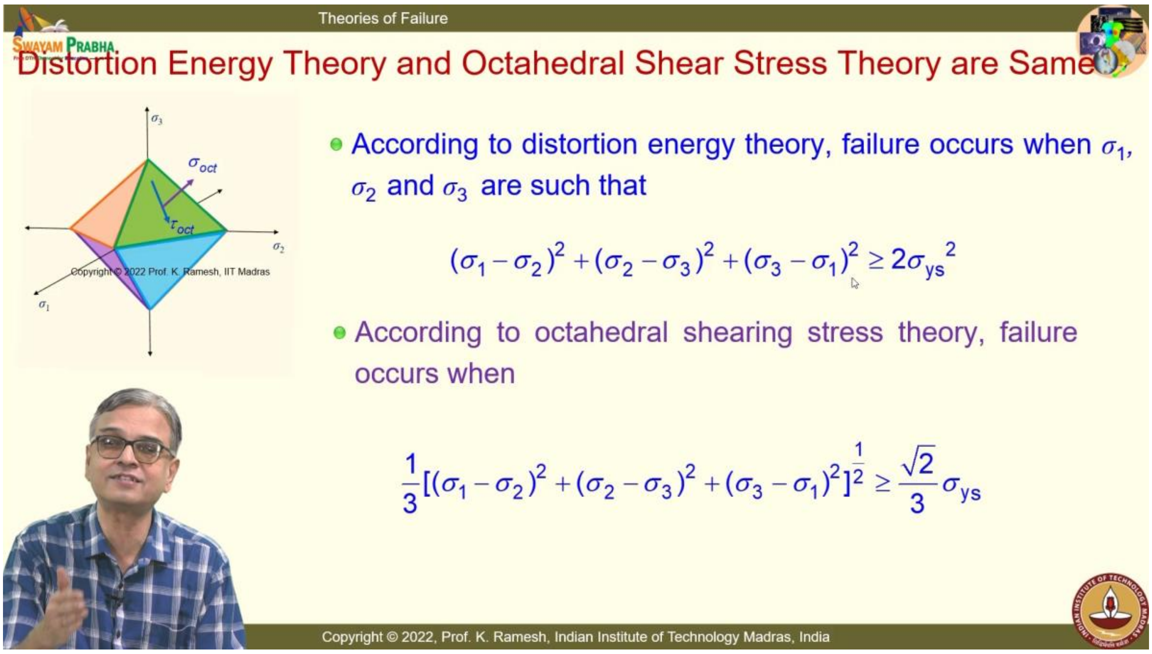

According to distortion energy theory, failure occurs when σ_1, σ_2 and σ_3 are such that

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_{ys}{}^2$$

In octahedral shear stress theory, failure occurs when this happens in this fashion √2/3. If you knock off this 3 and square this, are they not one and the same? So, that is the reconfirmation that distortion energy theory and octahedral shear stress theory, the final implementable criteria is one and the same. See, in practically solving problem, you will worry about this expression. You will not try to find out unless it is asked what is the octahedral shear stress. The final expression can be reduced in this form. Is the idea clear?

And this also provides the reconfirmation that mathematical development is in the right direction. So, in the case of brittle materials, we have seen only the simple maximum principal stress theory and for the ductile materials, Tresca was the oldest. And distortion and octahedral merged into one. So, you have two theories now for ductile materials. And we will continue further in the next class to find out a better theory for brittle materials and also solve some problems. Thank you.