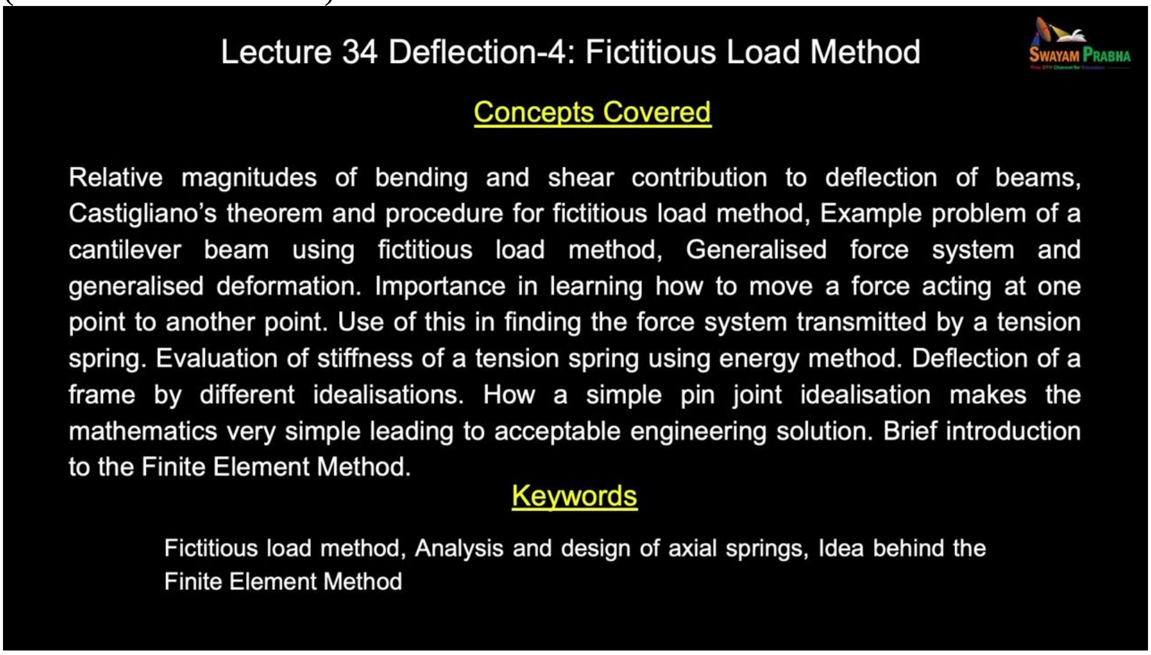
Strength of Materials
Prof. K. Ramesh
Department of Applied Mechanics
Indian Institute of Technology, Madras

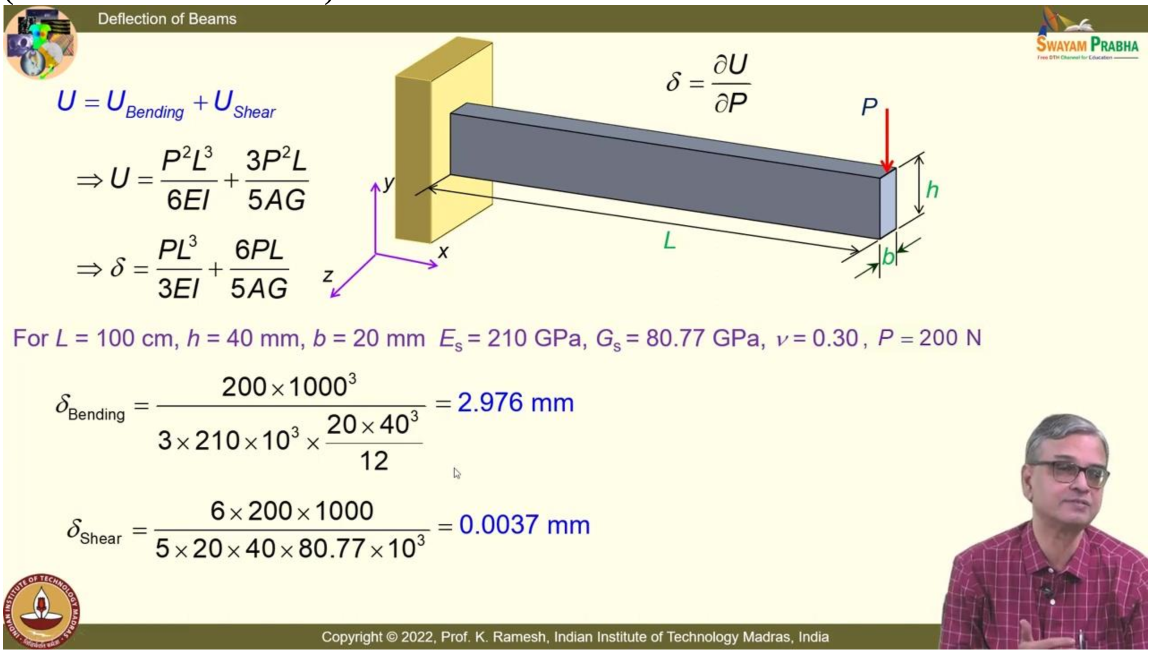
Lecture - 34
Deflection 4 Fictitious Load Method

(Refer Slide Time: 00:20)



Let us continue our discussion on deflection. In fact, I have circulated a problem sheet. The idea is, if you have gone through what is the problem statement, then you can easily follow the solution procedure that we are going to discuss today. And you know, in the previous class, we have looked at how to get the deflection including the shear deformation. If you look at the moment curvature relationship derivation, it was confined only to the bending moment. And a beam you analyze like this, you call this as a Bernoulli beam and then you are happy with it. The moment we started the energy approach, it provided a via media where I could include the effect of shear deformation.

(Refer Slide Time: 01:26)

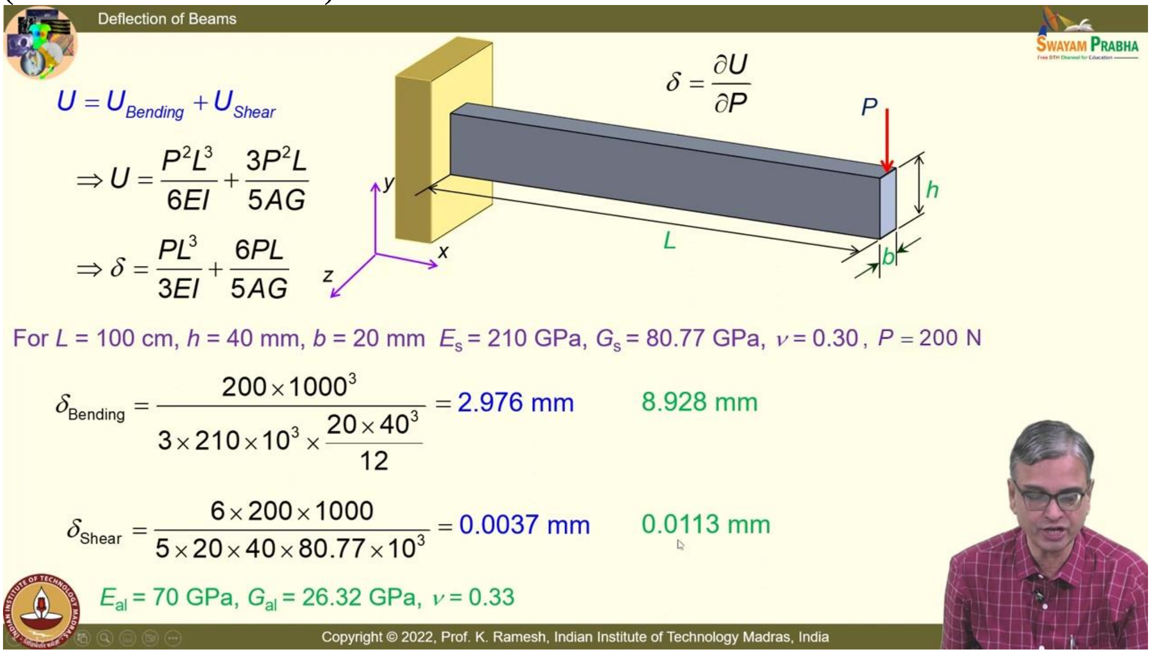


In fact, in the last class, we got the deflection at the load application point of a cantilever as $PL^3/3EI$; that you get from integrating the moment curvature relation also. In addition, you have an extra term, $6PL/5AG$. See, when you look at this as an expression, you do not get the feel, what is the relative magnitude of these displacements.

So, it is worthwhile to take up numbers for this and I have also taken a beam which has a length of 100 cm and your *h* is comparable to the length, it is 40 mm, fine. And *b* is 20 mm and E_s is 210 GPa and G_s is; when you say 210 GPa, you say that it is for steel. So, that is why it is put as G_s; suffix *s* is put. And when you substitute this, let us look at the deflection. It has two components; $PL^3/3EI$ that comes out to be 2.976 mm. And if you look at the shear, $6PL/5AG$, that comes to be a very very small number, 0.0037 mm. So, you have a relative appreciation. You have to worry about shear under certain circumstances.

You cannot neglect it. We have developed moment curvature relation; we have determined the deflection for a variety of beams neglecting shear. You have a justification when you substitute these actual values to this expression that shear deformation is very small.

(Refer Slide Time: 02:56)



See this is made of steel. Suppose I make the same aluminum, beam made of aluminum; Aluminum is 70 GPa and then you have these numbers change, but the cross-sectional dimension, the length remains the same.

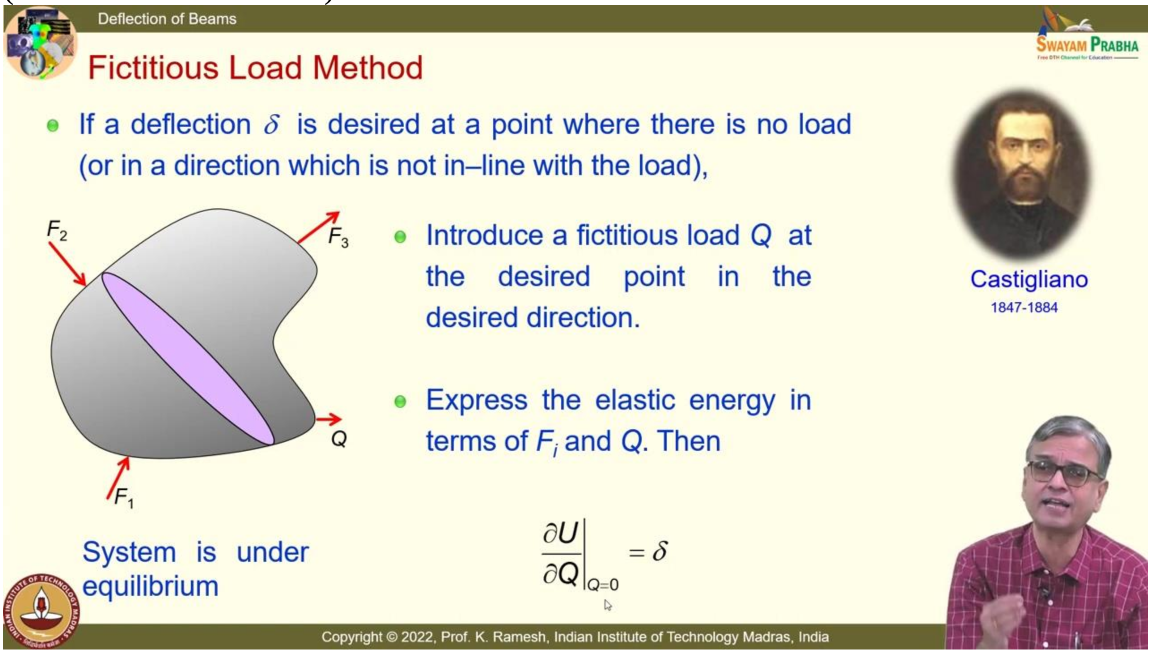
What do you anticipate? The deflection will be small or larger? You should appreciate from the Young's modulus. The Young's modulus is one-third of it. So, the deflections are going to be larger when I have an aluminum beam. So, it comes out to be 8.928 mm.

And when you look at the shear deformation, it is 0.0113 mm. And if you actually evaluate the percentage difference, it is only 0.12 % or 0.13 %. These are all very very small quantities. So, you do not make a significant error. See when the problems are very complex, if you go very close to the actual solution, engineers are happy. You must think like an engineer. From a mathematical perspective, it may not be exact.

That is not the way you can demand because the real-life problems are so complex. If you are able to go closer to reality; how closer to reality depends on the problem. If the problem is too complex, even 20 %, 50 % is alright. If the problem is simple, then you want to go within 5 %. In unknown situations, it can even be 200 %.

At least you know that you are in the domain of it, fine. So, that is how engineers handle practical situations.

(Refer Slide Time: 05:10)



And we have also developed a method called fictitious load method in continuation of the energy. And this was by the theorem propounded by Castigliano. And if you look at history, you know he lived for a very short while, only 37 years. He was too young in age to leave the world. And the theorem was available in his dissertation thesis. It was so powerful. It has really helped engineers to solve very complex problems. And we have discussed that I have a force system.

In the Castigliano's method, I have to differentiate with the force. then I will get the in-line displacement. Suppose I want to find out the deflection or deformation at different points other than the load application points, how do I go and do it? The method itself says fictitious load. So, you apply a fictitious load and you will be in a position to get the deflection. So, it can be interpreted in many ways.

If I do not have a force at that point, apply a fictitious load. Suppose I want to find out deflection other than the direction of the force, apply the force in that direction. So, that is what it says, not in-line with the load. So, I can have the fictitious load like this. I want to find out the deformation at this point.

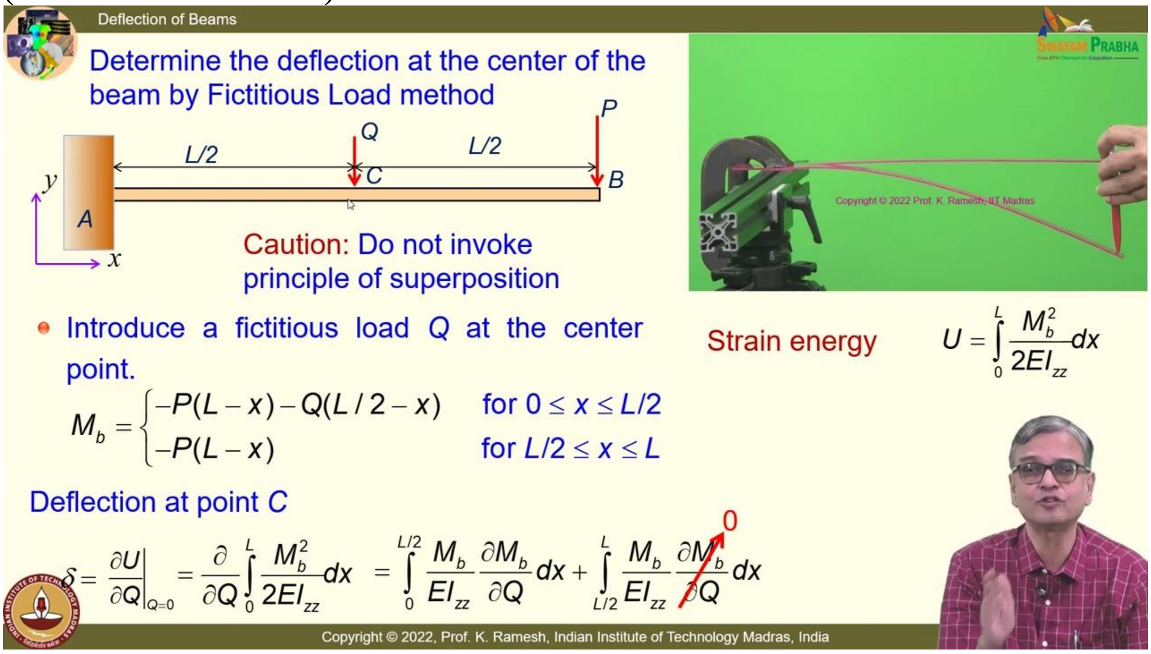
I can also find out the horizontal component with respect to this force at this point, introduce a fictitious load. But the moment you have introduced a fictitious load, you treat that as a load acting on the system, re-solve the problem. Re-solve the problem by finding out the reactions, all the aspects of it, get the energy. And only in the final end, before you calculate the deflection, after differentiating with respect to the load Q, you make Q equal to 0.

That is a subtle point. So, introduce a fictitious load Q at the desired point in the desired direction. So, that accommodates all aspects of the problem. So, I can put it at different points. Express the elastic energy in terms of the force system acting on the member as well as the fictitious load Q. See, at a time you apply fictitious load only at one point

Solve it, you get the deflection at that point along that direction. If you want it for another point, you apply it afresh. For illustration, I have said it can be any point, any direction. So, at a time, you will apply Q only at a particular point of interest. So, you differentiate the energy with respect to Q and finally, substitute Q equal to zero.

That is the procedure. You have to understand the procedure very clearly. And we will solve a simple problem for which we already know the solution from moment curvature, so that you can verify, whether the fictitious load method has helped you to get the deflection at that point.

(Refer Slide Time: 08:34)



So, I take a problem of a cantilever and the question is, you have to find out what is the deflection at the center of the beam. So, the method says since I do not have a load, I have to apply a load Q at the center and evaluate the energy stored in the beam.

You have to solve it afresh. So, I have two sections. I have one section here; I have another section here. So, I have to find out the bending moment in these two sections. You know from now onwards; we will not worry about the shear deformation. We have already seen how to incorporate shear deformation and we have also seen shear deformation leads to small changes in the final deflection.

Since they are very small, from an engineering perspective and engineering analysis, we can live with moment curvature relation. So, we will confine ourselves only to bending energy. So, you have to write the expression for bending moment in this section, expression for bending moment in this section, which you have the background. Please write that expression and check it with my slide because unless you write these expressions correctly, further calculations will not lead you anywhere. You have to write the bending moment precisely; and you measure x from the fixed end, that is how you normally solve the problem like this.

So, if I do that, I have the bending moment for the section 0 to $L/2$, I have

$$-P(L-x)-Q(L/2-x)$$

and you have for the rest of the section C to B, only P is contributing the bending energy, Q does not contribute the bending energy. So, you know, some of these expressions you need to remember. I know the bending energy as

$$U=\int_0^L \frac{M_b^2}{2EI_{zz}}dx$$

dx or ds, whichever the way you want to label it, integrated over the length 0 to L. So, I will have to write this for the first section as well as the second section. And you know, our mathematics becomes simpler if I differentiate it.

So, I have

$$\left.\frac{\partial U}{\partial Q}\right|_{Q=0}$$

So, I have

$$\frac{\partial}{\partial Q}\int_0^L \frac{M_b^2}{2EI_{zz}}dx$$

So, I express it for two different sections and then differentiate it. So, when I differentiate it, I will have this as

$$\int_0^{L/2} \frac{M_b}{EI_{zz}}\frac{\partial M_b}{\partial Q}dx$$

for the first section. And for the second section

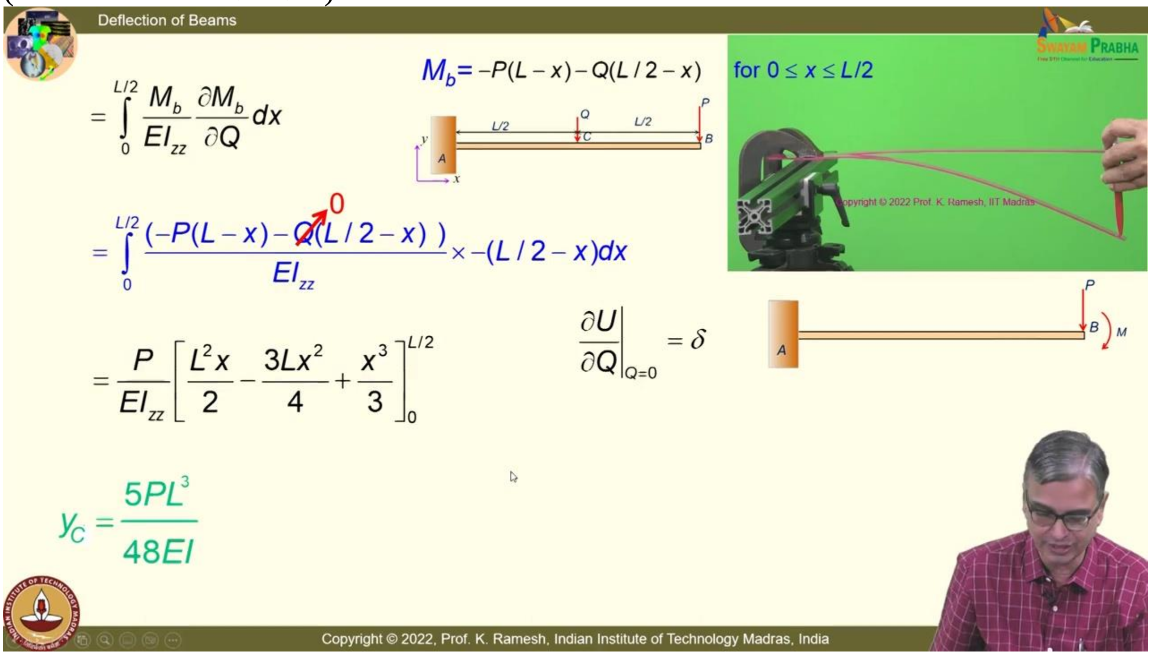
$$\int_{L/2}^{L} \frac{M_b}{EI_{zz}}\cancelto{0}{\frac{\partial M_b}{\partial Q}}dx$$

You can very well see in the second this one, M_b is not a function of Q. So, this is going to be zero. So, I can simplify it in the expression itself without getting into the actual differentiation. So, I have to find out only for this section, these expressions and you know what is M_b.

So, that you can straightforward do. Let me ask a question. Suppose I pose a problem that as if this load is about 500 N, the moment you come to energy method, rewrite that number as a symbol, then solve the problem. Only then you can do all these integrations and differentiations. So, you have to do that as a symbol and then do it.

Finally, substitute the values. And the caution is do not treat P independently and Q independently and find out the energies because if you do it like that, the basic procedure is violated. The basic procedure is, you have to resolve the problem as if Q is acting on the system. Only after differentiation, you make Q equal to zero. Until then, you cannot handle it separately.

(Refer Slide Time: 13:08)



So, I have this expression rewritten here and you also know what is the expression for M_b.

Can you differentiate this and then do the integration and substitute the values? You can differentiate this M_b with respect to Q. So, I have this expression reduces to

$$\frac{(-P(L-x) - Q(L/2-x))}{EI_{zz}}$$

And when you do $\partial M_b / \partial Q$, this expression reduces to

$$-(L/2-x)dx$$

Now, you substitute Q equal to zero after differentiation.

We have differentiated it. So, I will make Q equal to zero. So, when I do the final integration, I get

$$\frac{P}{EI_{zz}}\left[\frac{L^2 x}{2}-\frac{3Lx^2}{4}+\frac{x^3}{3}\right]_0^{L/2}$$

When I substitute the limits and simplify, you get the expression as y_C, that is the deflection at the point C as

$$y_C=\frac{5PL^3}{48EI}$$

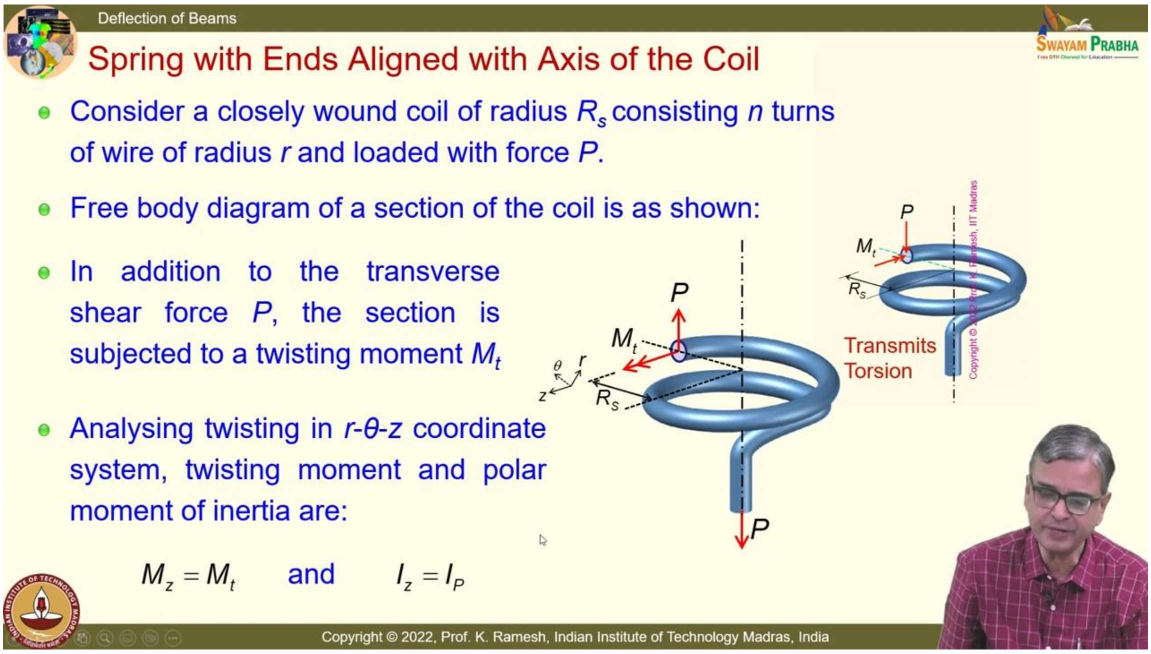
You can go back and see either by your moment area method or integrating the moment curvature relation, you will get the same result at the center of the beam. And you know, I also want to give you another homework.

You know, I have the load acting at this point, but I would like to find out what is the slope at this point by fictitious load method. If I have to find out the slope, what is that I have to do? What is the load that I have to apply? So, you have to bring in your appreciation of what is a generalized load and generalized displacement. If it is a concentrated load, it will give deflection in that direction. If I have a moment, it will give you angle of rotation. So, if I have to find out the slope at the point B, I have to apply a moment.

See, it may be desirable to indicate moment by M rather than Q. Q is a symbol. So, you can also treat M as a symbol and do the differentiation appropriately. So, if I use a fictitious load as a moment, I would be in a position to get the slope. Is the idea clear? So, you know how to find out the in-line displacement and in-line generalized displacement when I apply a moment.

When I apply a moment, in-line displacement is the rotation. So, you have to intelligently handle this method. It is a very versatile method. It is a very versatile method and you will find that it has many applications in your future learnings.

(Refer Slide Time: 16:22)



Now, let us go back to the classical problem. You know, you take a simple spring and if you find out what is the force that is acting, follow this animation very carefully because I would translate this by transmissibility. Nothing happens. Only when I move it to this point, when I move to this point, I also have a twisting moment acting on the spring. It is very important. See, one of the aspects I find is, how to move the force from one point to another point.

You have to master it. If you master it, you can write the bending moment diagram and shear force diagram easily. Just because you have not mastered it, you are taking time. That is why I am repeatedly showing how to move a force and if you have followed this animation, I have got this. Suppose, I want to draw the free body of this.

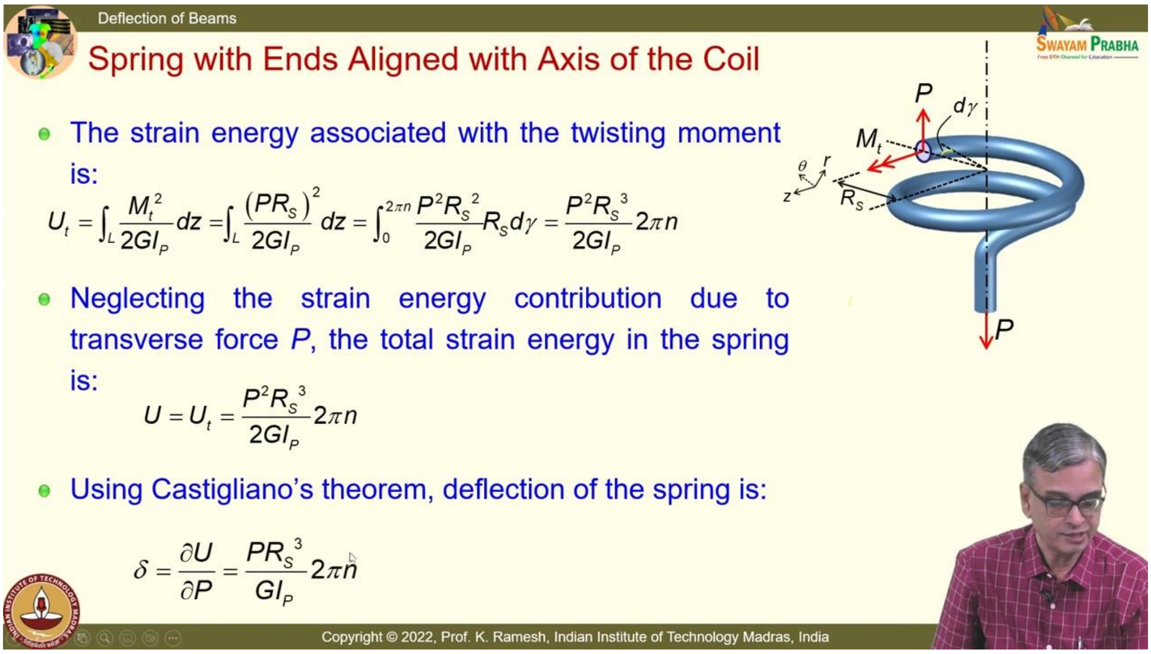
See, this is the force acting on it. Suppose, I put the free body, I will have just opposite of it. I have a force acting on it. So, when I cut a generic section, I have the twisting moment and shear force acting on it and we have already said shear deformations are very small. So, you can always neglect this.

Suppose, I want to go and design a spring. See, if you are asked to find out the stiffness, if a spring is given to you, you can do a force deflection experiment and then find out what is the stiffness. But if I have to design a spring with a particular stiffness, I need to know the basic expression. How the stiffness is developed in the spring? For that, I have to know what is the force it is transmitting. So, it is essentially transmitting torsion and it is much easier to find out the stiffness by the energy approach. And here, we are going to look at the torsional energy and you have seen that spring has a coil like this.

We will have the turns and what we will do is, we will say that they are circles. We will neglect the small helix angle. So, we will count the number of coils. So, that will give the length of the shaft under torsion.

So, you have to appreciate how to attack the problem. So, the spring cross-section transfers a twisting moment and a shear force and we will do it in r theta z coordinates. And you can easily write, very similar to what we have done for bending. It will be M_t^2; you will have, instead of *E*, you will have *G* coming into the picture. You have to remember those expressions when you want to come to energy method. So, I have M_z as M_t and I_z as I_P. Then, I want to do it in the flexure; I mean torsional formula.

(Refer Slide Time: 19:20)



The strain energy associated with the twisting moment is

$$U_t = \int_L \frac{M_t^2}{2GI_P} dz$$

So, you have to know these expressions whether it is axial load, bending load, torsional load or shear load. You should be able to write this expression and they are synonymic. Once you remember one expression, you have to substitute the appropriate quantities.

And I have to get the *dz*. How do I get the dz from this diagram? So, I have the M_t is PR_S and all these quantities are given. And dz, how do I get this is, I realize that I have to sweep across the coil. So, I take this angle as γ; $d\gamma$. So, I get dz as you have this distance R S.

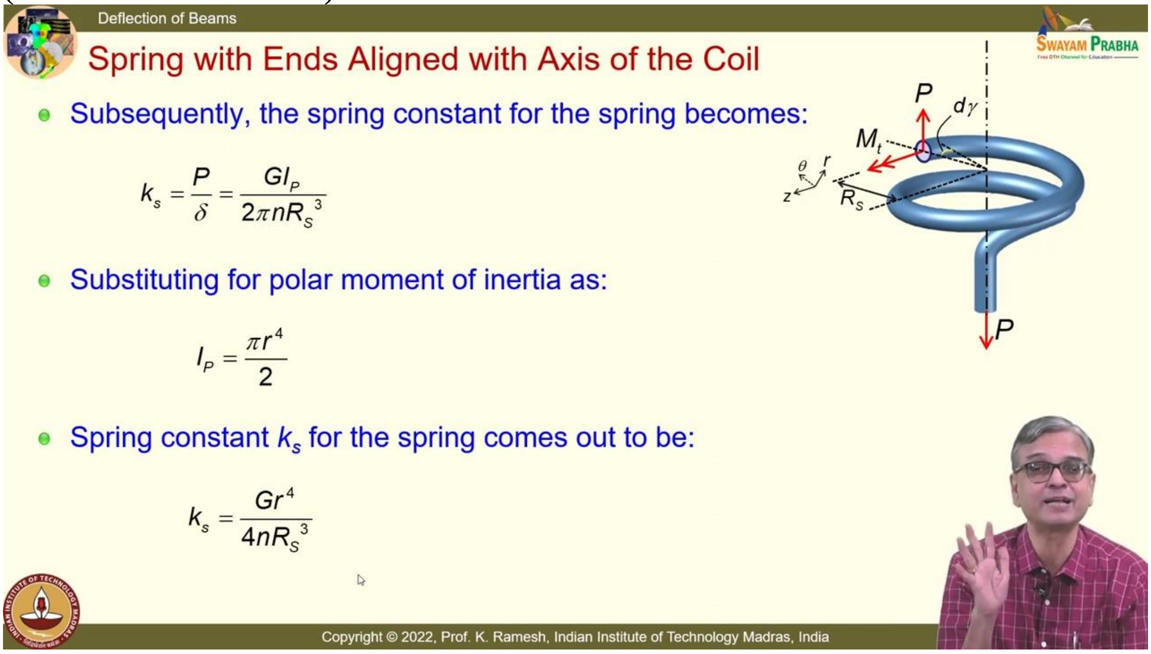
So, I have this as $RSd\gamma$. So, now, I put the length as 0 to $2\pi n$ because I am going to have n number of coils; *n* number of coils. I have to do that. So, the expression comes to be

$$\frac{P^2 R_S{}^3}{2GI_P} 2\pi n$$

And we will neglect the shear loading and you get this strain energy as this expression and differentiate with respect to the load *P*. So, I get this as

$$\frac{P R_S{}^3}{GI_P} 2\pi n$$

(Refer Slide Time: 21:20)



And you know how to define the stiffness. So,

$$k_s = \frac{P}{\delta}$$

So, that comes out to be

$$\frac{GI_P}{2\pi n R_S{}^3}$$

And if you substitute for the polar moment of inertia, if the radius of the spring is given as *r*; the wire is *r*,

$$I_P = \frac{\pi r^4}{2}$$

So, when you substitute this, I get a very famous expression

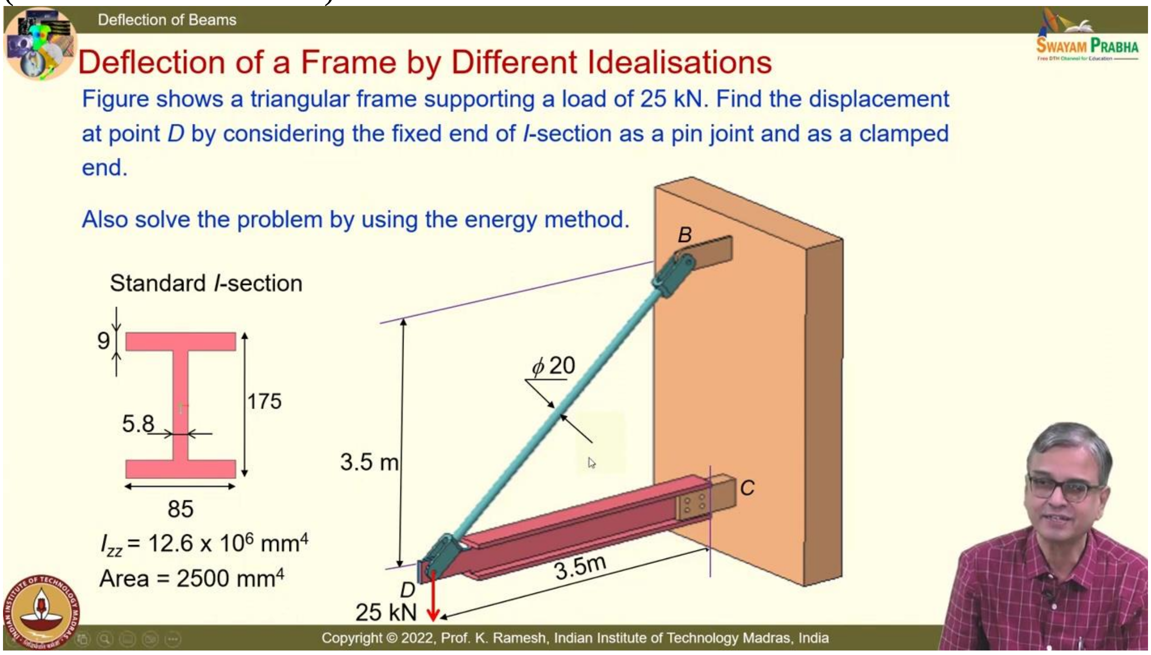
$$k_s = \frac{Gr^4}{4nR_s^3}$$

So, if you know what is the radius of the wire that you are using and the shear modulus; see, we have discussed tension test and we have done it for the bulk material. Am I right? And we have ignored a bulk material will have defects. By continuum assumption, we have ignored that. But the moment I come to spring, I get this diameter of the wire by extrusion process.

By processing the material, the number of defects are reduced. A thin wire is much stronger than the bulk material. So, if you go to spring design, they will not use the bulk material properties to get the G. They will test for this diameter of the wire; you have procedures available. These values will be higher than the bulk material values because it has less number of defects. And in contrast to your shaft design, springs are always made of alloy steels, whereas shafts are made of mild steel because you are looking at the stiffness.

You are not designing based on strength. So, you need to have; it is a sufficient that you use a mild steel for routine purposes. High end application, they may change the material. There are different considerations. So, in contrast to shaft design, you use alloy steel and you do not use the conventional tension test for using the properties. So, you utilize the advantage what happens in a small wire compared to a bulk material. You handle it this way from an engineering approach.

(Refer Slide Time: 23:48)



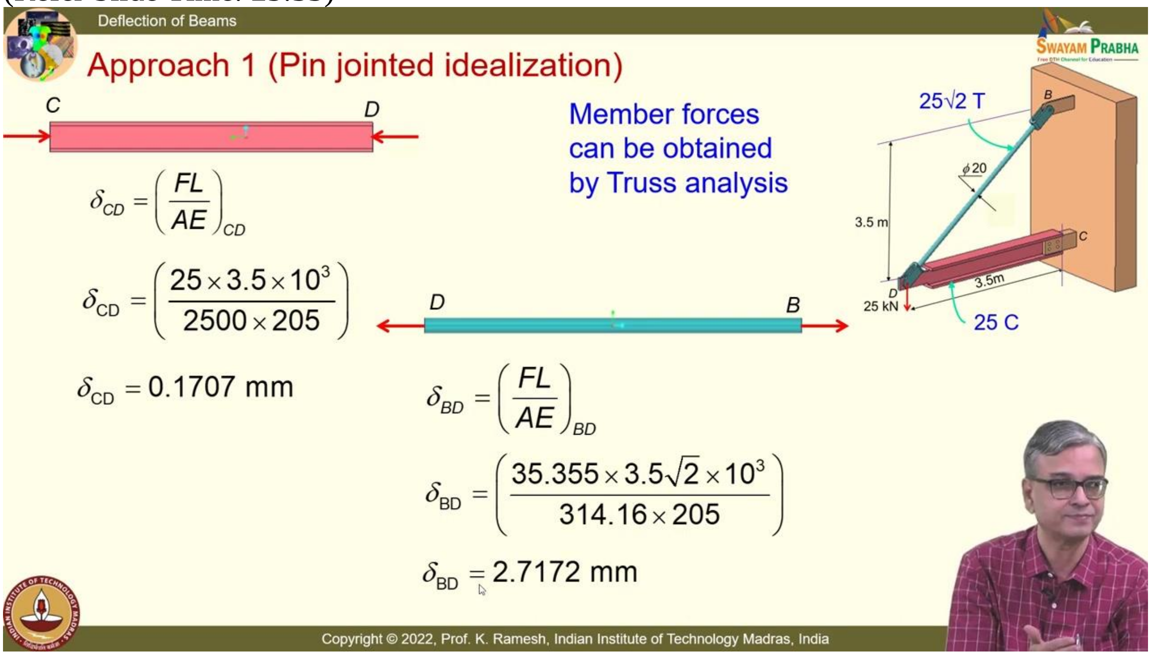
And you know, this class is mainly meant to open up and then find out that we have been making certain idealizations, certain idealizations look obvious, certain idealizations do not look obvious and you might have; ideally should have asked a question and resisted why do you make such an idealization? But you know you do not ask such questions, fine. So, I have to do that. See I have a frame and usually it is made like this. You are clamping this end very firmly like this with 4 bolts. So, you make this as a clamped end and you have a very heavy beam and you also have standard cross-sections available.

Designers just go to the table. They do not even calculate the moment of inertia; it is also given in the standard table. You have the standard dimensions. These are rolled sections available. So, you have to if your design says something, whatever that is close to that requirement, you take it from the rolled section because it is cheap.

It is available readymade. So, designers do all that. The moment you come to design; you do not have one solution. You have multiple solutions possible and you have this holding this by a tie rod like this. And you have been taught that I can idealize this as a pin joint and you can solve it by truss. Am I right? So, we will first solve it as a truss.

Can you get me the member forces? Fine. Try to get the member forces by the time I show you the next slide, fine.

(Refer Slide Time: 25:33)



The member forces can be obtained by truss analysis because you have only one pin joint. Isolate this pin joint and you have this load is given as 25 kN and you have this. You can also treat this as by symbol length L and this is also L.

So, when these two are equal, this angle is 45^o. So, you have all the essential information for you to find out the forces acting on it. See you must pick up speed. How do you pick up speed? Unless you see, how to simplify the problem quickly, you will not be able to do that. Now you can even do this by inspection.

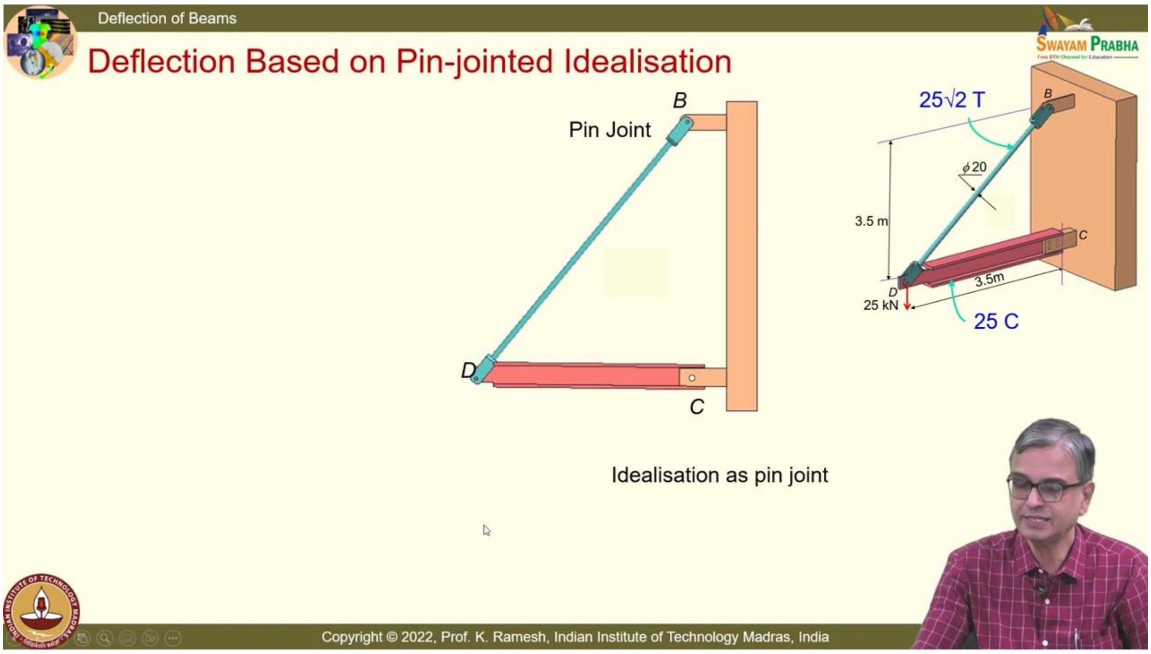
You must also develop that mental ability to do that. I have this force as $25\sqrt{2}$ and this is in tension and this is 25 kN in compression. Can you visualize this? Because I go and pull it. I pull this. So, this rod has to be in tension; because this is in tension, you have a horizontal force that has to be supported by this.

And you know, I have done this by analyzing this as a pin joint. I have not shown it here, but the moment I go and look at the deflection I will show that explicitly. So, once this is under compression, do you know how to find out the deformation? This is a famous expression; I have said that. What is that expression? Can you tell me? If the load is P, PL/AE, it is a very famous expression. You should not hunt for it.

So, instead of P, you supply the load that is acting and this is under tension. So, if I do that, I have FL/AE and if you substitute what is F and what is the length L and then area of cross-section and the Young's modulus, I get this as 0.1707 mm. So, the I-beam gets compressed by this amount and this tie rod gets elongated by this amount. I have given only the absolute value of the deformation.

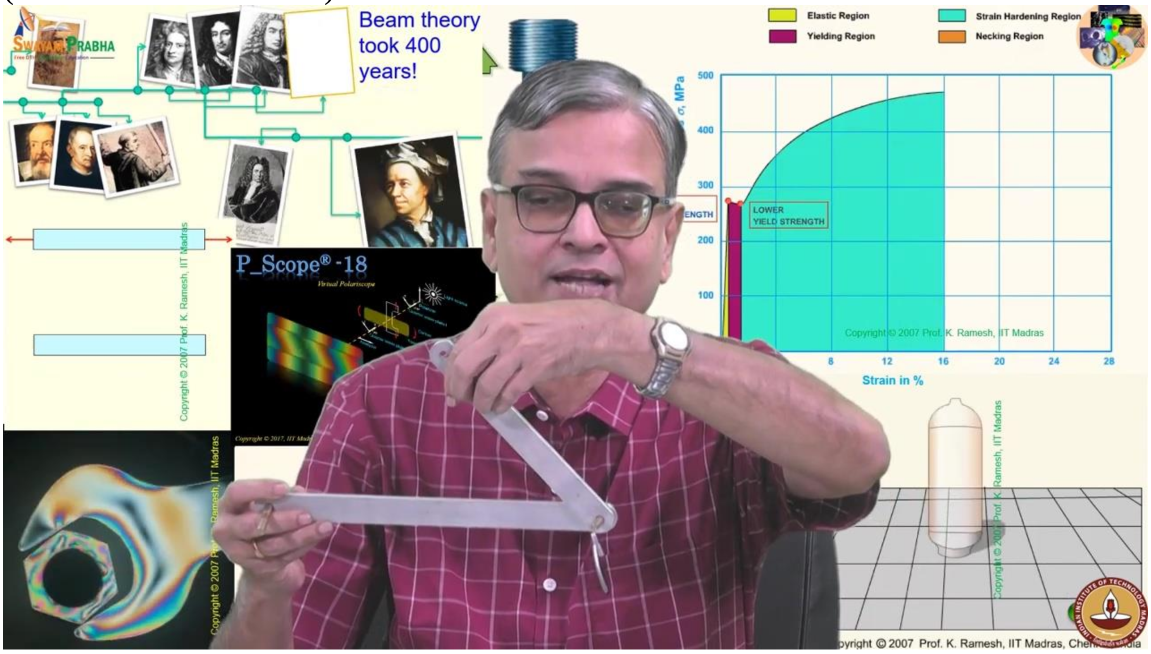
That is much more convenient when you handle problems like this. You attach the sign later, calculate the magnitude, attach the sign later. It has 2.7172 mm.

(Refer Slide Time: 28:16)



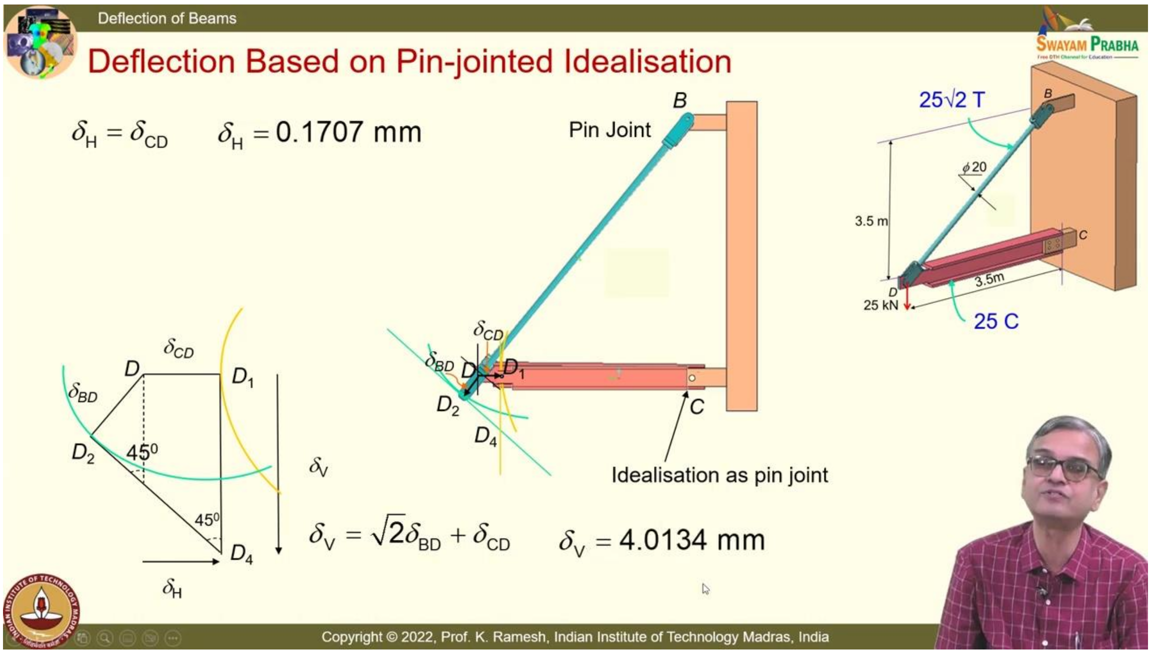
So, you have this as a pin joint. So, I will visually also make this as a pin joint.

(Refer Slide Time: 28:20)



See the moment I have this as a pin joint, you know, you have to appreciate what will happen. See, I have this as a pin joint. So, this can rotate like this and this can also rotate like this. So, what I am going to do is, I am going to make one member shorter, another member elongated and then find out, analytically, how do I locate the point? And I will also make approximations. See engineers have to do approximations, but do intelligent approximations. Now, I have this as a pin joint. I have this as a pin joint, visually also I have shown it.

(Refer Slide Time: 29:05)



So, now I will visualize this compression. This member is compressed. This is by distance δ_{CD}. So, the member is compressed and I have this point moved to a point D_1. So, I draw this bigger for us to visualize comfortably. So, D has moved to D_1 and we have seen that this rod has an elongation; and we have also seen, because it is a pin joint, it can move only in an arc of a circle. Is the idea clear? So, this would be the probable path, but you know mathematically this is difficult to handle.

So, I will replace an arc by its tangent. I will have a tangent. Now, I go and look at this tie rod and this tie rod gets elongated, that is shown and we have also calculated this elongation as δ_{BD}. So, this can have only a rotation about this. So, the meeting point of these two is going to be the final deformed position of the joint.

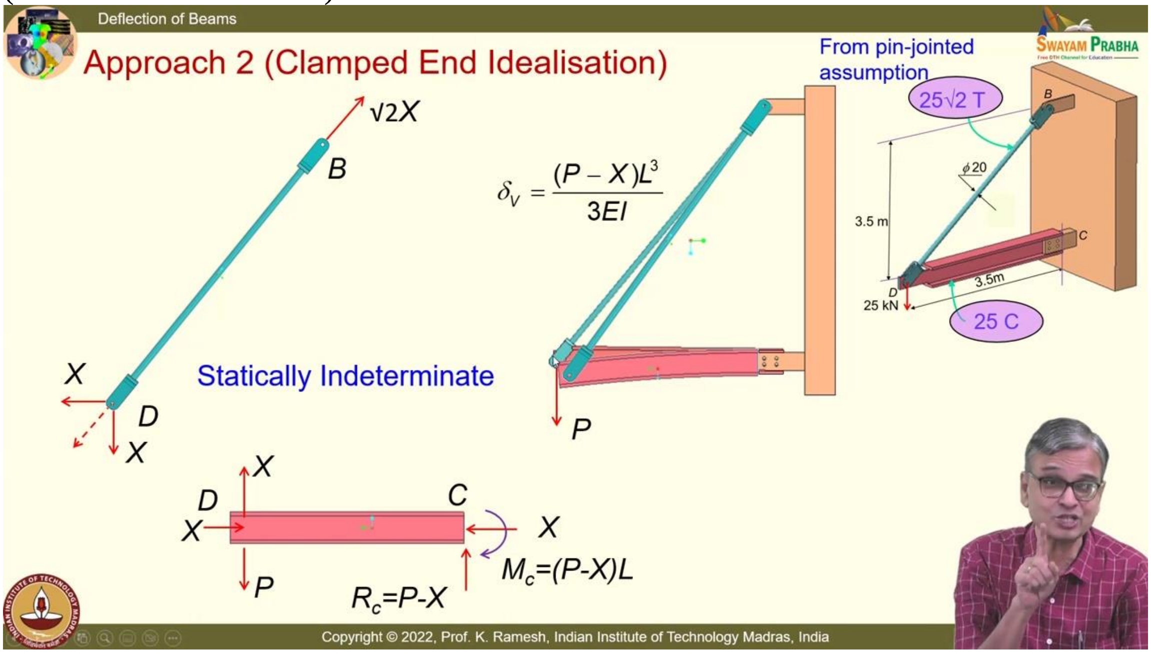
So, I draw this as a circle. I have shown this big here because this is too clumsy to see here. So, I have this. So, this will be the actual point. But you know I will replace this arc by its tangent. So, I would be essentially finding out this point which is far away from this, but mathematically, it is simpler for me to calculate this point rather than this because the entire deformation is so small, the error you introduce is also very small. How do you verify it? Only when you compare it with a more realistic analysis and more realistic analysis, what you can do? You can imagine that this is clamped and this behaves like a beam and go to energy method because the energy method takes care; you are not making any approximation.

You are finding out the quantities in a very nice fashion. We will postpone it for the time being. Let us understand, how do I calculate the horizontal shift as well as the vertical deflection of this from this diagram. I have to process this diagram. So, I can see from the diagram, your horizontal moment of point D is nothing but δ_{CD} and you know, you are given certain quantities that this is angle is 45^{o}.

So, that you should use it. So, from this diagram, I can find out what is this vertical deflection. It has a component of this distance plus this distance because it is 45^{o}, this distance is same as δ_{CD}. So, I will have some quantity like here. So, this I can put it as, from the property of the triangle, I can say this as $\sqrt{2}\delta_{BD}+\delta_{CD}$; is the idea clear? I can easily calculate.

This is simple geometry. There is no clumsiness involved. So, if I go and substitute these quantities, I get δ_V as 4.0134 mm, δ_H is 0.1707 mm. This approach was quick and if you visualize how the pin joint behaves, you have been able to calculate the deflections or deformations very comfortably.

(Refer Slide Time: 33:00)



Suppose, I make this as a clamped end, what is the first hit? First hit is, the problem becomes statically indeterminate.

You have to recognize that. You have to recognize; you will say that I am solving it so holistically. I want to realize that this is the clamped end, but the price you have to pay is, it is statically indeterminate. So, I have to bring in deflection for me to solve the problem. So, even the member forces will change. That was so straightforward. So, what I am going to have is, I am going to have this free body diagram, I am going to draw and I am also going to say that this is not going to rotate, but it is going to have a bending deflection.

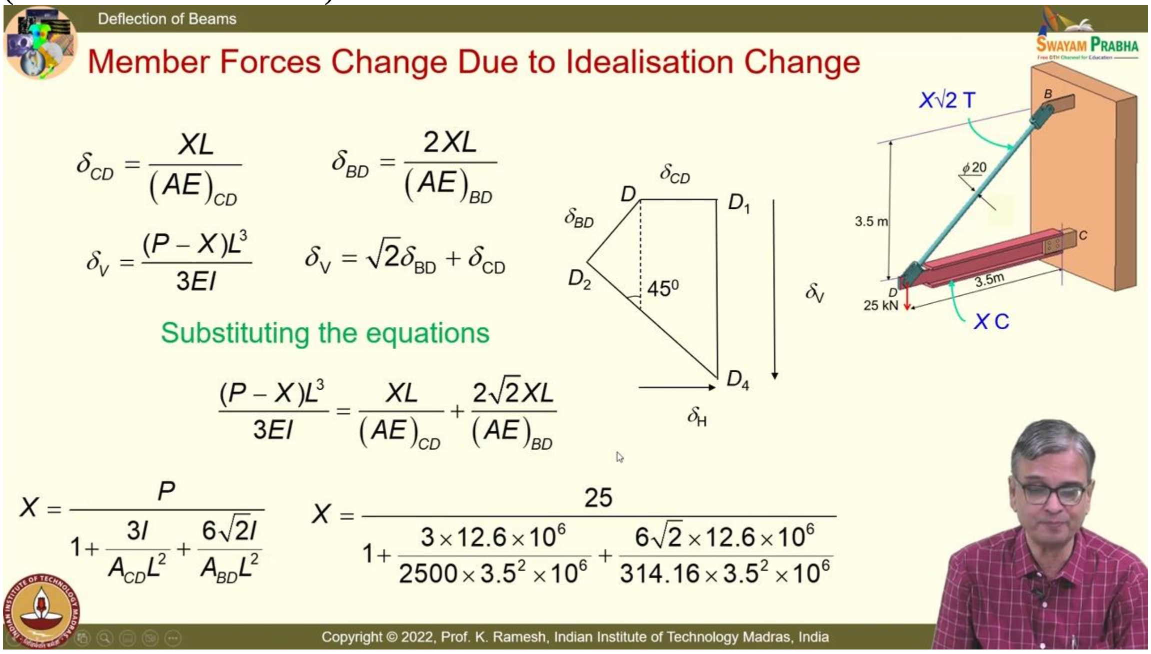
And you know, I will have a force here transmitted by this. Earlier, I was able to calculate this force. Now you know, I have to find out what is the force transmitted by these members? I cannot analyze this as a simple pin joint because the behavior is now different. So, I draw the free body diagram for this. So, I can only say that this is having a load X in the vertical direction and X in the horizontal direction. So, this instead of rotation, it will bend like this and you will have this member joined like this.

Is the idea clear? We have learnt bending in this course. So, we can afford to do this kind of analysis in this course. In the earlier course, if I had asked you deflection, you had only one choice. You would have simply made this as a pin joint and you would have analyzed. At the bottom of your heart, you will be thinking why I analyze this as a pin joint? It is an approximation I am doing it.

Engineering is a waste. Is it not? It is not confirming to what is happening in reality. Now we will see the calculation and see and justify, how engineers were very clever in making idealizations. So, I have to solve this even to get my deflection and I know what is the vertical deflection from my beam analysis.

Find $PL^3/3EI$. Here P is $(P-X)$ and X is an unknown. X was known very clearly in the previous analysis. Now X itself is an unknown. That you have to recognize first.

(Refer Slide Time: 35:43)

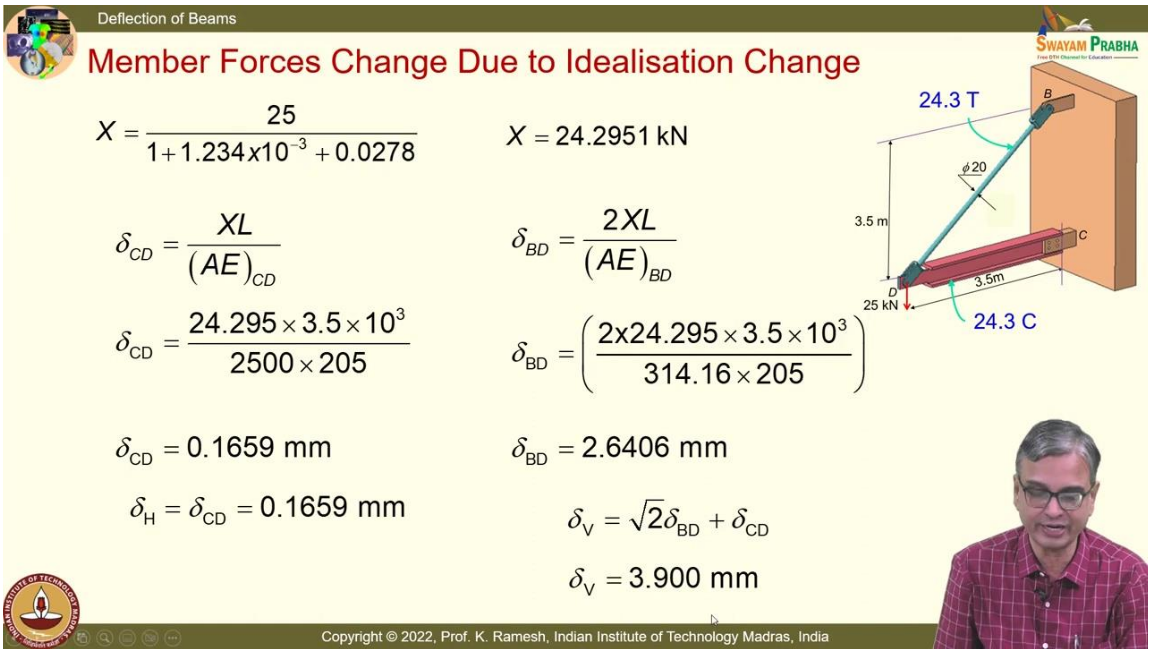


And when I go and look at, I have the deflection, δ_{CD} and δ_{BD}, I can write in terms of the assumed unknown X.

I know δ_V. So, I use delta v as the basis for me to calculate the value of X. Is idea clear? Because all statically indeterminate problems, you have to bring in the deflection behavior to solve for the forces. This we have already seen. We will also have this expression as $\sqrt{2}\delta_{BD} + \delta_{CD}$.

That will not change, but these individual deflections will change. And when I use this expression, I get one equation. It looks very complicated. You can probably take a photograph and ruminate over it in your rooms. And when I solve for it, I am in a position to calculate what is the force. We will see, what is the change that it has created in the member forces by making this idealization.

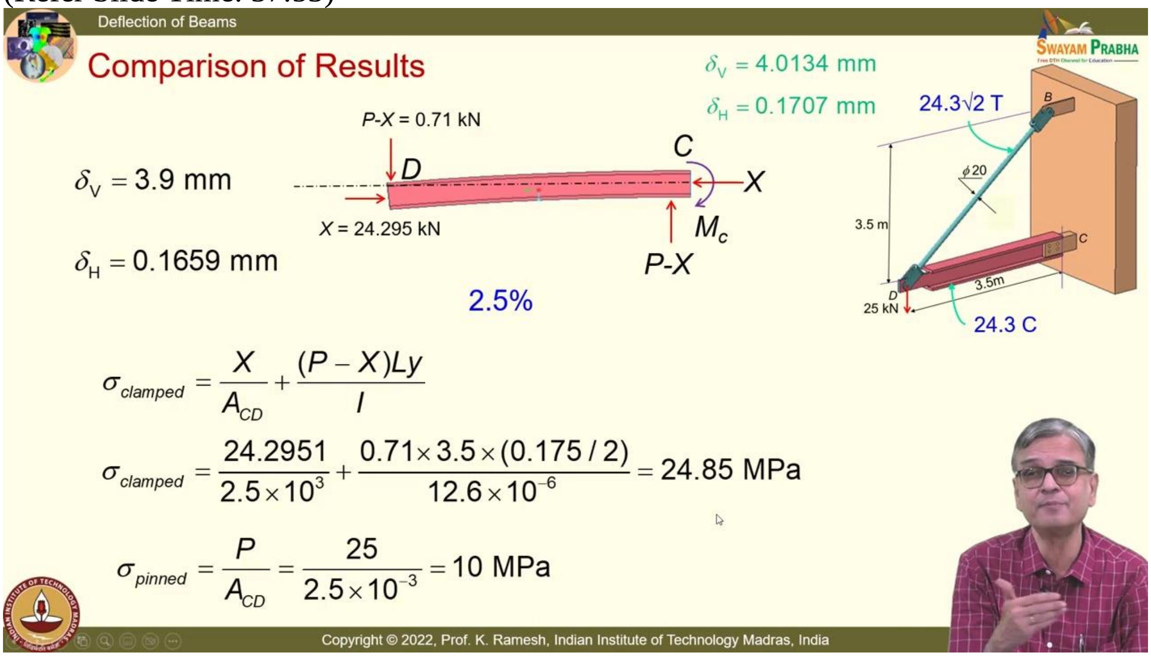
(Refer Slide Time: 36:53)



It has made our job difficult, but if you really look at, the force has changed from 25 to 24.3. For this small change, we have to do back breaking numerical calculations. So now, I have got this.

Now, I can go back and redo the complete deflection. Let us see what is the change; the horizontal one is 0.1659 mm. It was very close. And then the vertical one, I can do the calculation. When you substitute these values, I get this as 2.6406 mm. Let us go back and then do the comparisons. And δ_V comes to be 3.90 mm. And this was done by pin jointed analysis.

(Refer Slide Time: 37:53)

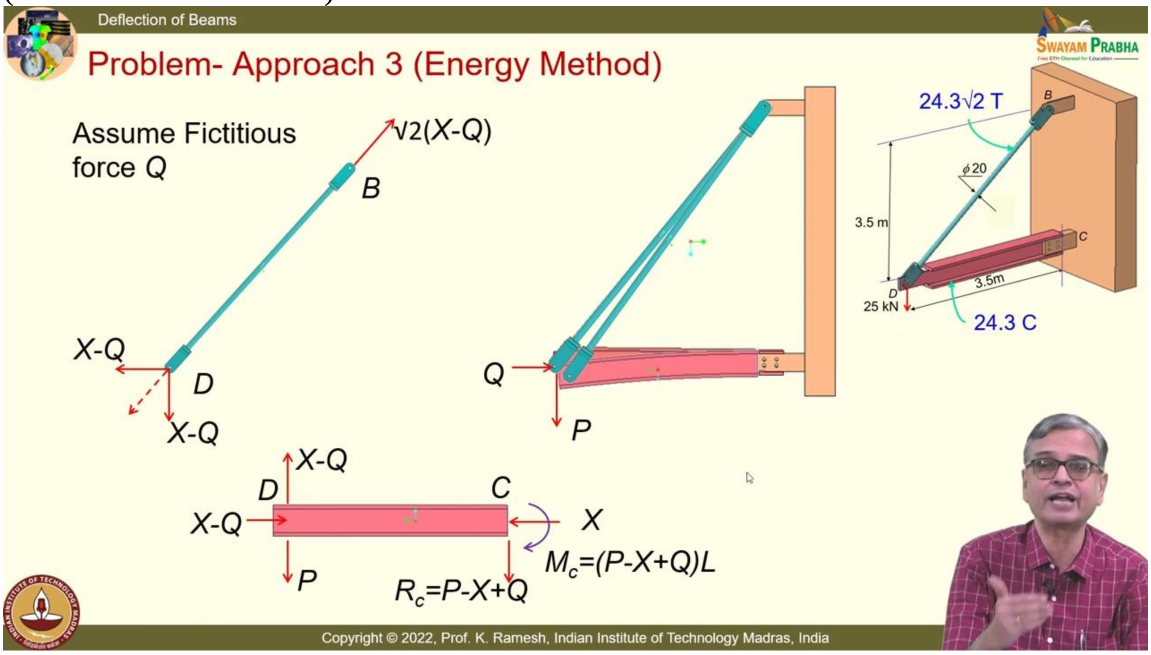


And when you have this analysis, I have this as 3.9 mm. What is the difference? 2.5%. For 2.5 % difference, you have to do back breaking calculations. So, pin jointed idealization is not far from reality. But there is one thing which is different. See, if you look at the I-beam, what are the stresses developed? The stresses developed are very high.

You know, I have bending that is taking place. And bending was ignored in your previous analysis. So, I have this stress as 24.85 MPa in the case of current idealization as clamped. If I have this as pinned, we would have just got this as 10 MPa. But if you know how it is designed, it is all designed based on stiffness. The stress magnitudes expected are much much smaller.

And you know this is mild steel and you know yield strength is around 255 MPa; 220 to 255 MPa. So, it is far less, even when you have the bending load. So, you are justified in making this as a pin jointed analysis. You get an estimate of what is happening in reality. Is the idea clear?

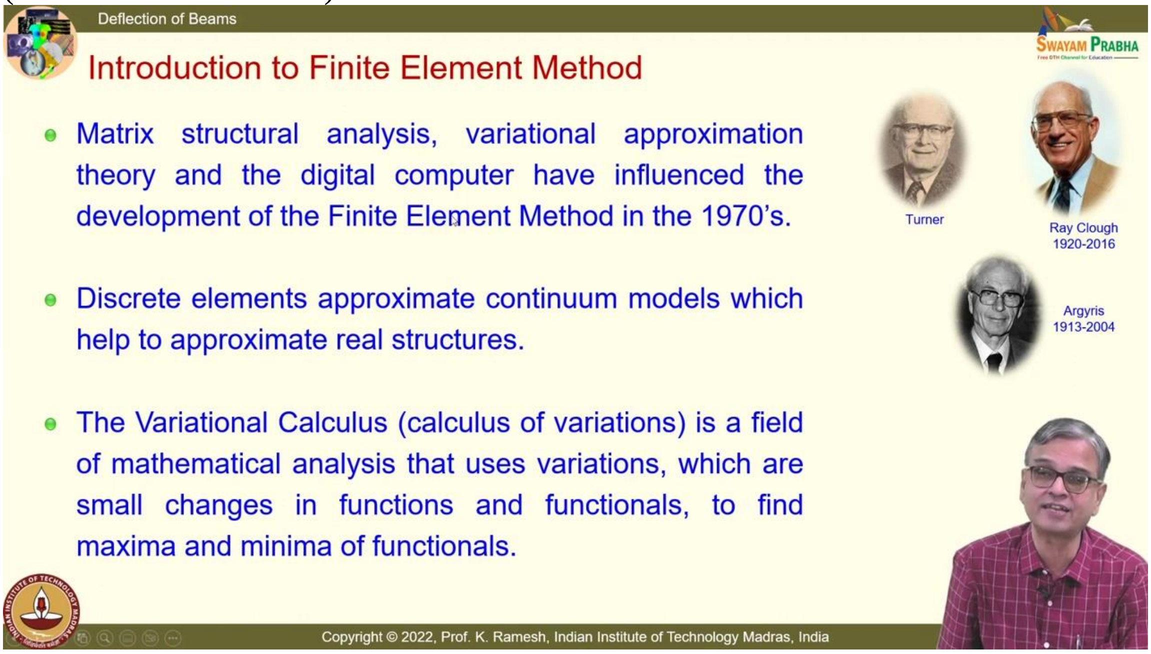
(Refer Slide Time: 39:32)



You know, if you go and do the energy method, I am not going to do it completely. I am going to just illustrate the procedure. The idea is, you know, I have to get the horizontal displacement also. Since horizontal displacement is not given, I have to introduce the certain *Q*. And I have also replaced this 25 kN as *P*. This is what I said. If you are not given as a symbol, put it as a symbol and do the energy method.

So, here again, I have to use the deflection for me to find out the forces. I am just showing this. From this on, you try to solve it yourself as a homework.

(Refer Slide Time: 40:13)



See, all along in this course, I have emphasized the role of experiments and only with the experiments, strength of materials developed. And whenever we looked at strength of material solution, wherever there is a contradiction, we also looked at theory of elasticity and determined the exact solution without assuming displacements and we compared. And we said whether the strength of material solution is valid or not.

And I have also said, if you go to theory of elasticity, the practical geometry that you can get exact solution are very, very limited. If I have to analyze a plate with a circular hole, I should solve it by polar coordinates. Suppose, I want to solve for an elliptical hole, you could not solve it with your Cartesian or polar coordinates. So, in the early development of theory of elasticity, people had done their PhD to develop coordinate systems. For solving an elliptical hole, they had to develop elliptical coordinates and elliptical coordinates are solved, used for solving plate with elliptical hole and elliptical hole becomes a crack in the limit.

So, the problem was to go to practical geometry. See, I have also said a simple spring which we have been seeing, only after learning torsion you could solve. And I have said you have seen a spanner; spanner even after one complete course on strength of materials, you do not have a solution for it because the geometry is very complex, neither I have solution from theory of elasticity. Then how do I solve it? The credit goes to the aerospace development; it was John Turner, head of Boeing's Structural Dynamics unit in the early 1950s, like you go for the internship, they invited faculty and it is Professor Clough from Berkeley to Boeing for faculty internship. So, only when the academicians go and sit with practical engineers, they understand what is their problem? What is the kind of mathematics required to attack the problem; that gave the birth of Finite Element Method. This was known as matrix structural analysis and you have Turner and he is a civil engineer, Ray Clough and it was developed in the early days by another professor, Argyris.

Not much detail is available about Turner other than that he was associated with Boeing. See, they were excited because in 1950s, when they want to develop an aircraft, they were all testing it before it is allowed for flying. But even before you test it, if you have a method by which you are able to predict its behavior, the number of test that you could do can be made lesser. Please understand, your numerical method will not help you to make an aircraft, it will help you to minimize the number of tests that is needed to certify the design. Ultimately, you have to bank on the experimental information. And if you look at finite element method, you had three different things that happened, one is matrix structural analysis; that is what John Turner did it in Boeing.

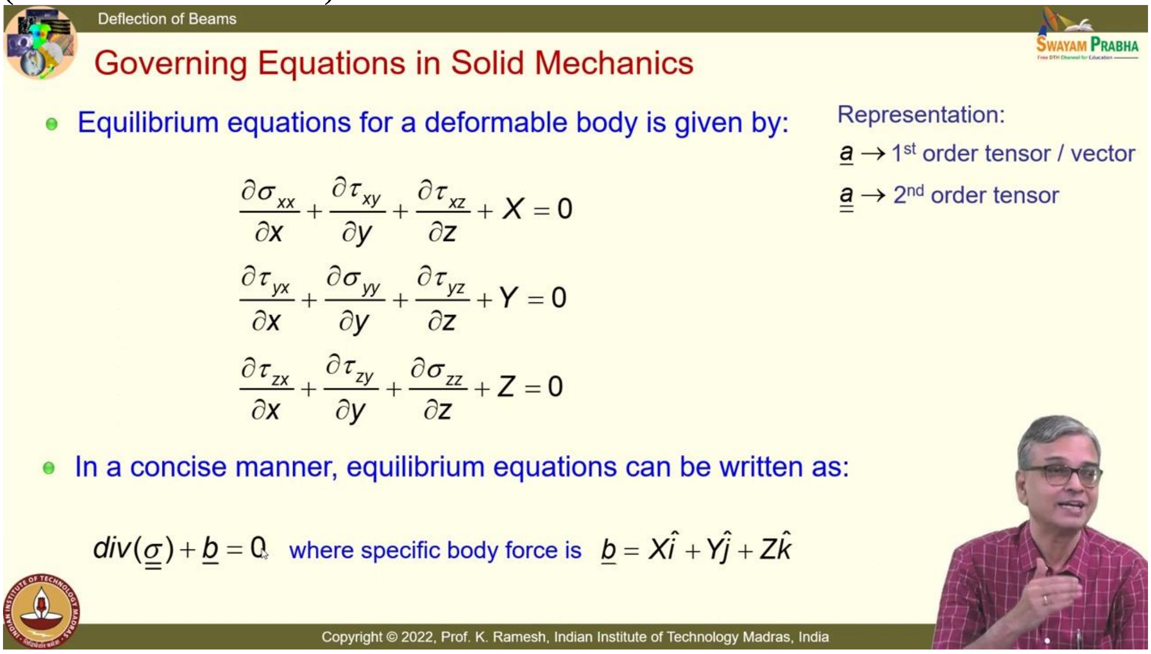
And later on, you had this proposed in terms of variational approximation theory, so that you can handle any type of differential equations. And particularly, the digital computer development has driven the development of finite element method because you need to solve thousands of equations and computers got developed only in 1960. So, only around 1970s you know, Clough called this method as finite element method. Originally, it was known as matrix structural analysis essentially solving your trusses.

You know, you have solved trusses where you have 10 elements. Suppose, you go and see a tower which is transmitting electrical cable, high voltage cable, you have thousands of elements, you cannot do it by hand computation.

So, initially they had truss elements and beam elements which are straight lines and they had conjectured how to solve this by a matrix analysis, it is called matrix structural analysis. Books have been written and they have used calculators, from calculators they went on to computers. And imagine, in those days, your simple PC will be occupying this entire room with 20 MB hard disk; please understand now everything is terabytes, fine. And only aerospace companies had the resources to have those computers. Academic institutions were not having that luxury, people have to go; that is why it all developed in an aerospace industry because there they have to do extensive analysis and testing before they roll out a plane.

And discrete elements approximate continuum models which help to approximate real structures and a larger generalization came from variational calculus. This is a field of mathematical analysis that uses variations which are small changes in functions and functionals to find maxima and minima of functionals. See we have already seen in Castigliano's theorem, if I take the strain energy and differentiate with respect to the applied load, I get deflection from the energy approach. Precisely the same thing is extended for finite element analysis. Here, I am going to solve and demand my differential equation is at least solved approximately, you are not demanding mathematical accuracy.

(Refer Slide Time: 46:24)

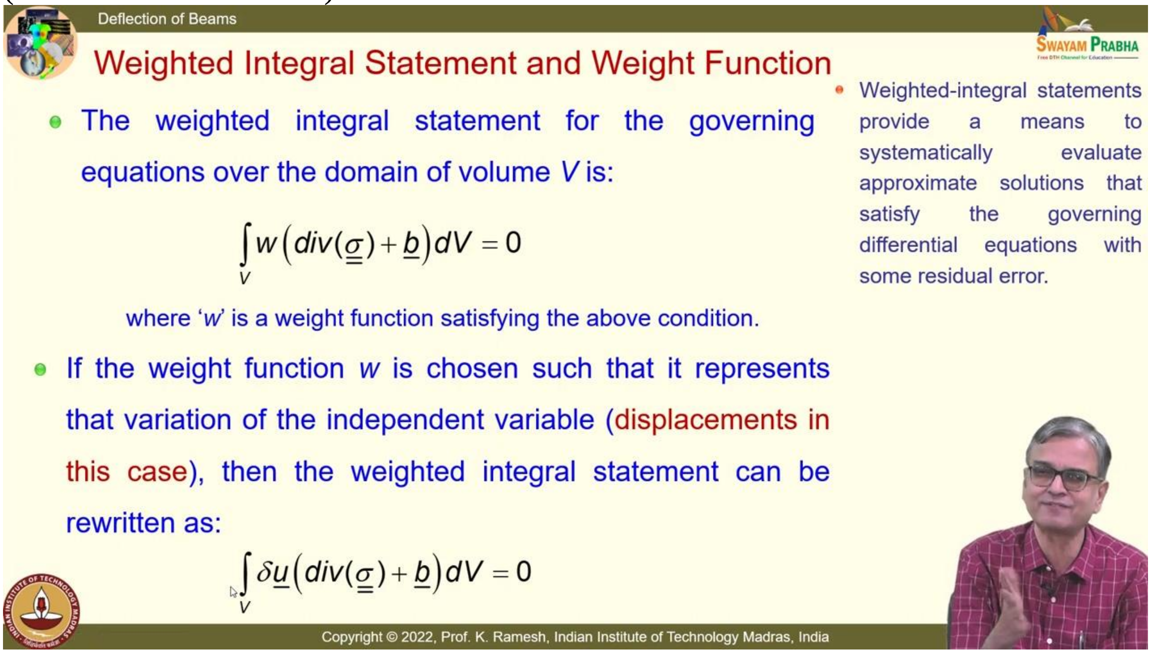


So, you have a formulation and you know, in the case of solid mechanics, the governing differential equation is essentially the differential equations of equilibrium which we have developed threadbare, you know how to develop it. Now what I am going to show is, you know to make our life little difficult to see the equations, this can be written in a very concise manner as

$$div\left(\underline{\underline{\sigma}}\right)+\underline{b}=0$$

where *b* is the body force and you have the symbolism if I have one bar it is a vector, it is two horizontal bars it is tensor of rank 2. And this is the basic governing equation, so you have to solve this governing equation to satisfy the boundary condition. The problem was the boundary is not well defined, so you piecewise solve this system in an approximate manner.

(Refer Slide Time: 47:23)

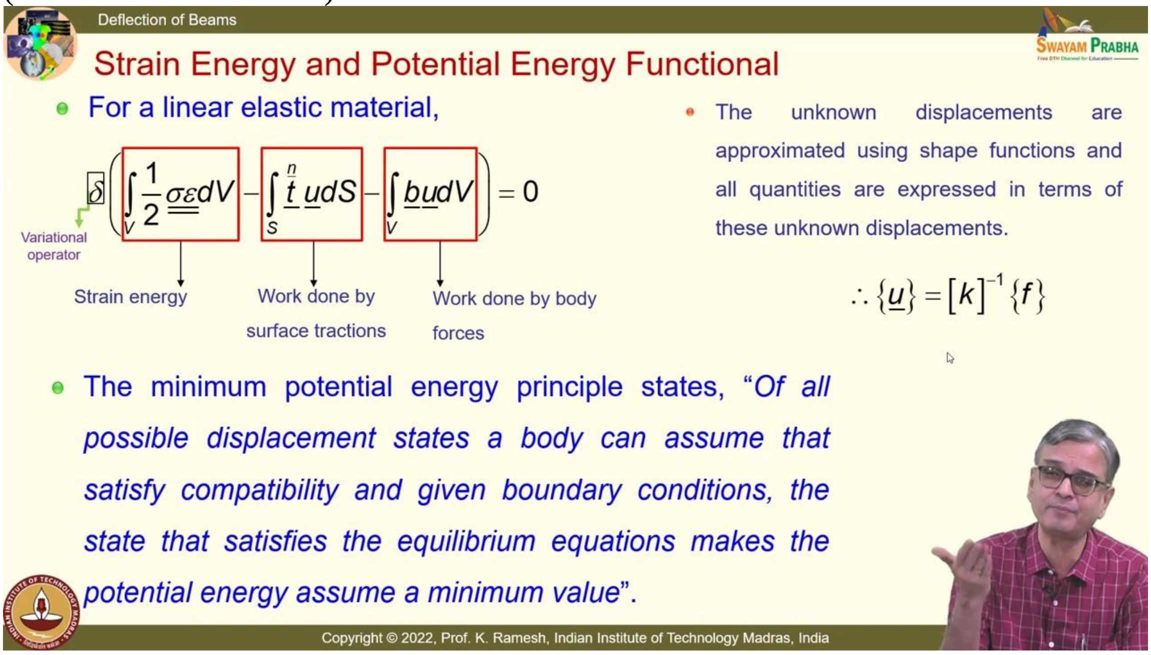


So that is what is important in finite elements, weighted integral statements provide a means to systematically evaluate approximate solutions underline this. Approximate solutions that satisfy the governing differential equations with some residual error, you have to live with it; if I am able to solve a very complex problem, some amount of error you can tolerate, but it should be done scientifically, it should follow certain mathematical principles.

So, I have a weighted integral statement and I have this governing equation is your equilibrium conditions and I have this weighted function and this weight function is also evaluated based on displacements. If the weight function w is chosen such that it represents that variation of the independent variable; displacement in this case. See in the case of strength of materials, what did what we do? We took cross- sections and loading in such a manner, you can visualize what is the displacement. In a complex problem, you cannot visualize the displacement, theory of elasticity helps you to solve the differential equations and evaluate the displacement by solving it as a boundary value problem. But the problem there was, when you took; when you take even a simple spanner, the geometry is complex, theory of elasticity fails; you cannot do it, but now I need to have a via media; that is provided by this approximate weighted residual type of approach and when I do this, I have this and this has a particular meaning, it is not delta, it is a variational symbol.

Because you know once you get into advanced mathematics, you have paucity of symbols; the same symbol will appear in different forms, you should understand the symbol based on the context.

(Refer Slide Time: 49:26)



For a linear elastic material, after simplification of that using the variational calculus, the expression turns out to be like this and this is variational operator. So, it is understood like this, if you have variational operator

$$\delta\int u = \int \delta u$$

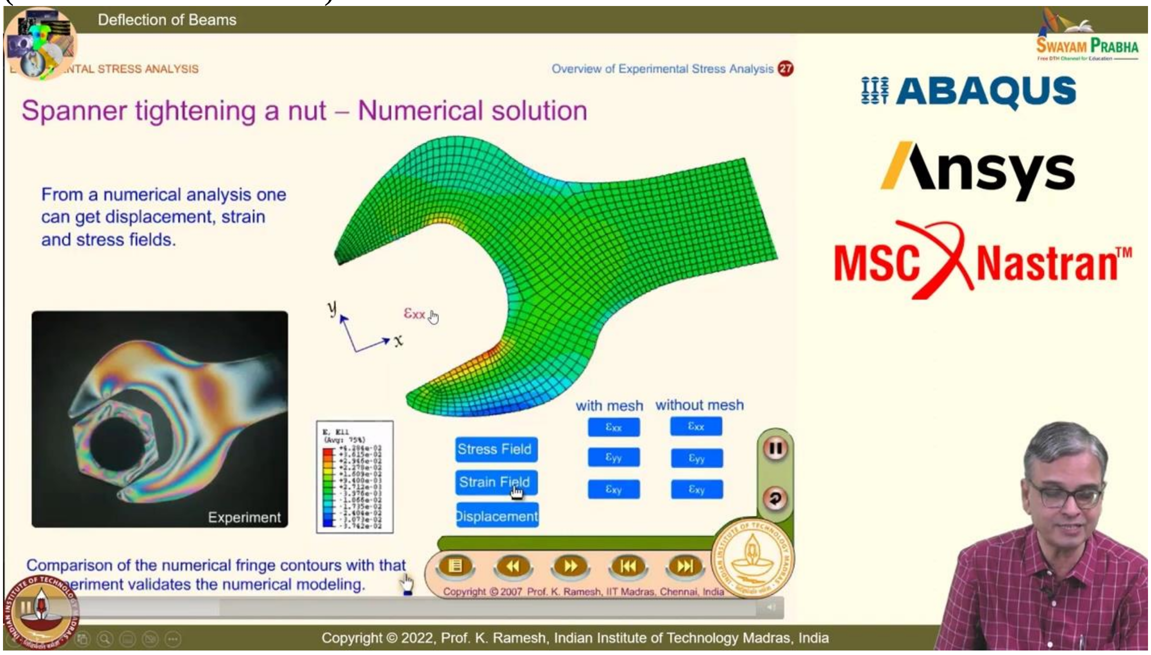
$$\delta\frac{du}{dx} = \frac{d(\delta u)}{dx}$$

This is how it is an operator. And they also use a different symbolism; this is nothing but your stress vector; instead of putting it as capital *T* they put it as a small *t*. And you know from Cauchy's formula, how to find out the traction on the boundary and I have also said, the discussion on free surface is very very important so that you are in a position to write the traction correctly.

So, even in finite element course, you have to do this. So, then what you do is, the unknown displacements are approximated using shape functions; all quantities are expressed in terms of these unknown displacements, all those details you will study in that course. I am only giving you a flavor; flavor is very important. So, what I do is, I apply the minimum potential energy; the principle states, of all possible displacement states a body can assume that satisfy compatibility and given boundary conditions, the state that satisfies the equilibrium equations makes the potential energy assume a minimum value; that is what we are doing it. What we are doing in a complex problem, we get this very close to reality with some small amount of error. So, you are doing the minimization and this is how finite element is made, you have this as a final expression; this is known as a stiffness matrix, you are accustomed to solving 3 by 3, 4 by 4, there it is 10000 by 10000 or depending if you have a super computer, it can be million by million.

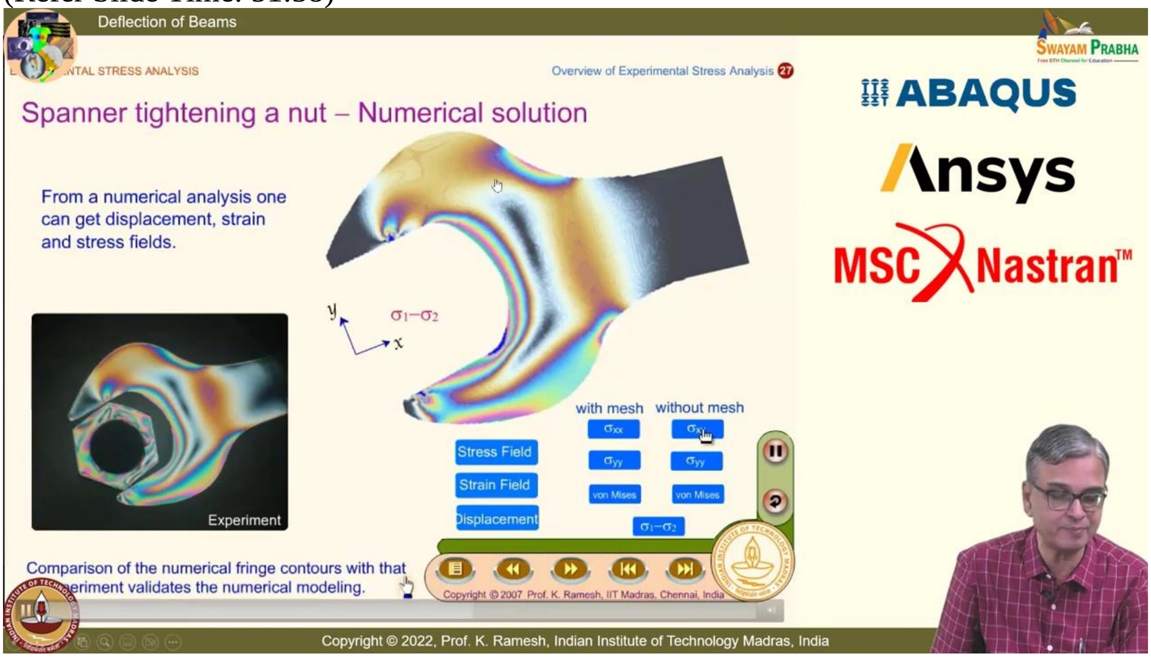
So, there are lot of challenges in how to get the inverse, it is not a simple task. So, mathematicians have a great role to play.

(Refer Slide Time: 51:28)



Now, let us take the simple spanner problem; you have seen the fringes, you have also seen these fringes in your laboratory course and if I use it by finite elements, I can get the *u* displacement, *v* displacement and this is how you discretize the structure.

(Refer Slide Time: 51:38)



I can get strain; ε_{xx}, ε_{yy}, ε_{xy}; I can also get the stresses, I can get σ_{xx}, I get $\sigma_1 - \sigma_2$ see this comparison between the experiment; I have shown because experiment can exploit a physical principle, it can give you only one type of contour. But once I solve it by numerical method, I can get all the contours and finally, you know, I am also showing $\sigma_1 - \sigma_2$, see the comparison even though you have a feeling that finite element does some approximation, if you handle it properly, there are many many criteria what should be the size of the mesh, how to have convergence, all those aspects if you follow very closely, you can have the fringe patterns very nicely match with experiment. And you have standard packages like Abaqus, Ansys, MSC Nastran and many many packages; and mind you none of the packages post process the results of finite element to plot fringe patterns; it is only our group has done it, we have published as papers; we hope that some of the finite element manufacturers take the cue from our papers and then incorporate this as a feature as part of the software.

Here, you can see the comparison is convincing that you have done a reasonable analysis. So, please understand, you also have a method; strength of material is not the end of it. You have to do a course in experimental stress analysis as well as a course in finite elements to solve day-to-day current problems in engineering. Thank you.