Strength of Materials
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Lecture - 21
Torsion 2 - Mathematical Development

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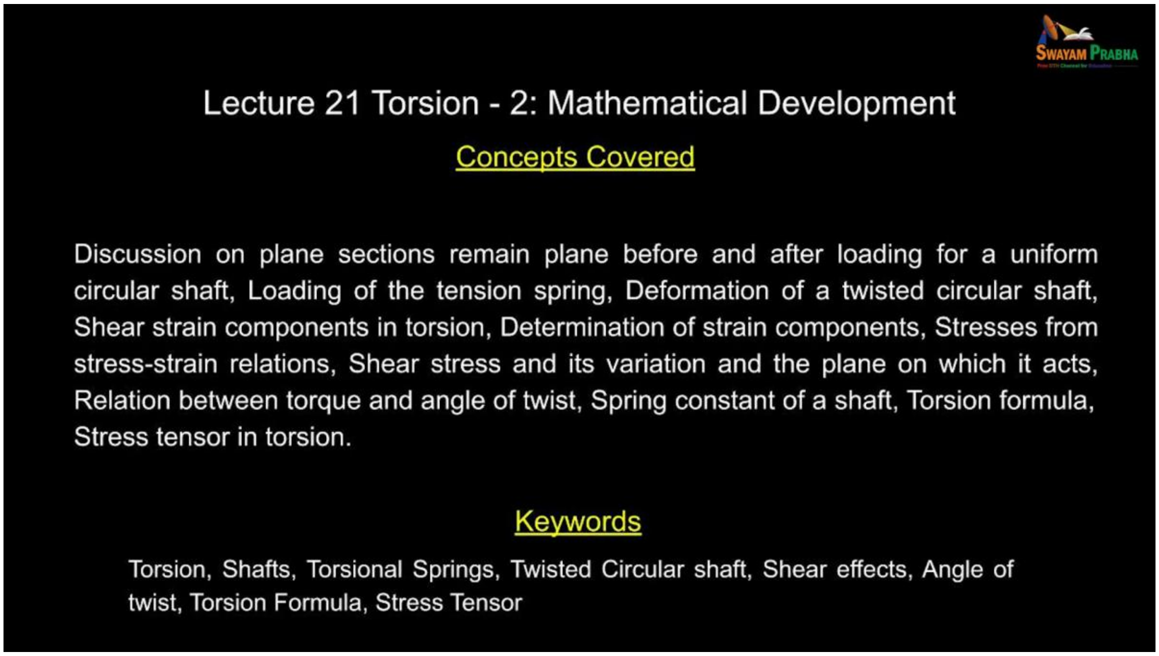

See, let us continue our discussion on torsion. In fact, we have done the thought experiment followed by an experiment and we have understood certain aspects of torsion, fine.

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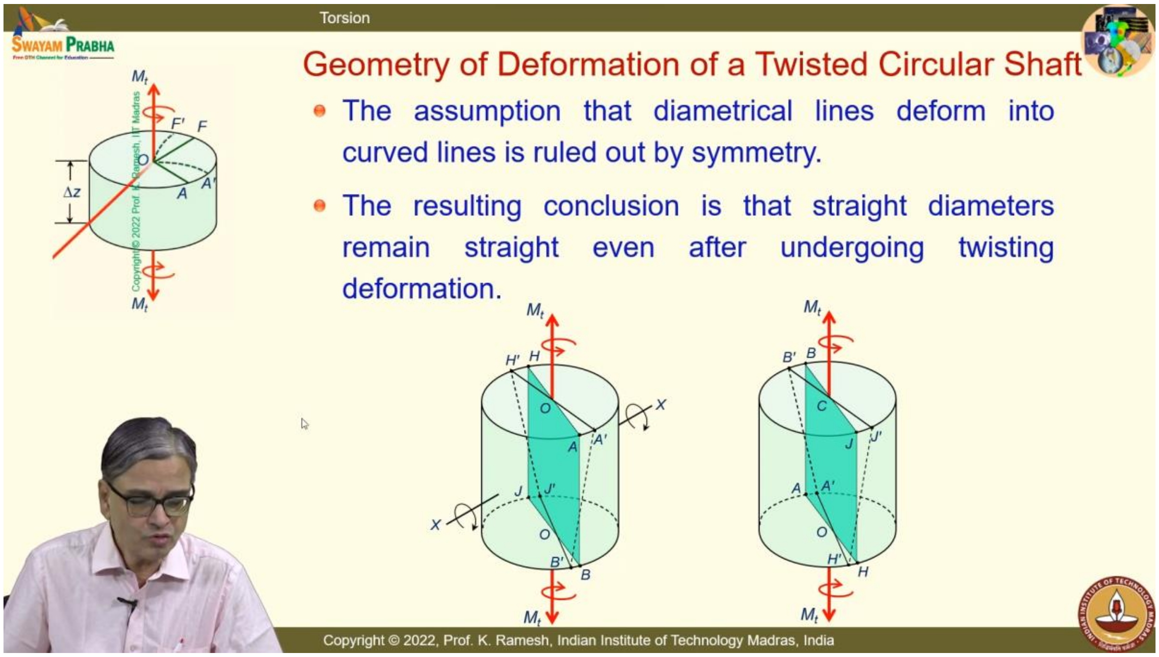

And see, what we did was, we hypothesized that I have a radial line that deforms into a curve. So, we started deliberately with a wrong hypothesis and then we looked at the geometry of deformation. And the main point that we had a discussion was invoking the symmetry. I was able to invoke the symmetry mainly because the cross section is circular. So, about this axis, geometry is symmetric. I consider the material as isotropic. So, elastic properties are identical along any direction and the boundary conditions are also symmetric about the axis. So, based on that, we have concluded that straight lines will deform into straight lines.

And we have also verified by the thought experiment when you rotate it by 180°. What you have at the bottom surface when I rotate it, it is identical to the top surface. We have not only done the thought experiment; we have also seen it in the experiment. So, first understanding is the radial lines will deform into radial lines. And we also observed from the experiment that these radial lines lie on the same plane. Nothing happens to that plane that we have not done any thought experiment. But we have done the actual experiment and seen that that plane remained horizontal.

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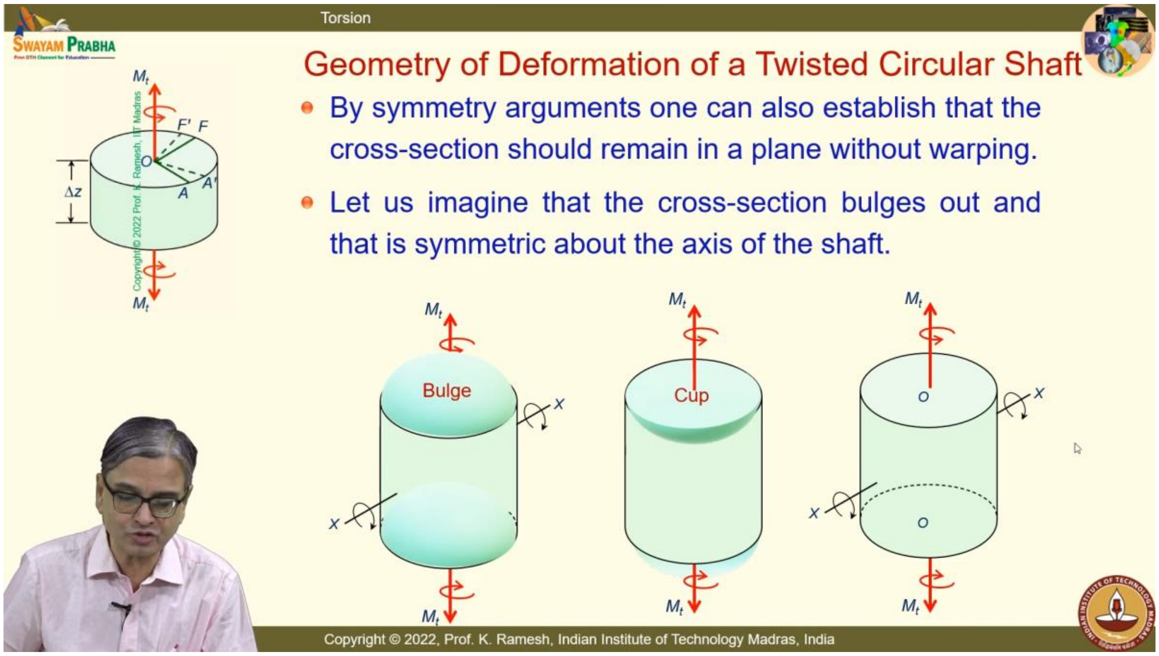

But I can also investigate that by a thought experiment. The only requirement is I should have the deformation symmetric about the axis. Any way you can take it and here I can consider that this bulges out and it is symmetric about this axis. This happens on every cross section. So, this cross section also bulges out.

And we know how to test symmetry. If I rotate the whole system by 180°, it should come to be the same shape which is not going to be the case. Because when I rotate it, because it is bulged, I would have this as a cup. So, this violates symmetry. So, from thought experiment also you can say that these planes remain plane. It is a very very important concept. It is also taken as an idealization. But when you perform an experiment, you see that very clearly in the experiments. So, it is no more thought experiment. We have captured reality in all its totality for a circular cross section which is a slender member and the deformations are small. All these attributes you should look at it.

So, when I rotate it, you will have the same this one that is very clear and obvious.

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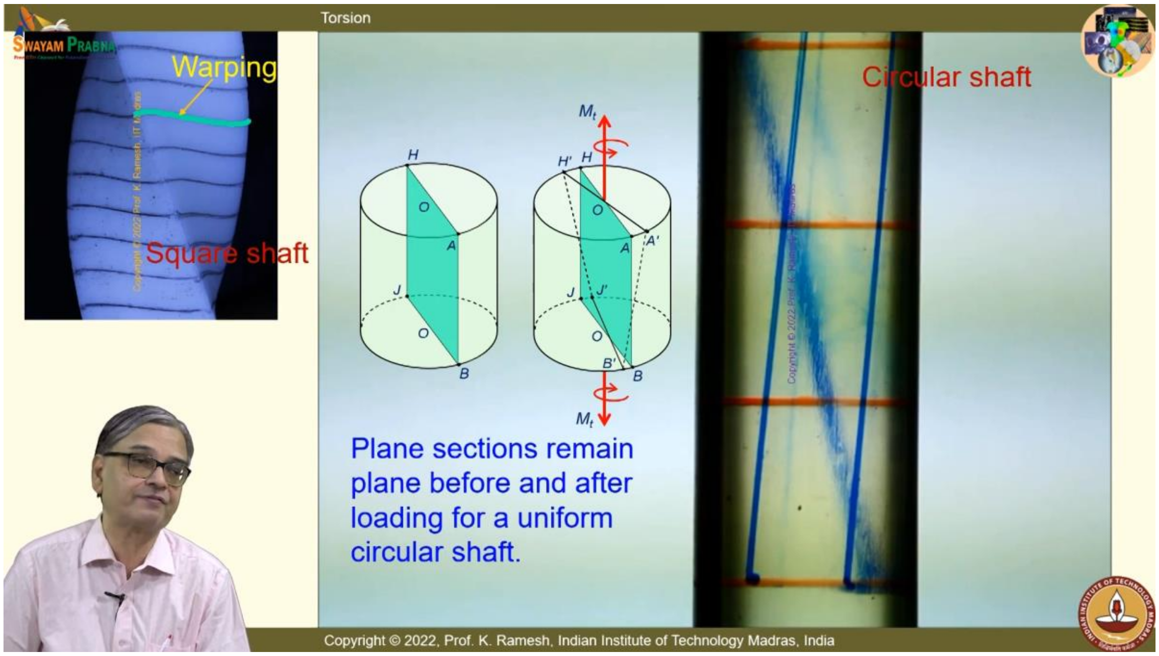

And now I have a circular shaft that is twisted. And you see the horizontal plane as this red line. And your vertical lines are drawn blue. See this is a cylindrical object, the line drawn at the back is magnified because this behaves like a lens. That is why you see that shade of blue thick here. And you could see very clearly that these lines remain horizontal. And then you have these two lines sheared, fine, and in the case of a square shaft, this line does not remain horizontal, but it has a shape like this. Fine. You know my hand was shaky. So, when I draw it with free hand, some kinks are seen. You can imagine that this is a smooth line. So, this has ups and downs and this is labeled as warping in the literature. So, we do not go to the analysis of square shaft in this course. And in the case of circular shaft, one of the important understandings is plane sections remain plane before and after loading for a uniform circular shaft. It is a very very important observation.

In fact, when we graduate to bending, we would use the same idealization and choose an appropriate loading which satisfies that and develop the bending theory. Whole of strength of materials hinges on plane sections remain plane before and after loading because using this assumption, we are in a better position to appreciate what is the deformation possible. Once you have captured the deformation, you do not require, you do not require to solve this by solving a boundary value problem. That deformation is understood. From the deformation, write the strain. From strain, you write the stress strain relation. So, the complete solution is obtained.

So, the key point here is how well you have picturized the deformation clearly. So, what we do in strength of materials is, we choose simple loading, simple cross section that satisfy this basic idea. Plane sections remain plane before and after loading which you can see for many problems later also. This would be satisfied. So, I do not have to solve for deformation by solving the boundary value problem of the differential equations.

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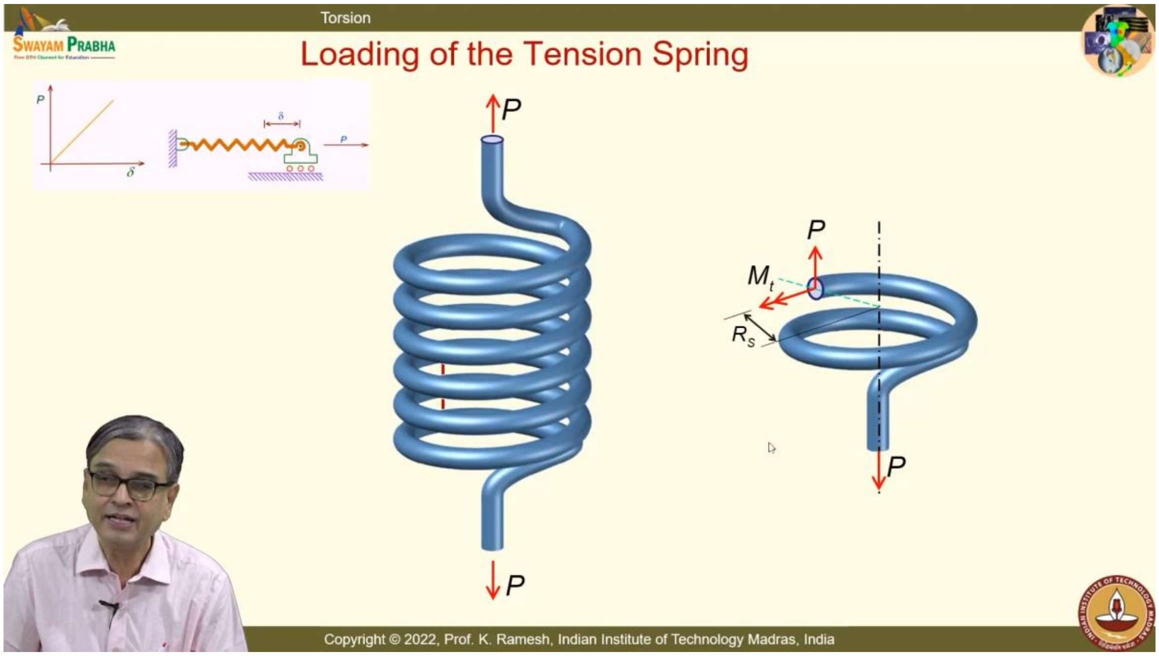

And you know, I have also asked you to exercise your thinking power to find out what happens in a spring because we all start teaching deformable solids by taking the help of a spring. See, I have said we use the spring only to illustrate the force deformation relationship is linear. And the other aspect of spring is also, you will, you can easily visualize that spring stores energy. So, all deformable solids will store energy. So, for these two aspects, we look at it.

And when you look at the sketch, you know, schematically it is easy to represent the spring like a zigzag line and when you axially pull it, it opens up and then when you compress it closes up. If you want to find out what way spring resists load, it is not tension or compression. Is the idea clear? You have to look at how the spring is constructed. I have the spring which is coiled like this and then you apply the axial pull. And in this case, you know, it is coiled in such a manner, this forms the axis of the spring. So, now what I will do is, I will cut at one of the coils here. I have come like this and then I have taken this and I have removed it and I have drawn the diagram here and this is the load that is applied.

Can you find out what is the resistance developed at this cross section because it is nothing but you have a load and then if you look at what is the way this cross section is located, this is located by the radius of the spring you can say, what kind of force system will exist on this cross section? Because you have the load that is applied which is along the central axis. So, this will try to, it is relevant to this discussion, it is trying to twist it. You get the point. A tension spring resists the tensile load by developing torsional moment. So, it is actually transmitting only torsion. In addition to this torsion, you will also have the axial force. So, you will have a shear introduced on this cross section as well as a resistance due to torque applied. So, if you have to design a spring, you should know what happens in the case of a torsion. It is no longer tension and compression.

In a similar manner, I have also raised another question. Go and open up your cycle carrier. Find out what is the load that is transmitted by this spring, fine. So, you should understand this. Make a neat sketch of this. When you want to find out how to design a spring, you should know torsion to design a tension or compression spring. Is the idea clear?

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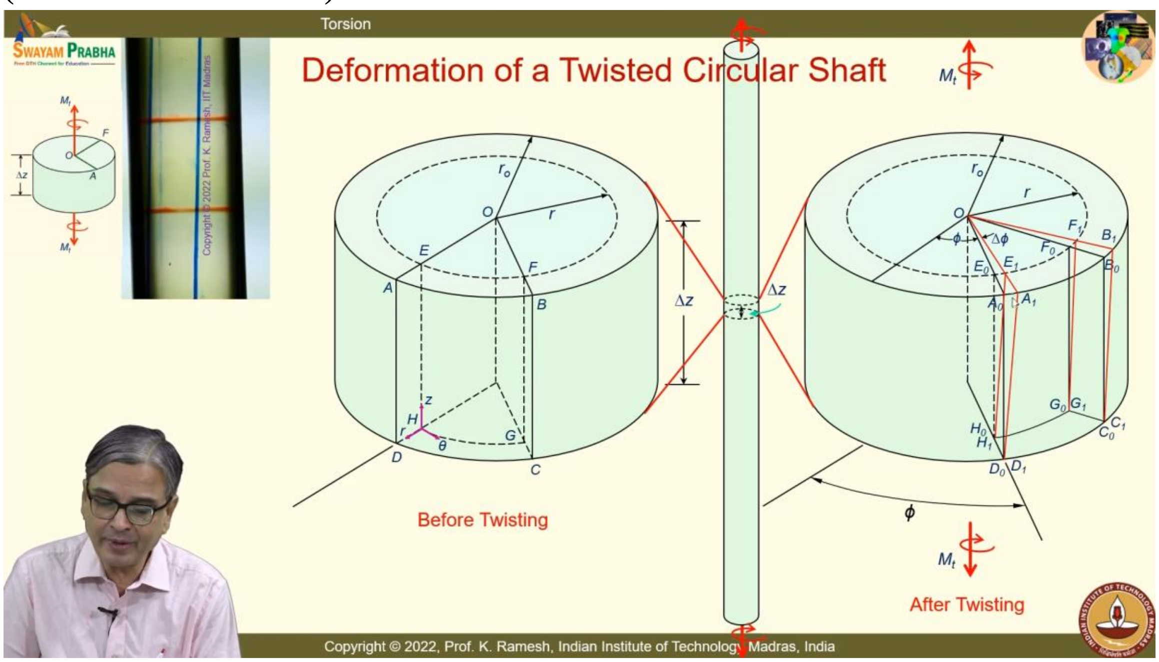

Now, you need to make neat sketches. I will show the animation slowly. And you have to keep in mind that we are again analyzing only a slender shaft. I take a very small section delta z. I have blown it up for me to have clarity on what is happening in this small section. So, do not get carried away by the size. It looks like a slab, huge slab, fine.

So, it is actually a very slender member. So, I have taken an arbitrary section. You can draw the lines and also label them the way that I have done it. Take your time. And what we are also going to do is, we are going to find out at an arbitrary radius *r*. I will also draw a arbitrary radius *r*. And I would like to find out what happens on this interior part of the shaft. Because we have already seen, even before developing torsion, we know that shear stress varies linearly from the center to the outer fiber. So, in order to develop that relationship, it is better that we take an arbitrary radius *r* and find out what is the kind of dotted lines that you have to start with. Then the axis of the shaft is taken as *z*-direction and you have the radial direction and theta direction because it is a circular geometry. It is better to go to a polar-coordinates.

And this is before twisting. Fine. And this is a section somewhere in the middle of the slender shaft. Now, what I do is, I apply a twisting moment. When I apply the twisting moment, I have the twisted shape here. That is what we have to understand and then find out what kind of changes that happen. And you also have the nice animation which is continuously running, which will tell you what way I have to draw the deformed lines. The first thing is, you will have to appreciate, I have a section far away from this end. So, this section would have twisted by an angle ϕ .

And I have another section which is at resistance Δz away, that is the top surface. It would shift by, twist by $\Delta \phi$. I have ϕ and $\Delta \phi$ is what you will have to understand. So, I have this and I have drawn the same lines here. Whatever the lines that I have drawn it earlier, I have drawn it. Because my idea is, because of applying the twisting moment, how do these lines deform? Because once I understand the deformed position of these lines, we already have definition of the shear strain.

Shear strain is nothing but change in the original rectangle. So, we will have to investigate whether it reminds us original rectangle after deformation or is there any change? So, please draw this. First you have to appreciate, I have taken a section somewhere in the middle of the section, because we do not want to study anything close to the load application point, because we all know the Saint Venant's principle, fine.

So, we want to stay away from the end effects. So, when I apply the moment, the bottom of this section would have had an angle of twist of ϕ and just the slice away, that is Δz away, it will have, which we have not drawn. Now we will draw this. As far as the small section is concerned, only now I am going to put the twisting moment. Because the twisting moment, this line shifts like this, that is drawn in red. And I have this line as, you know, I have $E_1F_1A_1B_1$ is what is the new position, D_0 and D_1 are identical and, C_0 and C_1 are identical. And what you have is, I have this original line E_0A_0, that is the black line and the deformed line is the red line, because we have already seen radial lines remain radial, so the line remains straight. And this small angle is $\Delta \phi$. And you can find out what is the distance $E_0 E_1$, I can find out the distance $E_0 E_1$. Can you tell me what is the distance $E_0 E_1$? $r\Delta \phi$, very good.

And we will also have to investigate and you know, very clearly you have this. You have this sheared; these blue lines are sheared and that is what you see here. Here it is drawn as red, so that you can distinguish between the undeformed and deformed position. And also label them, we will use this drawing subsequently. You are going to see this drawing again and again in the slide. So, even if you have missed some portion of it, you can fill it up.

We are going to investigate this deformation and find out which are the angles remain 90°, which are the angles do not remain as 90°, that contributes to the appropriate shear strain. Because we have already ruled out existence of normal strains, you would not have ε_z, ε_r and ε_θ, all three are 0. Among the three shear strains, which of the shear strains are 0 is what we will have to look at. But before that, we will first calculate the existing shear strain due to torsion.

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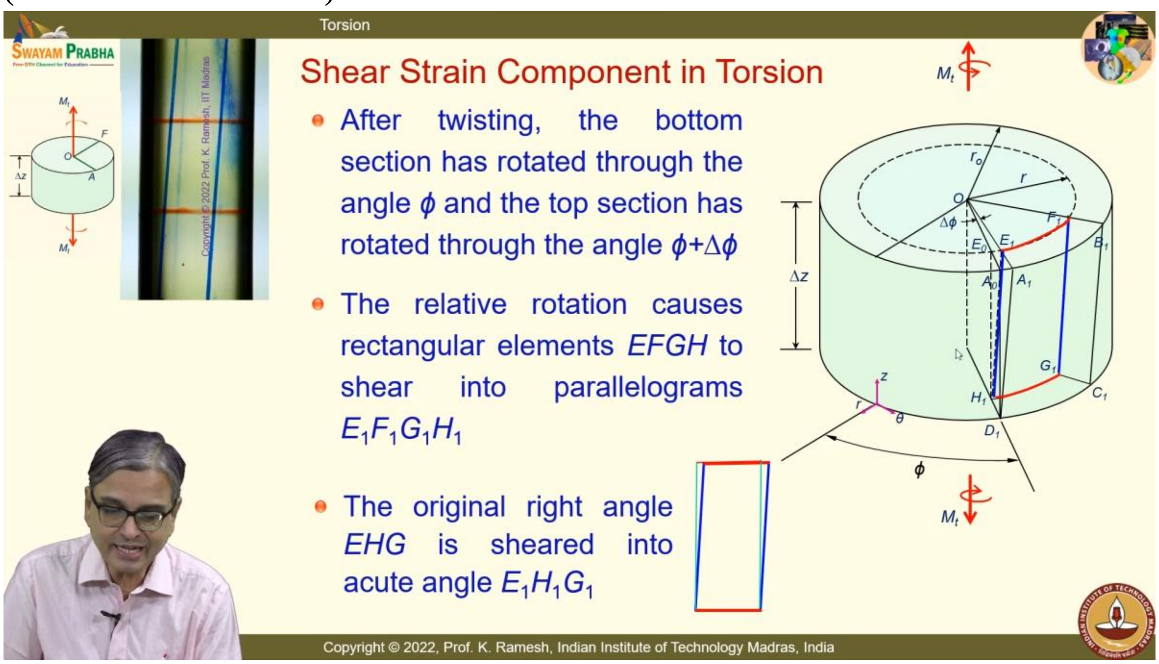

And this is again explained, whatever I have said is again explained here. You have to appreciate that why I have shown ϕ here, why I have shown $\Delta \phi$ here. You should appreciate that because this is taken somewhere in the middle of a slender shaft. So, I have angle of test is ϕ and $\phi + \Delta \phi$. And we will see that $EFGH$, $EFGH$, it has gone into $E_1F_1G_1H_1$.

You know I have followed the same color code, I have the circles as red and then the straight lines as blue, so that you can easily visualize, you can easily visualize from this diagram, what has happened? I have these blue lines sheared, which we have seen by the experiment. Same thing has happened here. So, the original right angle EHG is sheared into acute angle $E_1H_1G_1$, which is also shown in a two-dimensional sketch here. You have already calculated what is this, that is nothing but $E_0\ E_1$, you know that is $r\Delta\ \phi$. And you know what is this height, what is this height? Δz.

So, you have all the quantities, so you can find out what is the shear strain, fine. And you should also be able to say, how will you label this shear strain? I have the axis $r\theta z$ label like this, fine. Think about it. We have seen that this has sheared, and we are in a position to identify this angle and also this distance. This is happening between what axis?

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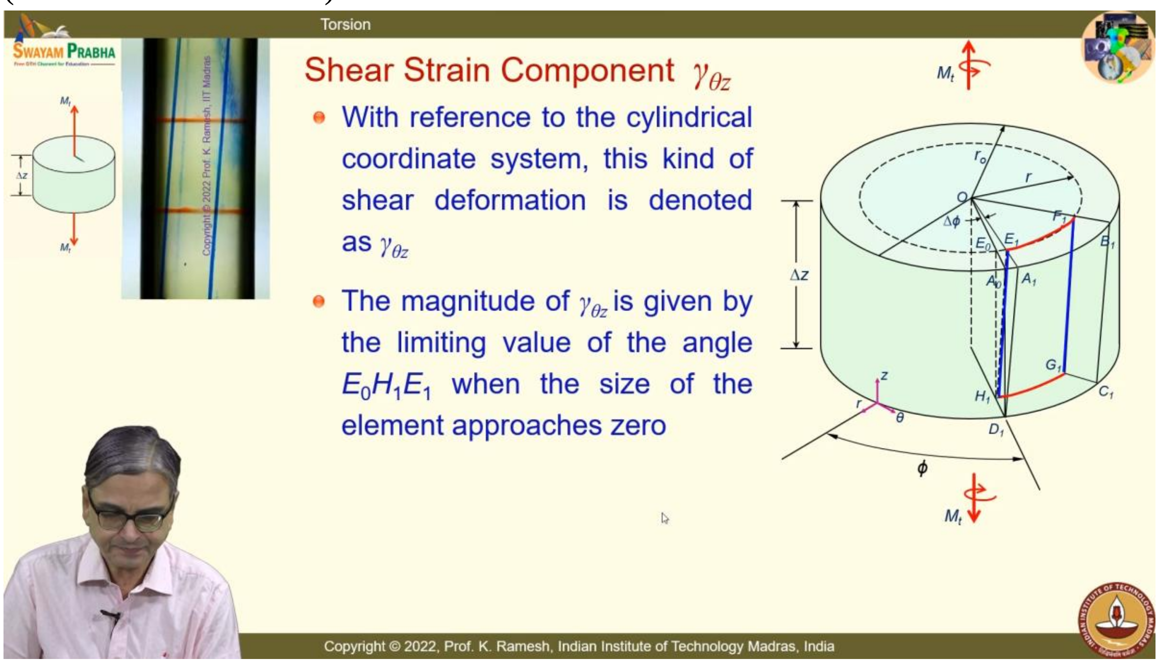

θ and z. So, I have this as $\gamma_{\theta z}$. See tensorially, we want to have $\varepsilon_{\theta z}$. If you want to write a transformation relation, we want to write $\varepsilon_{\theta z}$. But all the development in the early part of the strength of material, they have done it only with gamma. And you know $\varepsilon_{\theta z}$ is one-half of $\gamma_{\theta z}$. Now, we have to find out what is the magnitude of $\gamma_{\theta z}$. You have already done half of it. We will fill in the mathematics and get the values. And we will also have to understand in the limit Δz tends to 0. That is what we are going to approximate. And again, this is shown for you to give appreciation that the blue lines have sheared.

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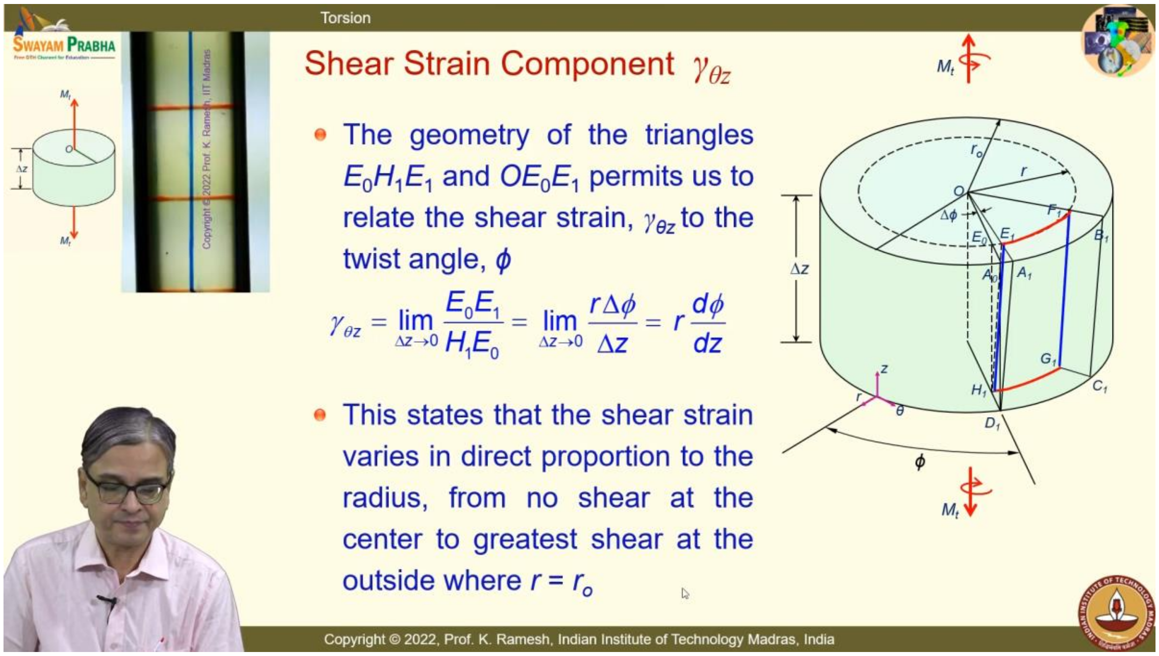

And we have already looked at the geometry and you have already told me what is $E_0\ E_1$

$$\gamma_{\theta z} = \lim_{\Delta z \to 0} \frac{E_0E_1}{H_1E_0} = \lim_{\Delta z \to 0} \frac{r\Delta\phi}{\Delta z} = r\frac{d\phi}{dz}$$

So, I can have $\gamma_{\theta z}$ as limit Δz tends to 0, fine. Because we write $\tan\theta$ and then θ is small, we write the ratio, the appropriate distances. So, $\frac{E_0E_1}{H_1E_0}$ will give you and $E_0\ E_1$ you have already said correctly, $r\Delta\phi$ and then H_1E_0 is Δz .

So, this gives me $r\frac{d\phi}{dz}$ as your shear strain component $\gamma_{\theta z}$. You will have to appreciate that this is a function of r and how we have looked at r? This is 0 here and you have the generic r here and this is $r_{\max}$. So, r changes from 0 to $r_{\max}$. So, your shear strain also will change linearly over the cross section from the center to the outer radius. Outer radius has the maximum value of shear strain. So, that is what is documented here.

The shear strain varies in direct proportion to the radius from no shear strain at the center to the greatest shear strain at the outside when r equal to r_0, r_0 shows O means outside diameter, outer diameter, whichever way you can take it.

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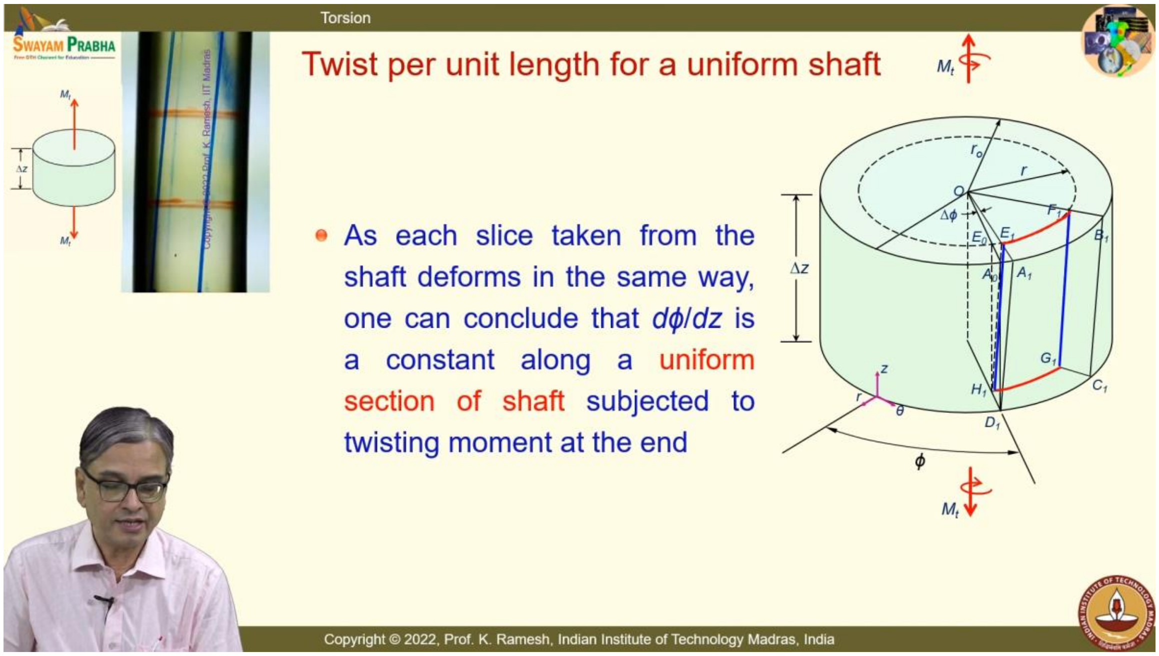

And let us look at another aspect. One can also conclude if I have a cylindrical shaft, $\frac{d\phi}{dz}$ is a constant along a uniform section of shaft subjected to twisting moment at the end. So, I can have a twist varies linearly over the length.

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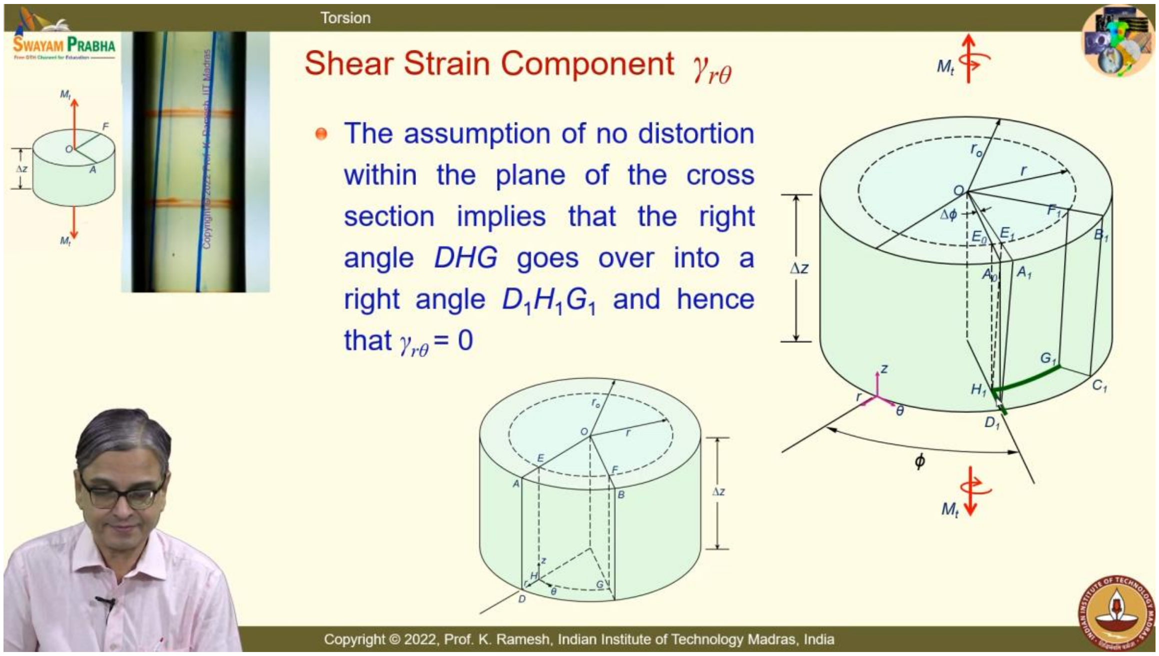

Now, let us look at what happens in the r and θ directions. So, r and θ directions means I have to worry about what is H_1D_1 and H_1G_1. So, we have originally it was DHG. Now, this is $D_1H_1G_1$. And if you look at, if I draw the tangent to these lines, you will find that included angle is 90°. There is no distortion there.

If the original angle is preserved as 90° after deformation, you say that component of shear strain is 0. So, I have $\gamma_{\theta z}$ goes to 0. So, you see these two lines, radial line and the circumferential line. And if you draw the tangents to this circumferential line and find out what is this angle, this angle will still remain as 90^o.

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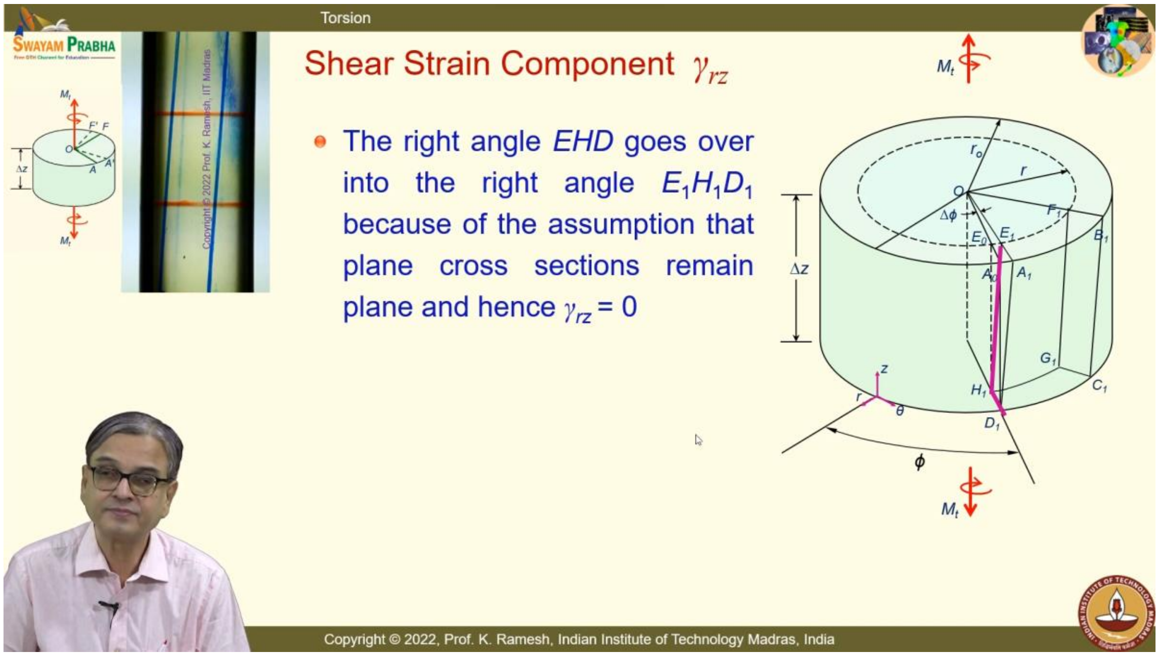

Then we move on to the shear strain γ_{rz}. So, when I have to look at r and z, I have to look at what happens to $E_1H_1D_1$, that is $E_1H_1D_1$. Here again, if you draw the line, it is only sheared like this. You can see very clearly here, sheared like this here. It is only sheared like this. The blue line is shown as pink here for the purpose of this. And if you look at what is the angle between D_1H_1 and H_1E_1, here it will be 90°.

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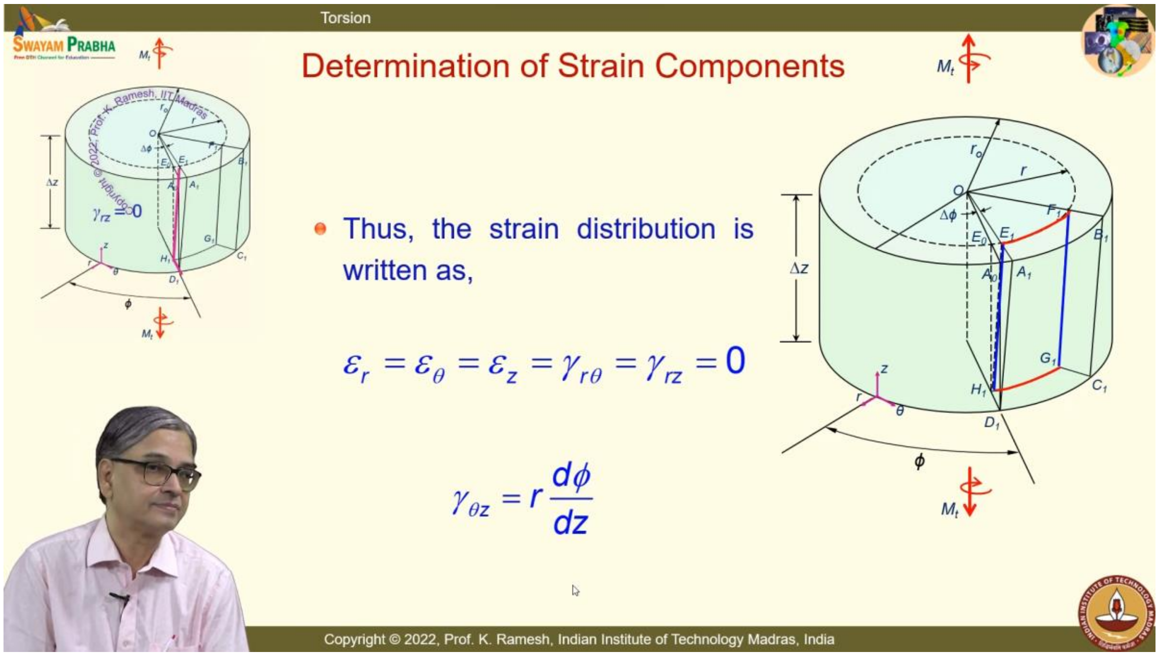

So, γ_{rz} is also 0.

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0$$

So, I have the strain components. 5 strain components are 0, ε_r , ε_θ , ε_z , $\gamma_{r\theta}$, γ_{rz} , they are all 0.

$$\gamma_{\theta z} = r\frac{d\phi}{dz}$$

The only shear strain that is existing in torsion is $\gamma_{\theta z}$ and that is given as $r\dfrac{d\phi}{dz}$. Is the idea clear? See, we have taken a very simple problem to illustrate. Understanding developed here can help you to solve even complex problems. I said that if you have a stepped shaft, definitely in the region of a step, there will be disturbance. So, you can either bring in stress concentration factor or bring in a factor of safety or bring in some other empirical relation if the structure is too complex. So, wriggle out of from that zone, but rest of the zone you have solution, exact solution. So, that is how engineers operate.

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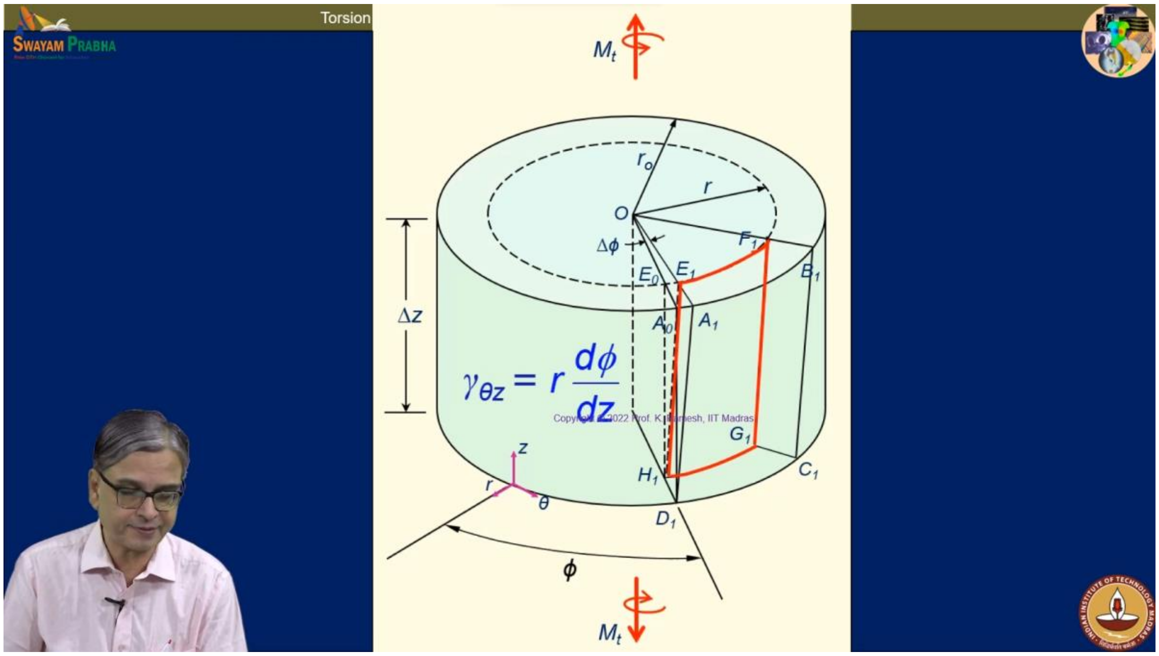

So, you whatever we have discussed, it is repeated as animation here. So, you can ruminate and then see stage by stage how we have looked at the deformation. From the deformation, we have figured out what is the non-zero strain that is present. We have also written it in terms of the geometry of the problem. So, it is obtained as $\gamma_{\theta z}$ and then $\gamma_{r\theta}$ is 0 and then γ_{rz} is 0.

So, you have the strains given. The only restriction placed on the material is it must be isotropic. Only then I can invoke symmetry conditions. If the material is not isotropic, all my discussions go for a six. Fortunately, many of the engineering materials can be modeled as isotropic.

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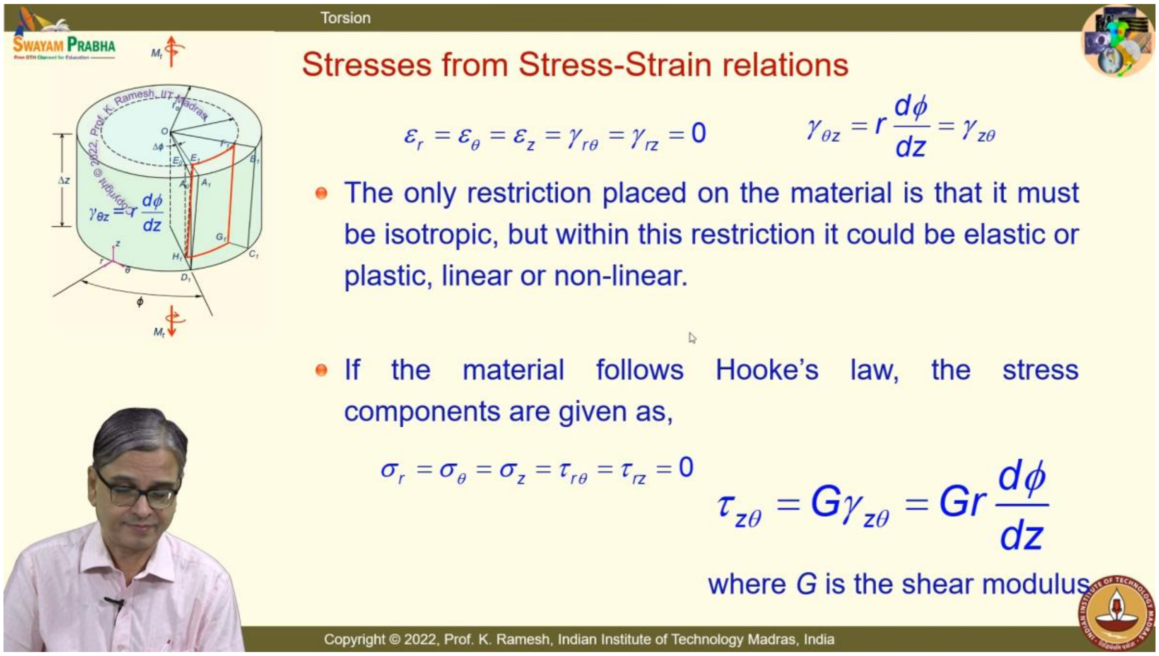

So, whatever the relationship that we have developed can be easily applied to elastic conditions, plastic condition, linear or non-linear as far as the strains are concerned.

So, if the material follows Hooke's law, the stress components are given as when you have five strain components are zero, five stress components are also zero. We have already seen in the case of isotropic material, normal stresses produce normal strain, shear stresses produce shear strain. There is no coupling. The coupling happens in other than isotropic materials. So, from the knowledge that we have already gained in stress strain relation, I can find out $\tau_{z\theta} = G\,\gamma_{z\theta}$.

And you would have noticed, I have used the idea of equality of cross shears or cross shear strains that is what is shown here. $\gamma_{\theta z}$ is identical to $\gamma_{z\theta}$. Similarly, $\tau_{\theta z}$ equal to $\tau_{z\theta}$. And now you have an expression that is $Gr\frac{d\phi}{dz}$, where G is the shear modulus you all know. So, now we have the exact solution even though we started with a hypothesis, hypothesis is verified by experiment. So, what you get the relationship now are exact solutions.

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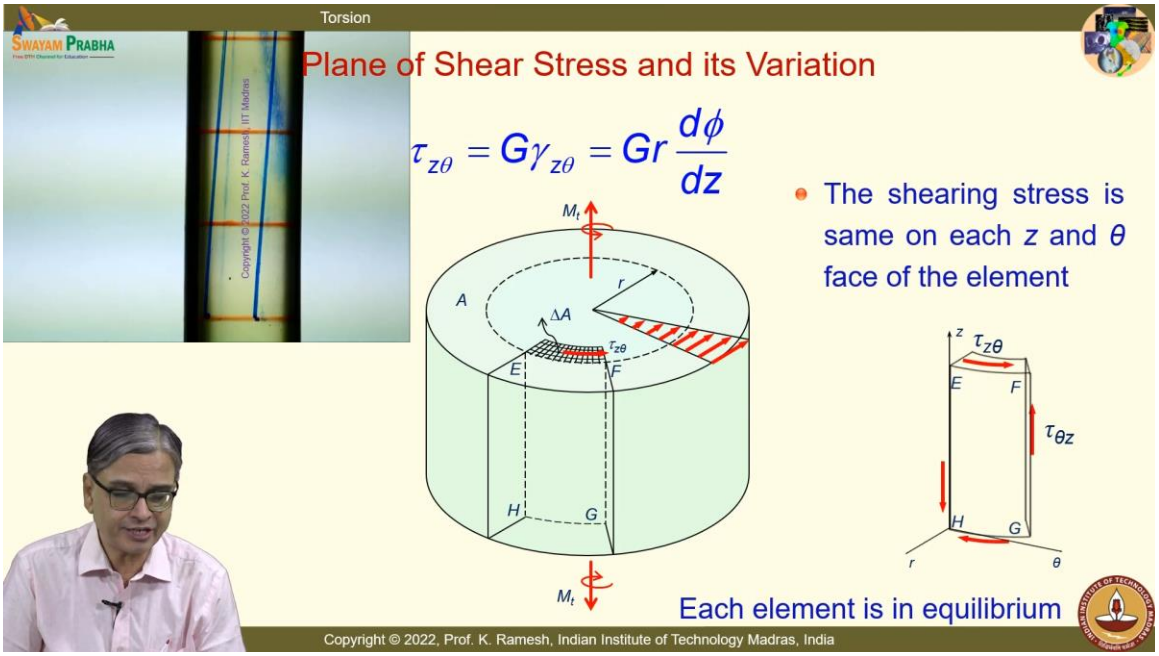

And you know I have always noticed whenever you learn the subject because you take the shaft and then twist it here. See initially when I hold the shaft, there may be some kind of relative motion between my hand and this member. But once I do it, I am only applying twisting, there is no relative moment of my hand and whatever the twisting moment developed is transmitted by the section and this complete surface is free. You have to appreciate that.

So, what I have here is you have seen how the shear strain is developed. Now, we look at the shear stress. I take a small area ΔA and I have this shear component because you know we have the symbolism. First symbol should tell the plane, this is the z plane and the direction is θ, so $\tau_{z\theta}$, and when I take out an element like this, this element is taken out and you put the reference axis.

You can make a neat sketch please. I have shear stress component developed and you have the complementary shear. From your equality across shear, you know that these two shear components are equal in magnitude. And I have this labeled as $\tau_{z\theta}$ and this is $\tau_{\theta z}$. From your understanding of solid mechanics, cross shears are equal.

So, this is the shear stress that you have. What is what is summarized here, the shearing stress on the plane z and θ are of same magnitude and each element is in equilibrium. We have not verified the equilibrium condition. We will go and verify the equilibrium condition in a short while. And you know shear stress varies linearly. We will also picturize this. So, I will have maximum shear stress and then it tapers down to 0 at the center. And how do you establish equilibrium? See whatever the shear stress developed should balance the applied torque. Is not it?

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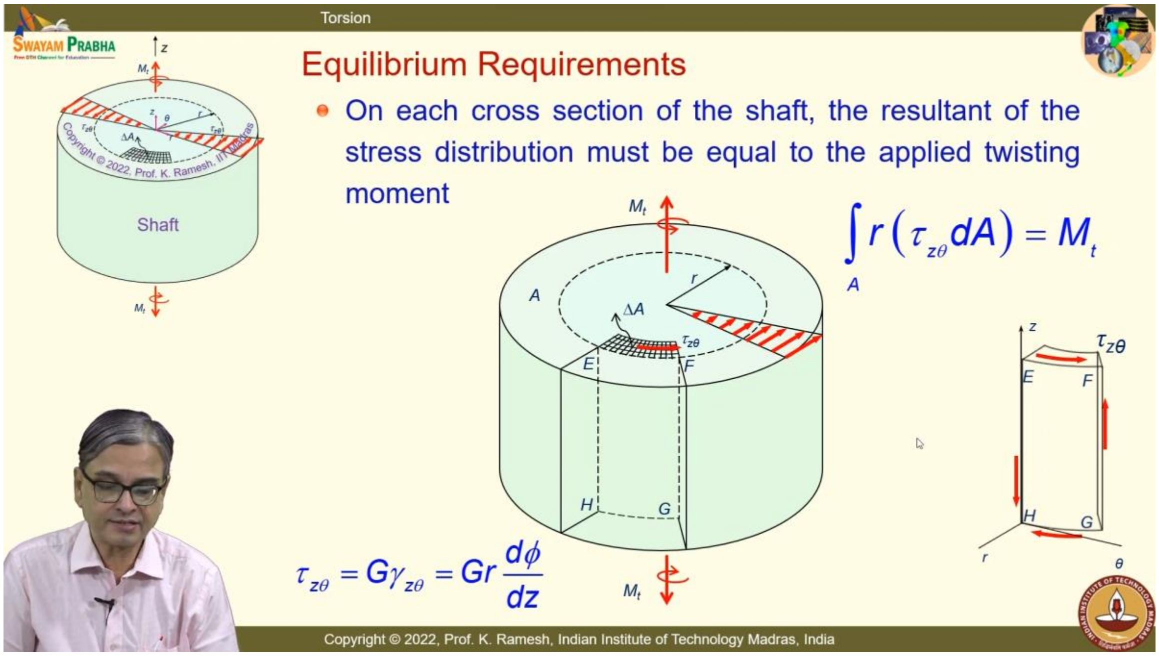

So, when I write the equilibrium condition, I have repeated the same figure. So, that if you have missed out, you can correct your figure.

I am going to write only the equilibrium condition. And the equilibrium condition should satisfy, I have also shown that this is varying as a triangular variation. On each section of the shear, the resultant of the stress distribution must be equal to the applied testing moment.

$$\int_A r\left(\tau_{z\theta} dA\right) = M_t$$

So, your integral $\int r(\tau_{z\theta} dA)$, this is what is shown. See I have the shear stress acting on an area. So, I can find out what is the force. And once I take the moment, the moment arm is r and I do it for the complete area. A represents the complete area of this circular cross section. That should be equal to twisting moment. And when I do this, I also get an interesting expression.

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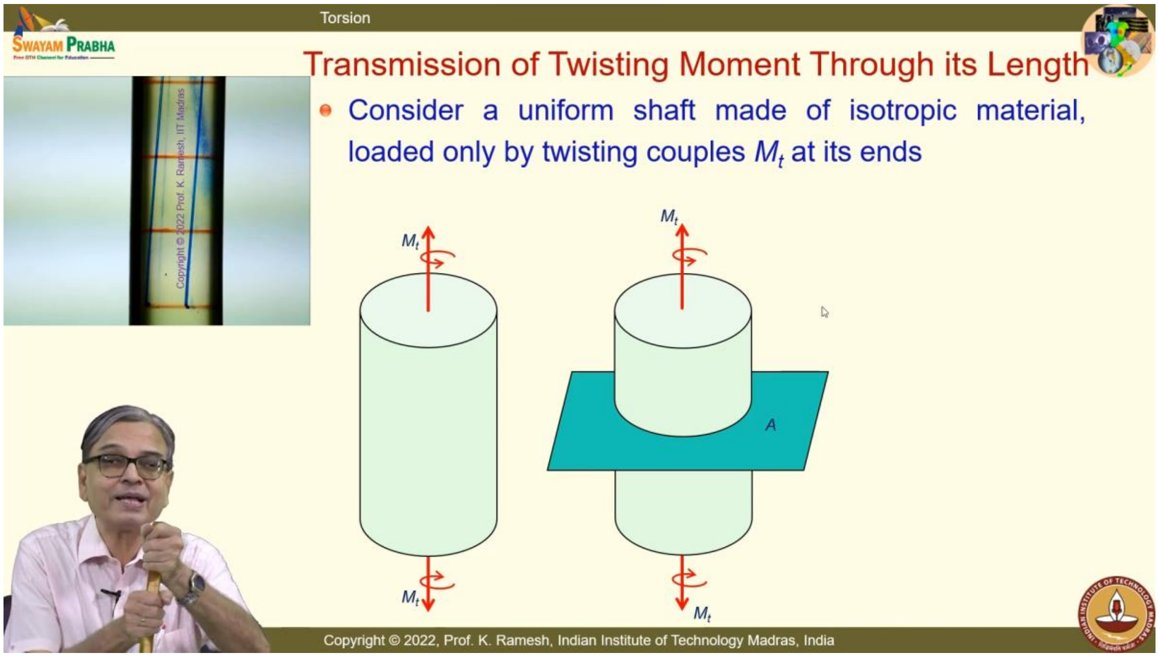

When I substitute the expression for $\tau_{z\theta}$, you know what I want to emphasize here is, suppose I take the model and apply the axial load, you can easily visualize that each cross section transmits the same axial load.

Somehow when people change the problem to twisting, they have a doubt that you still have, because we have shown the shear as tangential to this on this cross section. People have a feeling that you are applying something on this surface. We have done a detailed discussion on free surface.

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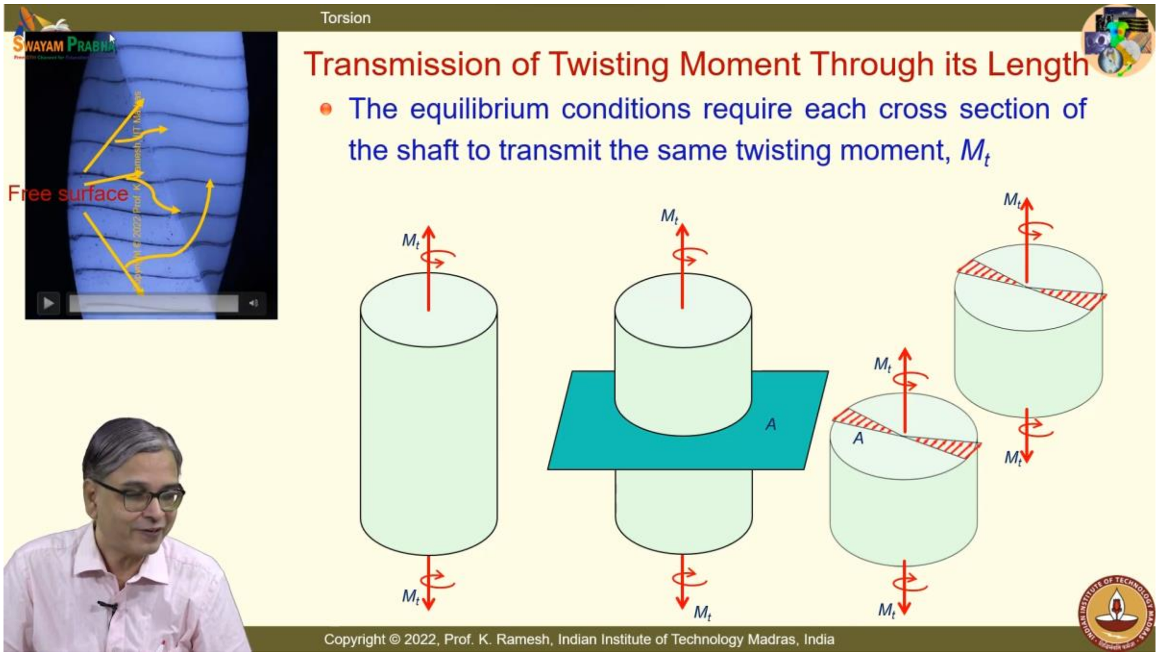

So, what way you have to appreciate is, when I have a section like this taken out, you will have shear stresses developed on this surface that would balance the twisting moment. If I look at the lower section, this will also have the shear stress developed on this surface and balance the twisting moment and the entire shaft will be transmitting the same twisting moment along its length. That aspect has to be appreciated clearly. So, what you have, whatever I have told you earlier, I have also summarized this in the sketch.

So, all this portion what you see is a free surface. You know what is the definition of free surface. If you go and write the tensorial nature of stress developed in torsion, find out the stress vector acting on this plane, stress vector will go to 0. That is what you call it as a free surface. So, now what you have to appreciate is, when I take a slender shaft and put any arbitrary cross section, you should visualize that this cross section what you see, develop the necessary shear stress. So, there is nothing on this surface. So, when you take a point on the surface, you should look at which plane is free surface and which plane you can have shear stress acting. That clarity has to come.

Now, let me ask one another interesting question. See I said that twisting of a circle or a square shaft is difficult from mathematical analysis point of view. When I have instead of a circular shaft, a square shaft, you agree that all of these places is again free. From your knowledge of the solid mechanics development so far, you can find out definitely what is the stress tensor on four lines in a square shaft.

Do you agree with me or not? I have given you the clue. I am not talking about stress. I have said stress tensor and you have to connect it with, we have learnt that this is a free surface and you have to imagine that you have a square shaft. So, what do you have in a square shaft? You have outward corners. So, you know from your earlier discussion that if I have the outward corner here, this is twisted here and it is also twisted here, what should be the stress tensor? I am happy. So, the knowledge what we develop should help you to answer certain questions, if not all questions. Is the idea clear? So, even though you have a square shaft and a torsion, you may not find out the stresses completely in this course. From the knowledge you gain, you can at least say what happens in the outward corners. Excellent.

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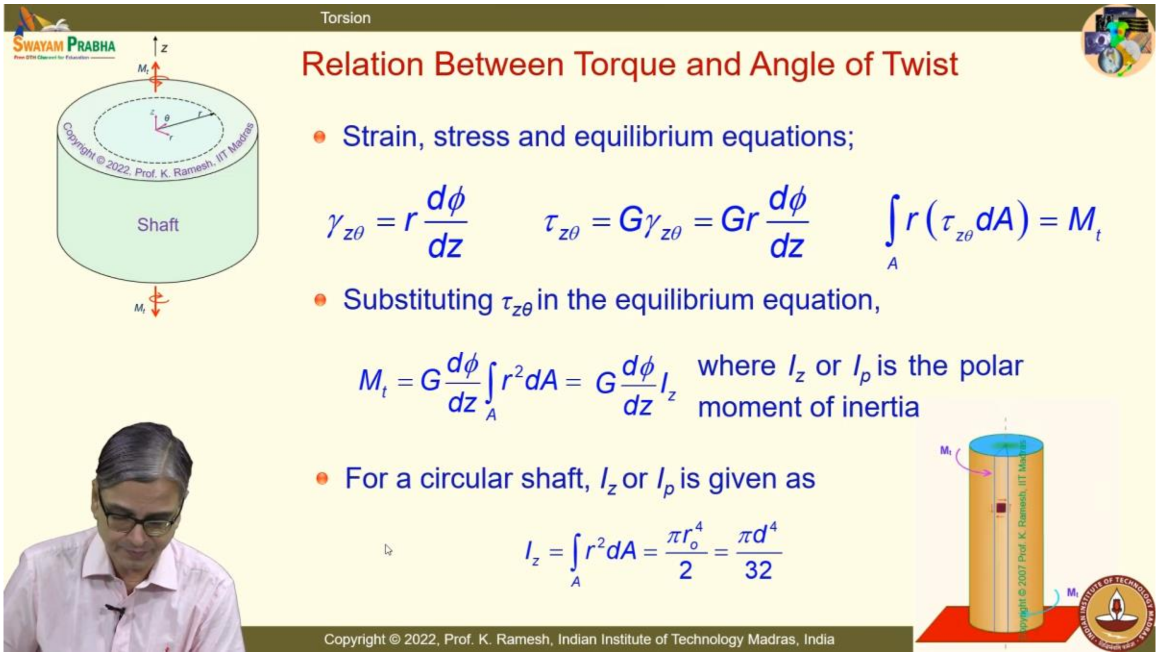

Now, whatever we have discussed, a very nice simple animation is put here. I have radial lines remain radial and longitudinal lines get shifted like this and this summarizes the complete picture of what happens in torsion beautifully. And we will just summarize the expressions.

$$\tau_{z\theta} = G\gamma_{z\theta} = Gr\frac{d\phi}{dz}$$

So, I have $\tau_{z\theta}$ equal to $r\frac{d\phi}{dz}$ which we have seen as shear strain by looking at the deformation. From stress strain relations, we have got what is shear stress $\tau_{z\theta}$. From equilibrium, I have got this. Now I can substitute for $\tau_{z\theta}$ in terms of $Gr\frac{d\phi}{dz}$ in this expression and you get a mathematical entity. And I have also asked you a class back in mathematics, what is that it recognizes the cross section? I raise the question. I have asked you to think about it, fine.

So, when I substitute here, I get this twisting moment as $\int r^2 dA$. What do you learn from your property of cross sections? This. Fine. This is also the polar moment of inertia.

$$M_t = G\frac{d\phi}{dz}\int_A r^2 dA = G\frac{d\phi}{dz} I_z$$

So, moment of inertia understands the cross section because if I change the nature of the cross section, moment of inertia will change. So, if you say how does the mathematics recognize as your cross section, you can say that when I look at the moment of inertia, it understands the shape. If I look at area, it does not understand the shape. I can have cross sections of various geometry having the same area of cross section, but moment of inertia will change. So, it is another way of looking at it.

See while developing this, we have taken the axis as *z* along the shaft. And I have also said that this is I_z. And you know when you go to bending, you may also have different axis where the I_z will have a different connotation. In order to avoid confusion, it is better that we replace in our problem solving I_z as I_p indicating its polar moment of inertia. I may not have done that in all my expressions.

$$I_z = \int_A r^2 dA = \frac{\pi r_o^4}{2} = \frac{\pi d^4}{32}$$

So, for a circular shaft I_z or I_p is given as I_z equal to $\int r^2 dA$ $\frac{\pi r_0^4}{2}$. If you write it in terms of diameter, it comes to be $\frac{\pi d^4}{32}$, which will be repeatedly using it, fine.

So, you should understand that this is also the definition of I_p because if you write in this chapter the twisting moment as M_t and your moment of inertia that is necessary for the calculation as I_p, you will not make a mistake when you go to bending. There should not be confusion between the two.

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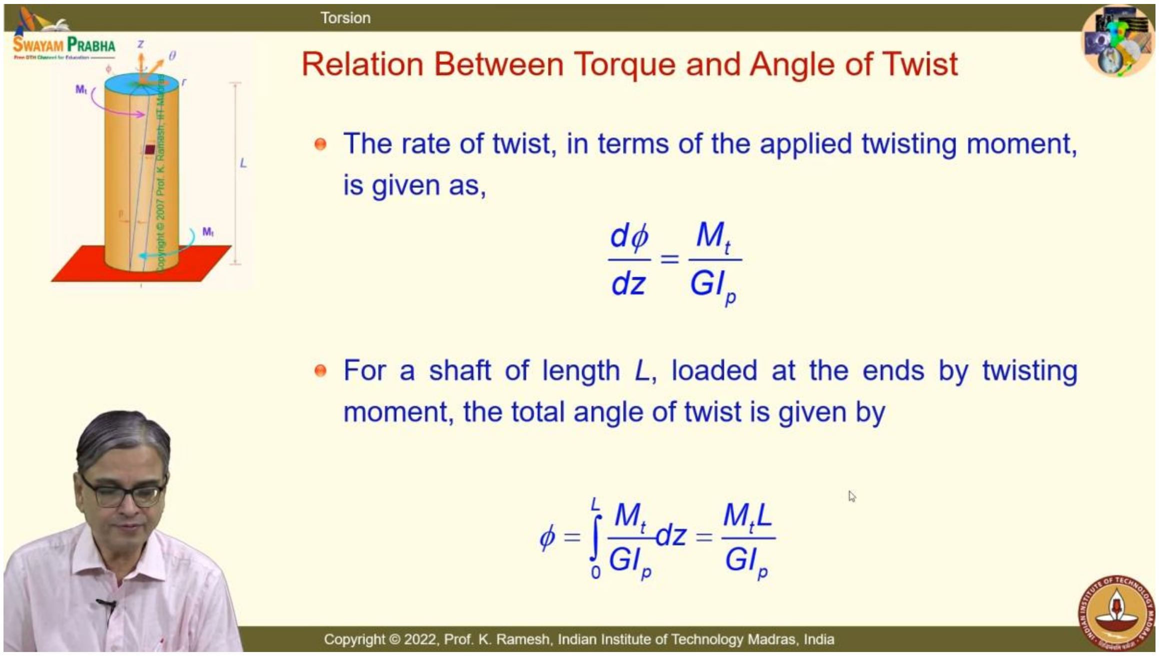

And we can also establish a relationship between torque and angle of twist. The rate of twist in terms of the applied twisting moment is given as,

$$\frac{d\phi}{dz} = \frac{M_t}{GI_p}$$

I have this expression $\frac{d\phi}{dz}$ equal to $\frac{M_t}{GI_p}$.

And GI_p also has another name. This is also known as torsional rigidity. The product GI_p is also known as torsional rigidity.

$$\phi = \int_0^L \frac{M_t}{GI_p} dz = \frac{M_t L}{GI_p}$$

Suppose I have a shaft of length L loaded at the ends by twisting moment, the total angle of twist is given by ϕ equal to $\int_0^L \frac{M_t}{GI_p} dz$ equal to $\frac{M_t L}{GI_p}$. So, for a cylindrical shaft you can find out the angle of twist comfortably.

You know we also have one problem which we have discussed on the variation of twisting moment along the length. We also had one portion where twisting moment was varying as a function of length. So, there you can use the integration properly. So, I can also find out in a varying twisting moment how to find out the angle of twist. All that is embedded in the expression.

This expression can be used to estimate that also. But in a simplistic case, you can simply say $\frac{M_t L}{GI_p}$.

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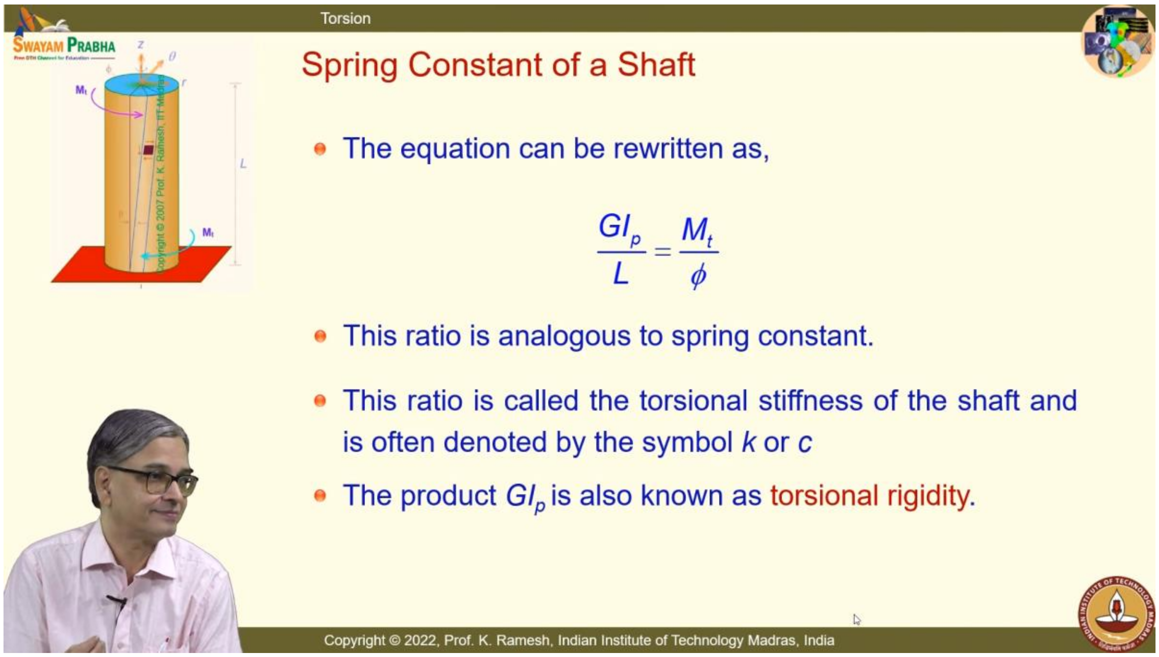

When I have a cylindrical shaft and a constant twisting moment, I said that you also have a torsional spring. The moment you say spring, you can also visualize what is a spring constant. In the case of a tension spring, you talk about axial force and axial deformation. In the case of a torsional spring, you have a corresponding force, generalized force and generalized displacement. So, instead of the axial force, you have a twisting moment. Instead of axial deformation, you have a twisting deformation, angle of twist. And you have to appreciate that the angle of twist in all our mathematical development is only in radians. So, you have to convert it appropriately from radians to degrees while solving the problem. Because the definitions what we have applied, all our discussion till now, phi is always expressed in radians. So, the ratio is analogous to spring constant.

$$\frac{GI_p}{L} = \frac{M_t}{\phi}$$

And so, you have $\frac{GI_p}{L}$ can be called as torsional stiffness. And I have already said the product GI_p is known as torsional rigidity. See this idea of rigidity comes because we graduate from rigid body mechanics to deformable solids. And when you go to deformable solids, we go for small deformation. So, when you looked at the basic solid mechanics, we felt that when there is a shear, it distorts. So, that is why you call that as also modulus of rigidity. So, somewhere or the other, you want to hang on to your rigid body mechanics. Fine. So, if the shaft was rigid, it would have been very very stiff, you cannot twist it at all. So, this is also a measure of product GI_p is also a measure of what is the way it can resist torsion. If the torsion rigidity is higher, you will have angle of twist will be smaller. It is higher value indicates it is more and more rigid.

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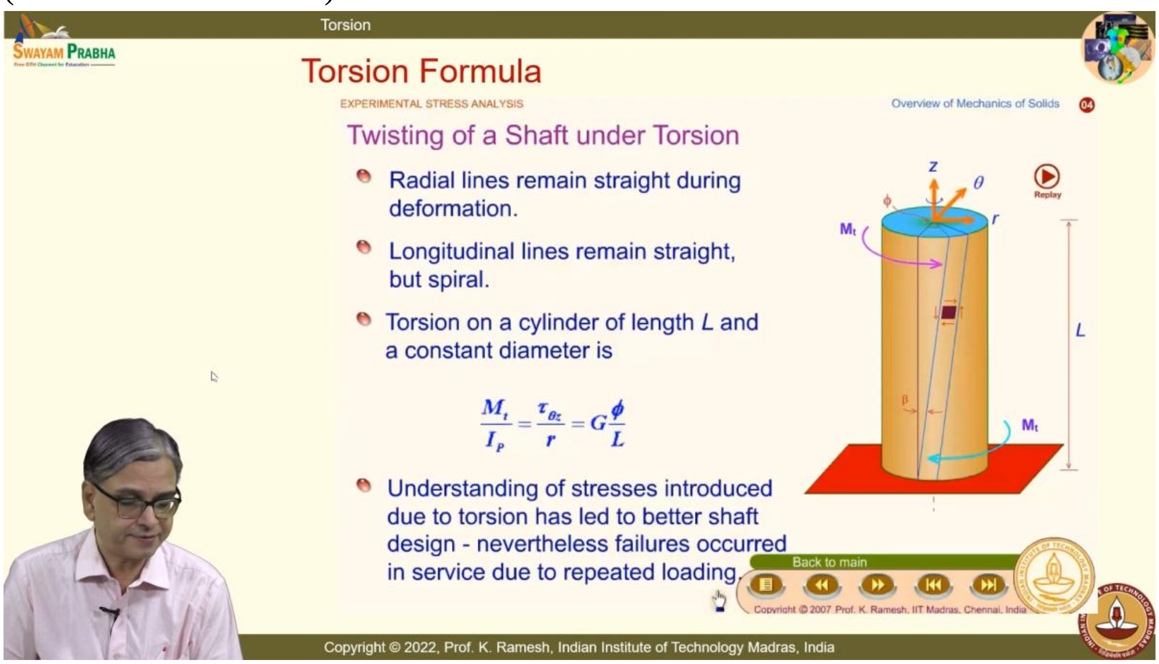

Now, we should also summarize this as expressions. So, you have radial lines remain straight that is what is shown here. I have a radial line, this remains straight and longitudinal lines remain straight, but they get spiral, they get twisted like this.

$$\frac{M_t}{I_p} = \frac{\tau_{\theta z}}{r} = G\frac{\phi}{L}$$

And this is a famous expression. This is valid for a cross-section of circular shaft of uniform cross-section with only n moments applied, which is also constant. We have already derived that this angle of twist is $\frac{\phi}{L}$. We have already derived that is ϕ equal to $\frac{M_t L}{GI_p}$. We have already derived that. So, this is known as a torsion formula. You will have a similar expression appearing in bending. We call it as flexure formula.

And these are valid for uniform cross-section and only constant twisting moment is transmitted. This expression is valid. Instead of twisting moment, you will have a bending moment and appropriate moment of inertia and also you will have appropriate normal stress. You will have an equivalent formula in bending. So, it is better to remember this formula, fine, and also understand its limitation. It is for a uniform circular cross-section and constant twisting moment.

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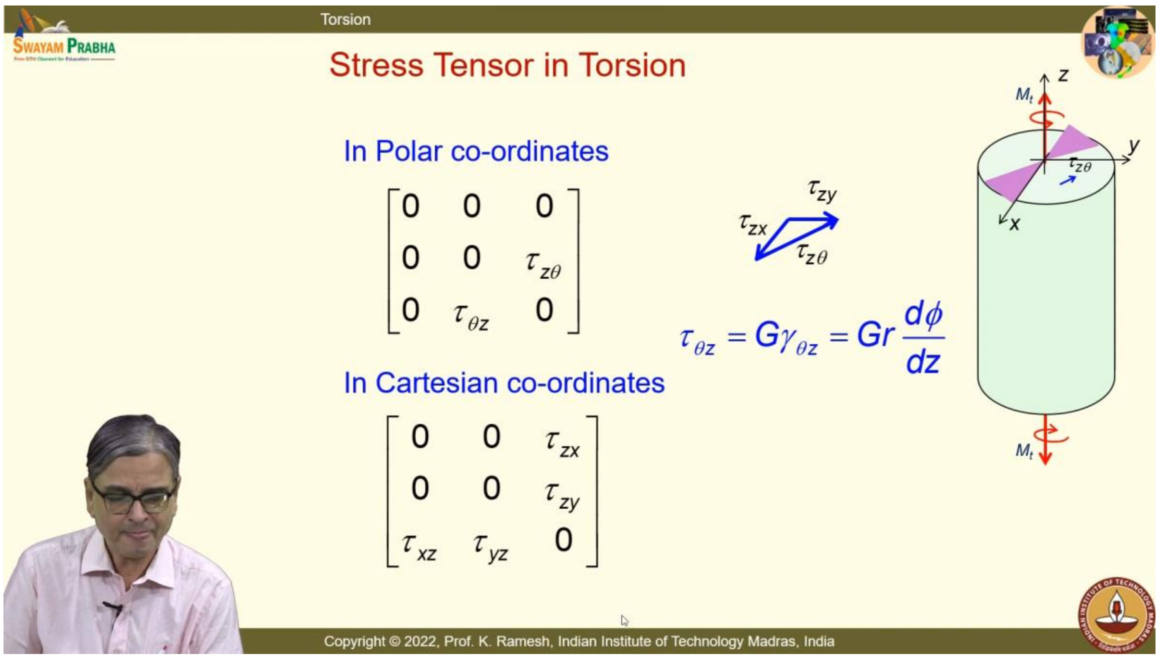

And you know, I have always told you in this course, when you look at stress component, it appears like a scalar. Do not get duped by it. While developing what are the stresses developed in torsion, we have also looked at the deformation, component wise we have looked at it and we have evaluated the component $\tau_{\theta z}$. It appears only like a scalar, fine. Only when you put it in a matrix and call it as a tensor, when you do the transformation, you will recognize that it behaves like a tensor and also you will add. When I have a combination of different loads, if I have to add them, principle of superposition if you have to do that, you have to look at the axis, fine. And if you are in a correct set of axes, then you will also be in a position to add them directly or transform it and then add. So, it is better that you put it in the polar coordinates in one form. I will have this as $\tau_{z\theta}$ and $\tau_{\theta z}$

like this. So, I have $r\theta z$ so you should recognize the shear stress developed in torsion is also a tensorial quantity because we developed and reasoned out only for a component it appeared like scalar, but it is a tensor of rank 2 and it is a pure shear stress state.

You know you will also have to recognize; suppose I take an arbitrary $\tau_{z\theta}$ and then I change this reference axis to *x* and *y*. I should recognize that I have this as $\tau_{z\theta}$. This can be split into two components along τ_{zy} and τ_{zx}. So, I have this as τ_{zy} and this as τ_{zx}.

So, when I look at the same stress state in Cartesian coordinates, it will appear little more populated. So, I have this as τ_{zx}, τ_{xz}. I have not shown equality of cross shear property utilized here. Just to drive home the point that you have a stress tensor with populated numbers τ_{zy} and τ_{yz}. See we know a recipe whether it is a pure shear stress state by looking at the stress tensor. What is that you have to verify? We have looked at utility of stress invariants.

Which invariant you have to worry about for establishing this as a shear stress state? I_1, I_1. What is I_1 here in polar coordinates? And what is I_1 here in Cartesian coordinates? So, you have done the transformation correctly. So, you also understand that is whatever the stresses developed in torsion can be represented as a tensorial notation both in Cartesian and polar coordinates. So, in this class we have developed graduated from looking at the deformation to find out what are the strain components.

We found among all the strain components only $\gamma_{\theta z}$ exists. All other strain components are 0. From stress strain relation we could find out $\tau_{\theta z}$ and I said equal to a cross shear says $\tau_{\theta z}$ equal to $\tau_{z\theta}$. Then we also developed and verified whether the equilibrium is satisfied. The distribution of shear stress when we find out what is the moment about the axis should balance the twisting moment.

And we have also learned that the entire surface is free when I twist it. And then this complete surface is free. And we have also utilized this knowledge when we have a square shaft and twisted even though we have not found out the stress distribution in the case of square shaft, you can find out on the four lines which form the corners. Stress tensor is zero.

Thank you.
