Inverse Methods in Heat Transfer
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Lecture – 32
Bayes Theorem -- Simple Examples

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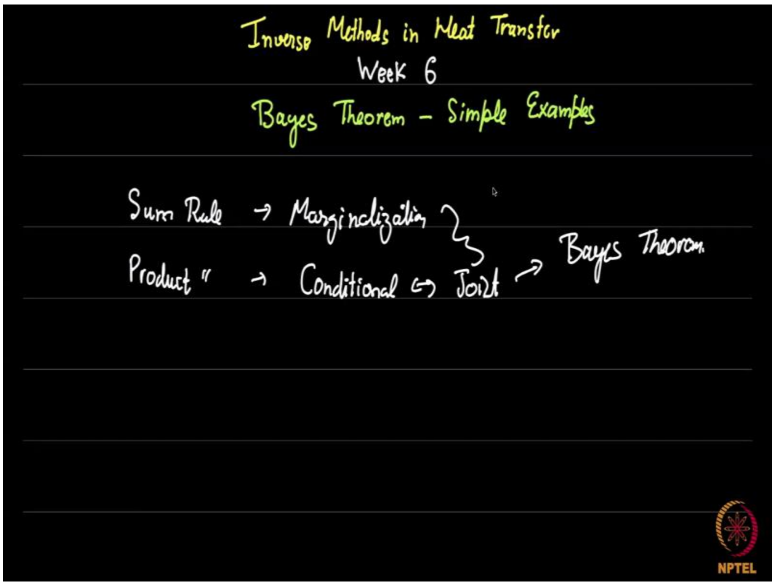

Welcome back, this is week six of Inverse Method in Heat Transfer. In the previous video we saw these two rules of probability the sum rule, which helps in marginalization and we saw the product rule and the product rule connects the conditional probability to the joint probability. So, you can switch between the two using the product rule of probability. We saw that a direct consequence of this was bayes theorem.

In this video we will basically see, in fact you can see base theorem as a consequence of both these as I will talk about briefly in this video. but the main formula is thanks to the conditional sorry the product rule.

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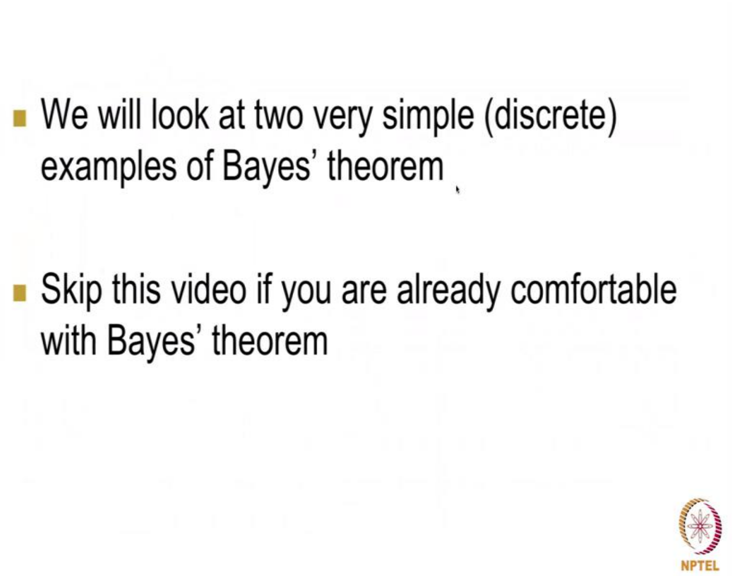

So, we will see a couple of simple examples. Now I want to point out these are really simple we are looking at very simple discrete examples of Bayes theorem. Next week we will look at continuous examples directly applied to the inverse problem case. I would recommend that you skip this video if you are already very comfortable with uh Bayes theorem you need not go by my locations of course next week we will see more complex examples and that might be a more pertinent to your case if you are already something somebody who is very familiar with.

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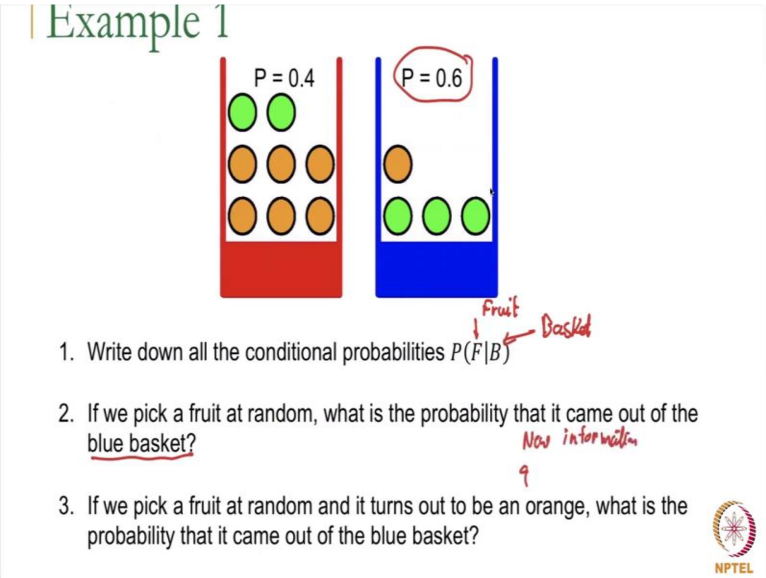

So, let us go back to our original problem that we were looking at in the previous video. once again, a red basket and a blue basket and different distributions of fruits within that and we want to write down all the conditional probabilities, remember F here stands for fruit and this here stands for the basket. We had made the joint probability table on the last time and we have a few questions. So, the next question is if we pick a fruit of random; what is the probability that it came out of the blue basket?

So, I picked up a fruit and I want to know what is the probability that I picked up, I picked it up from the blue basket. Now you can obviously see that has to be 0.6 but we will calculate it in a slightly different way here and next we ask the same question but a little bit more specific. If we pick up fruit at random and it turns out to be an orange what is the probability that it came out of a blue basket?

Once again please think about the intuition of this problem if I gave you nothing if I gave you no information about that fruit you would say 60 percent that came out of a blue basket, if it turns out to be an orange what is the likelihood that it came out of a blue basket rather than a red basket. This you have to think of will it increase will it decrease uh from 0.6. How is the probability affected by this new piece of information that we have.

Again, when we come to the next week, we will see how this is relevant there as well in the inverse problems the same only.

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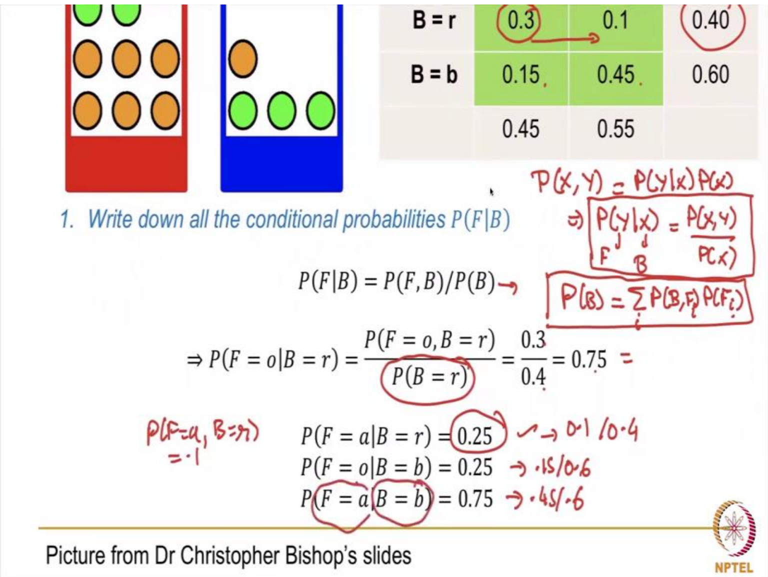

So, let us. Now make write down all the conditional probabilities. this is straightforward but nonetheless let us first make a joint I had a joint trial table but. Now this is a probability table. So, remember previously we had 30 here 10 here, 15 here and 45 here, I have put that back and divided by the number of trials. Now we have made a giant probability table. Now what does this number mean? this means what's the probability that it the fruit we picked was an orange and the basket was created so, that is what it means.

Now here we are going to use our product rule. The product rule remember was,

$$P(X,Y) = P(Y|X)P(X)$$

which gives us P of conditional probability is,

$$P(Y|X) = \frac{P(X,Y)}{P(X)}$$

Now notice how intelligently this notation has been chosen it is almost like a divide and you can see that that is where it kind of reminds you that you have to divide by X. So, that is just as a useful mnemonic you can use this fact $P(Y|X) = \frac{P(X,Y)}{P(X)}$.

So, we can use the same thing here Y here could denote the fruit and X here could denote the basket. So, you would basically have P of fruit given basket, is P of fruit and basket divided by the probability that that basket was chosen. Another way of saying it is what is the chance that I picked an orange given that it was a red basket, well how many cases did I have basket of orange and a red basket by how many cases did I actually choose the red basket.

So, it is easier to see it when it is number of cases. So, we write these numbers down. So, for example let us take this first case $P(F = o, B = r)$ is here. So, that is 0.3 divided by $P(B = r)$ which is this plus this where did this come from $P(B)$ came from the sum room and remember,

$$P(B) = \sum_i P(B,F)P(F_i)$$

So, you sum over all possible fruits which is it is an orange and it is an apple.

So, once you do that, I am going to recalculate this once more below. but please do remember this $P(B = r)$ can actually be calculated through the sum rule. So, some of you might have seen Bayes theorem also in this form I will show that later but $P(B = r)$ also was given to us originally as 0.4 so, 0.3 by 0.4 0.75. So, similarly we can calculate all these other cases. this would be $P(F = a, B = r)$ which is 0.1 divided by 0.4 which is 0.25.

So, similarly these two this one is 0.15 divided by 0.6. So, this was 0.1 by 0.4, this is 0.15 by 0.6, this is 0.45 by 0.6. So, put these together you can actually write all the possible conditional

probabilities. So, for example given that the basket was Blue, the probability that you pick an apple is obviously 75 percent this straightforward. Given that I am already in this box, I am obviously going to get 75 likely that I am going to pick an apple.
Now we have just done it in a more formal way here right out of the table because really speaking that is how we operate for large cases we cannot do them intuitively.

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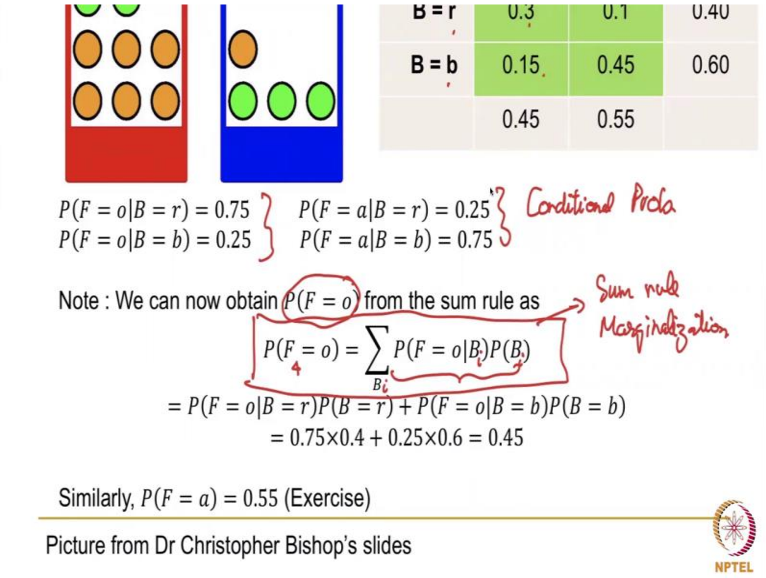

Now we have these four conditional probabilities, suppose I ask you what is the probability that the fruit is an orange? this you would calculate using the sum rule. So, we would ask the question what is the probability that the fruit is an orange and I am going to sum over all possible baskets, the fruit was an orange it came from the red basket fruit as an orange came from the blue basket.

So, if we do that so you can do this in greater detail using the product rule. So, I am going to write it this way. fruit is an orange, red basket and what is the probability of picking a red basket. fruit is an orange; basket is blue what is the probability of picking the blue basket. So, first we pick a red basket the probability 0.4 from that we pick an orange with a probability 6 by 8 it is 0.75. So, 0.75 into 0.4 Plus 0.6 which is the probability of between a blue basket multiplied by one fourth of those are oranges 0.25.

So, summed up 0.45 and you get this 0.45. So, similarly you will get this other probability which you can do as an exercise you will get that as 0.55. all these are reasonably simple calculations but notice this. So, this probability can be calculated as from the sum rule by a marginalization. Again, all these are fancy terms most of you would have done this intuitively but these fancy terms become useful when we come to the continuous case and it is kind of not obvious how to do these integrals etcetera.

So, which is why we are doing it in detail even though you know this in this simple discrete case.

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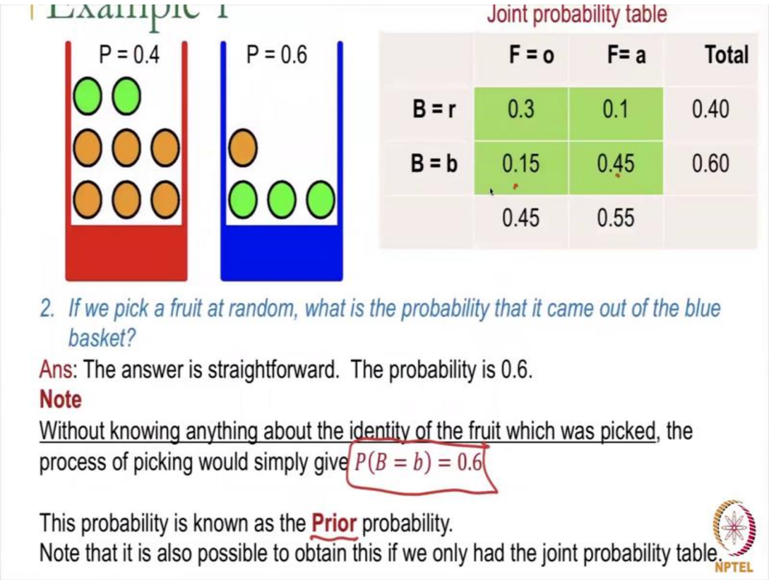

So, the second question was if you pick a fruit at random, what is the probability that it came out of a blue basket? this answer is of course a straightforward. So, you automatically know it is 0.6 because that was given to us. This given thing is what is known as the prior probability what is the probability that the blue basket was picked is 0.6. So, suppose I similarly say that I am within the rainy season in Chennai, which is where I am right. Now what is the probability that it will rain and it rains practically every day here.

So, if you go by last year's weather pattern you will say that in a such and such month it will rain with probability nearly one on a particular day. So, this would be the prior probability. Now I say something like you are in Chennai in the monsoon season but there has been a dry spell, given that currently there do not look to be any clouds, which are being shown on the satellite images what is the probability that it will remain. Now your probability comes down.

So, the prior is what you know before and the posterior is what your condition tells you. So, the recent condition tells you. Now this probability also could have been obtained as we just saw by just summing these up. So, $P(B = b, F = o)$ and then summing up with via marginalization is also something that you may get as 0.6. So, just prior just purely through the joint probability table without these numbers given also you can actually calculate these numbers like by summing these up.

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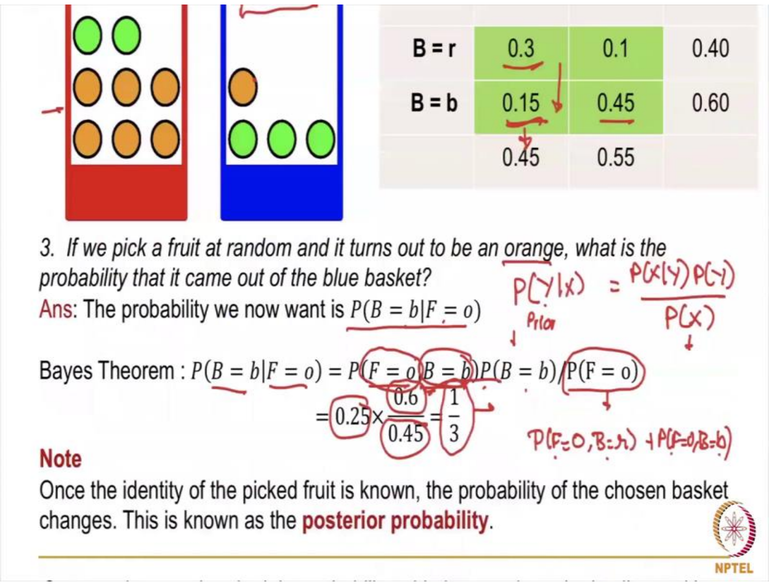

Now let us look at the question which we are leading to which is what we will use base theorem, which is if we pick a fruit at random it turns out to be an orange, then what is the probability that came out of the blue basket color. Basically, compare it with the previous question, there was a random fruit I do not know anything about the fruit what is the probability it came out of the blue basket 2.6.

Now I tell you something more I tell you it is an orange then suddenly the probability changes. So, this probability is now what we are going to calculate. this is the conditional probability that the basket is blue given that the fruit is an orange. So, that is the probability let us calculate that right here. So, this is basically a straightforward application of Bayes rule. we will say I am of course making this more explicit we can say something like,

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Of course, as we will see this $P(X)$ has further components has been seeing sharply. So, here Y is the basket, X is the fruit. So, probability of the basket being blue and fruit being orange is the other way around probability of the fruit is orange given the basket is blue which we calculated just. Now multiplied by the probability that the basket is blue, this is what is known as prior. If I gave you no information what is the probability basket is blue divided by the probability that the fruit is an orange.

Now notice how this was calculated the fruit is an orange was calculated as fruit is an orange, given that the basket is red, plus the fruit is an orange, given that the basket is blue, also not given that the fruit is an orange and the basket is red divided plus fruit is an orange and the basket is blue which is here 0.15 plus 0.45 is 0.6. the other variable point three plus point one five which is 0.15 sorry I added the wrong column yeah.

So, let us calculate these what is the probability that the fruit is orange given the basket is blue that is straight forward one fourth 0.25, we had also calculated it earlier the prior was 0.16 and the probability that the fruit is an orange regardless of which basket it came out of is 0.45. So, the net probability now becomes one third. So, what is this one third if we pick the fruit at random and it turns out to be an orange what is the probability that it came out of the blue basket it is one third.

Now notice the blue basket itself would have been picked up lightly at 60 percent but we have now reduced the probability to 33 percent because if the fruit is an orange, it is more likely that it came from the red basket rather than from a blue basket, where it would have been very unlikely. So, this is where base theorem helps you it gives you non-intuitive results. it is hard for you to say you know you we might just hear and I will show you a next example where it becomes even less intuitive this this was going making the rounds during Covid.

So, Bayes theorem is particularly useful because humans have very poor intuitions about fractions our intuitions are good about numbers. we are very poor at understanding fractions which is why we tend to get really bad results. If somebody asks this question, I mean there has been a large-scale study and you know people some people have effectively got the Nobel Prize for showing that our intuitive or intuition will probabilities really poor which is why we automate as you will see even in the next week.

We automate all systems where we have to calculate probability because our intuition for it is very good. This probability is known as posterior probability, that is before I knew anything I said the basket was blue with probability 60 percent after I knew that a fruit is an orange, I can Now give you a posterior probability that is only 33 percent again it is sufficient to just make the joint probability table to obtain all quantities and which is what we will be doing for the continuous cases.

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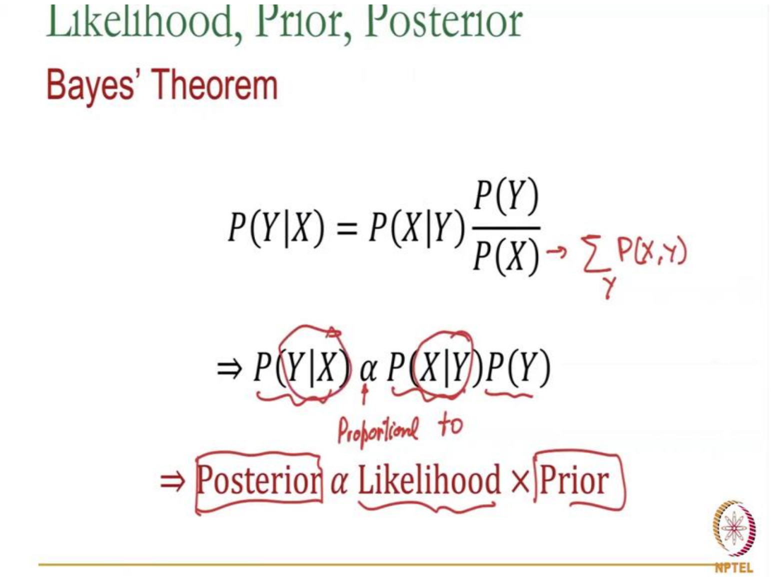

So, here is bayes theorem as we just wrote down is,

$$P(Y|X) = P(X|Y)\frac{P(Y)}{P(X)}$$

P(X) of course is $\sum_i P(X)$ we are considering I am going to make this informal $P(X) = \sum_y P(X,Y)$. All possible values of Y you sum over in this case it was basket was red basket was blue except for you some overall responsibilities.

This is not Alpha this is proportional to and this is the form in which we will use it in the next week, which is,

$$P(Y|X) \propto P(X|Y)P(Y)$$

This is known as the posterior, this is known as the prior and this is known as the likelihood. Now what are X and Y, I mean X and Y typically are used in a form where X given Y is a little bit more obvious and this is less obvious.

But they do not have any specific meaning one or the other. of course, they will have a specific meaning when we come to inverse problems. So, inverse problems you will see that there is actually cause and effect, effect versus cost. So, one of these is the cause and one of these is the effect and we will see how it makes sense to see both the likelihood as on supposedly when we come to inverse columns.

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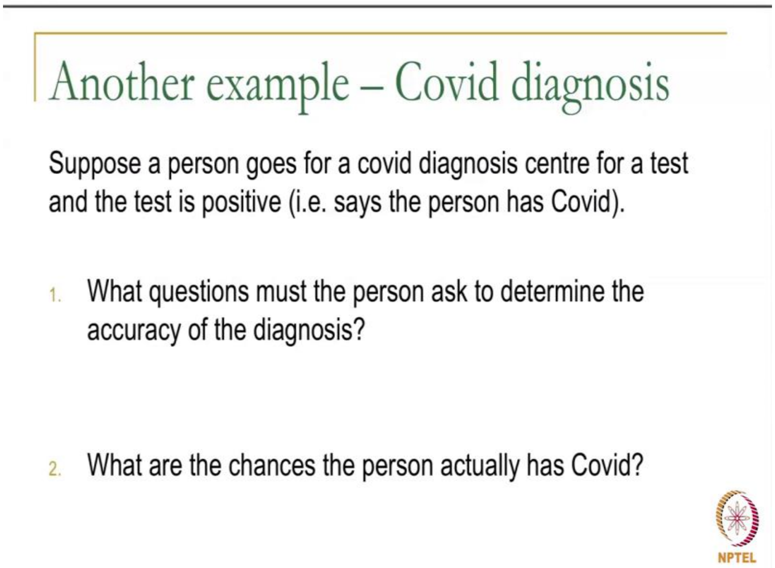

So, let us come to a second example which is a more recent example. Usually, this example is given the cancer diagnosis right, just mildly changed it to Covid diagnosis because that is what went on in the world this horrible thing over the last couple of years. So, this became very relevant and we started looking at the results of RTPCR tests as you might remember many of us went underwent these tests.

And there would be certain numbers which were given that the RTPCR test is so, accurate and not and you want to find out what is the likelihood that you have Covid and there is also the phenomenon of us doing multiple tests one after the other. So, let us say you're going for a diagnosis what are the questions we must ask to actually determine the accuracy of the diagnosis. as it turns out it is not sufficient to just ask for one number this is the first non-intuitive result that you will get here once you do a proper Bayes theorem analysis.

And given some numbers of the accuracy what are the chances that the person actually has code. So, I am going to make up some numbers like I said these numbers are not so, good for Covid tests. Covid test numbers are actually a little bit worse and you might have heard right now as we speak about it in 2023 that there have been a lot of controversies about these tests and the efficacies of vaccines and a lot of it actually genuinely has to do with bayes theorem.

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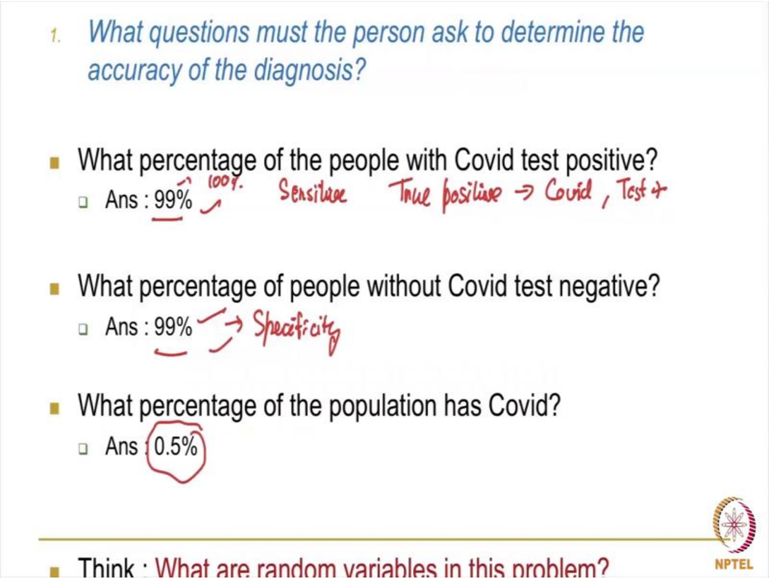

So, if we go and ask, somebody takes an RTPCR and we ask what is the accuracy of this test. it turns out there are two spare separate questions one can ask. So, the first question you can ask is you did a test and what percentage of the people with Covid test positive. So, this person says 99 percent and let us say you have got a positive result you are like oh there is a 99 percent chance that I have Covid but that would be wrong, that is because let us say I have a dummy box and this number is even 100 percent.

So, let us say I have a dummy box for every person it says Covid positive. Now hundred percent of people who are Covid positive will still stroke of it positive because it always shows perfect positive. So, that test is meaningless even if it is 100 percent what we will call sensitive, that is if I am person or I have a voice box which keep on saying your Covid positive or Covid positive. of course, it will be right for all the people who are perfect positive also, but it will be wrong for a lot of the people or in fact all of the people for which it is covered negative.

So, it is not sufficient to ask only the question about sensitivity which is what are the true positives. A true positive means the person is positive or the person has covered let me use that word and the test is positive. but that is only one of the four possibilities the person could not have covered and it could feel so positive etcetera etc. So, you have to ask a parallel question which is what percentage of people without Covid test negative.

Or you can even ask what percentage of people without Covid test positive. But now let us say this also turns out to be 99 percent. Now you have a more interesting case. So, now you say 99 percent of people who have Covid test positive, only one percent of the people that is from here this is called specificity, only one percent of the people without Covid test positive or 99 percent of the people without Covid test negative.

But it turns out and if we do base theorem properly, we actually need a third number, which is what percentage of the population has covered it and let us say I gave the number 0.5 which was true about at a certain point of time. you can increase this number to 3 percent 5 percent 10 percent whatever you want and your numbers will say accordingly. but let us say at a certain point only 0.5 percent of the population has covid and you go and you check and you get Covid positive in an RTPCR.

The RTPCR turns out to be really good in that 99 percent of the cases, it will test positive, if you are positive 99 percent of the cases, it will test negative, if you are negative. Now if the question is I got positive what is the chance that I have covered you might say it is 99 percent obviously I mean these two numbers are 99 turns out we will be really far away, because of this number and that is where the non-intuitive part of Bayes theorem comes in. So, it is important to think what are the random variables in this problem.

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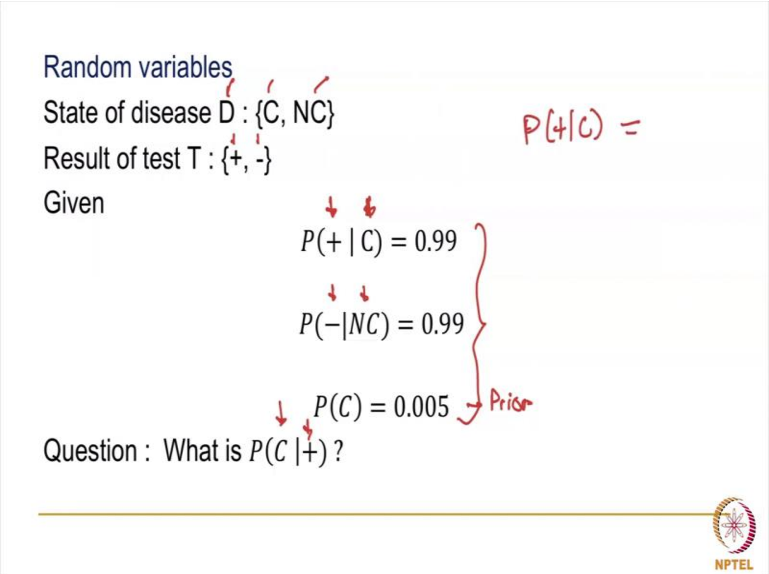

So, here is the question what are the chances that the person testing positive actually has covid. So, here we now carefully define our random variables. Now our random variables are two things, the state of the disease, the person either has Covid or has no Covid and the other random variable is the result of the test positive and negative. Why are these random variables, because the result of the test need not always be positive need not always be negative.

As I said if you remember a random experiment is something that you can do a few times and you might get different results. remember many of us repeated our Covid test. So, now what is the data given. Now let us look at our data given very carefully. it says probability that you are positive given that you have Covid is 0.99, probability that you are negative given that you did not have Covid, is also 0.99.

Now no other information is given, this is the prior. I picked a random person off the street and said Hey what is the probability this person has covid, you will say well it is 0.5 percent, I have no other information about this person so it is 0.005. So, the question now is given these three pieces of information remember if one of these three is missing you, actually cannot give a good numerical answer, which is why we have to be very careful as intelligent consumers of a statistical data. it can be very confusing even to the expert this has been checked.

Even Nobel Prize winning people make mistakes about probability, Elementary probability which is why we have all this machinery in there. So, the question is not what is the probability that you will be positive given that you are Covid? But probability that I tested positive what is the chance I have over it. Now notice,

$$P(+|C) \neq P(C|+)$$

As I told you earlier this is an important distinction.

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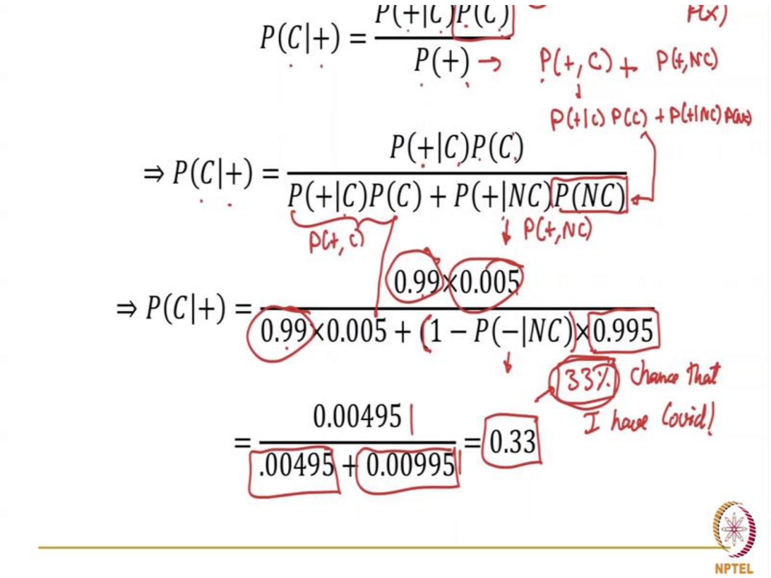

So, here is Bayes theorem a simple Bayes theorem. So, Bayes theorem says,

$$P(C|+) = \frac{P(+|C)P(C)}{P(+)}$$

same formula $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$. Notice how powerful such simple expressions can be. this is just remarkable the power of mathematics here, it is remarkable that it can clarify. So, many things which we would have gotten confused about so, let me go further $P(C|+)$ is this divided by P(+), remember it is summation over all the possible cases.

So, what are the possible cases you could be positive and you could have had covid or you could be positive and you might have not had it these are the only two possibilities. Now this further by the product rule is your positive given Covid by the probability that you had covid, positive no covid multiplied by the probability that is what we have in the denominator here once again to clarify this is positive with Covid is positive.

So, now let us plug in the numbers, P(+|C) was 0.99, P(C) was 0.005, P(+|C) 0.99, this is 0.005 and P(+|NC) is 1 minus the possibility, that you had negative with no Covid. This should be straight forward. If I had no Covid I would test positive when 0.01 times. So, suppose this was 0.95, then P(+|NC) which would be 0.05.

So, the condition of the denominator has to be same only, then we can do the one minus, I should put a bracket sign here, multiplied by what is the probability that you had no Covid at all. 99.5 percent chances are without any other information, I do not have Covid. Now I put these numbers, you will see that if you do the calculation, this number is approximately twice the symbol and you get 0.33. So, there is only 33 percent chance that I have Covid despite the fact that my test is really accurate.

99 percent chance that only positive people test for Covid positive and 99 percent chance that negative people will test negative. Even with that kind of accuracy because as you will see shortly, I will show you it with numbers you have only 33 percent chance. what does that mean? That means, this number has to be made a whole lot higher. that is this 99 percent is not sufficient. The more as you will see you can check this out the more you keep increasing this number the more likely that you have.

Now a few other things because it might be a confusing example. but then how did we like very quickly the doctor said this guy has Covid and they isolated all of us the reason was there were other priors. The priors were does this person have cough, does this person have fever, is this person showing other symptoms, that is why they collected all that information. Another way of creating a stronger prior so, here the number that really killed us was this.

This is 0.005. if this is 50 percent you will get 99 a percent here if the prior possibility of you having Covid is 50 percent you will get exactly 99 chance that you have Covid or not. But since the you are giving no information, the prior is that 99.5 percent, I have no chance I have just because of test set. So, it does not mean I have it the test unless it is really accurate can really mislead as I will show you with numbers.

Now what you can do which is why people did this you can repeat your RTPCR test. Now once you repeat your RTPCR test. Now you have a stronger prior your prior. Now is you have 33 percent chance of Covid. So, this number increases then you repeat this. Now you repeat a second time that number might go up to let us say 66. Then you repeat the third time it might now go up to 99. So, that is how you make surer and that is why we repeated Covid test.

So, the point here is the prior is shifted by the probabilities that we have calculated and we will see use this again powerfully when we come to inverse problems. So, the prior is a very important quantity that you require if you have no prior or let us say a 50 percent prior then we have got nothing other than simple Bayes theorem.

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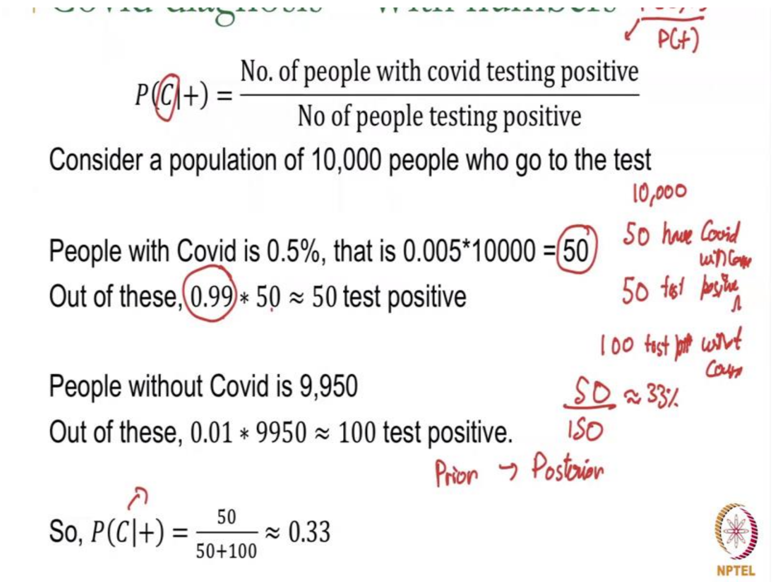

Now we are going to use numbers so, just to DE confused or reduce the confusion in this example. So, let us say you have you want to check this out you want to find out what is the probability that I have Covid given that I tested positive. So, how should you calculate it, what you should calculate is the number of people with Covid testing positive divided by the number of people actually testing positive.

Another way of just saying it is this is number of people with Covid and testing positive divided by the number of people testing positive period. So, let us say we have a 10000 people who go for the test, only 0.5 percent which is only 50 people actually has Covid. So, now we had 10 000 people 50 have Covid. So, these 50 people go for the test how many will test positive. Remember we have a 99 positive test rates of, 99 multiplied by 50 is 49.5 approximately all 50 tests positive.

So, this is really good. So, we have tested of 50 people are tested positive, but in these 10000 people 9950 did not have Covid. And out of them a very small number only one percent test positive which is 99.5. So, but 100 test positive without Covid. So, swiftly test positive with Covid 100 test is positive without Covid. So, now what happens what is the probability. Now you collect these people who have tested positive only 50 have covered but there were 150 people.

So, 33 percent is the chance that you lay among the 50 people who had Covid. Now if we increase the number. So, how many of these people testing positive had cough etcetera then the numbers change. So, I hope it is clear that there is the prior which is really important and the prior is what gives you this probability which is the posterior probability. So, depending on how strong the prior is or how weak the prior is the posterior is strongly affected.

And this is why this is both the positive as well as the negative part of Bayes theorem. in that the prior can strongly affect what our results are and like I said there are arguments between frequencies and Bayesians about which one is correct which one is not correct. Nonetheless you can see that in this example it is very important for us to know the prior before we make an accurate judgment whether it is in cancer diagnosis or covid diagnosis on what we have and we cannot simply make this decision without Bayes theorem.

So, I will stop here and we will continue with our journey with probability theory in the next video, thank you.