**Inverse Methods in Heat Transfer**
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**Lecture – 19**
**Simple Nonlinear Inverse Problem: Transient Heat transfer**

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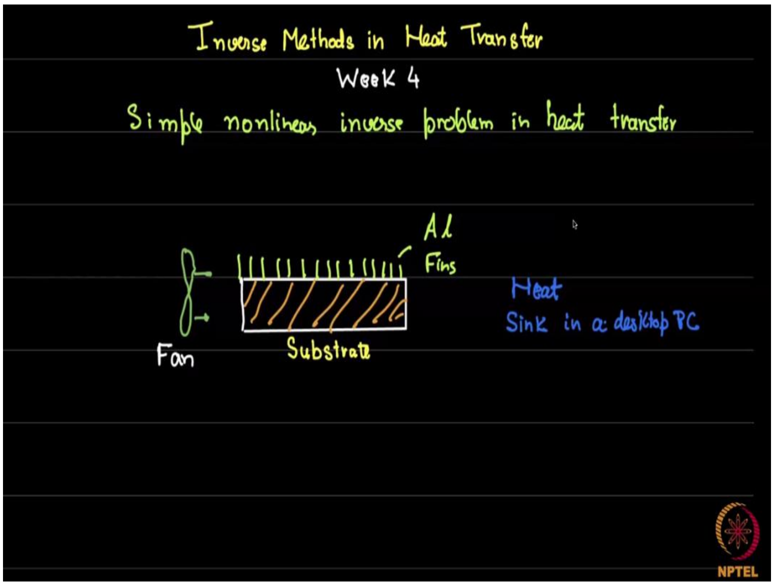


Welcome back. This is Week 4 of Inverse Methods in Heat Transfer and we are going to look at in the short video, simple nonlinear inverse problem in heat transpose. This is supposed to motivate what happens for the rest of the week. As we had seen last week all the problems that we had considered were simply using linear models. We are going to look at a practical case where we will actually have a nonlinear problem.

That is nonlinear in parameters please remember. We had discussed in last couple of weeks, extensively the fact that nonlinearity means nonlinearity in parameters. So, what we want to look at is a case where the forward model naturally throws up a nonlinearity within heat transfer. So, here is a simple case. It is a sort of constructed case, but nonetheless even in the simplified model you can see that this kind of nonlinearity can occur very naturally even within heat transfer.

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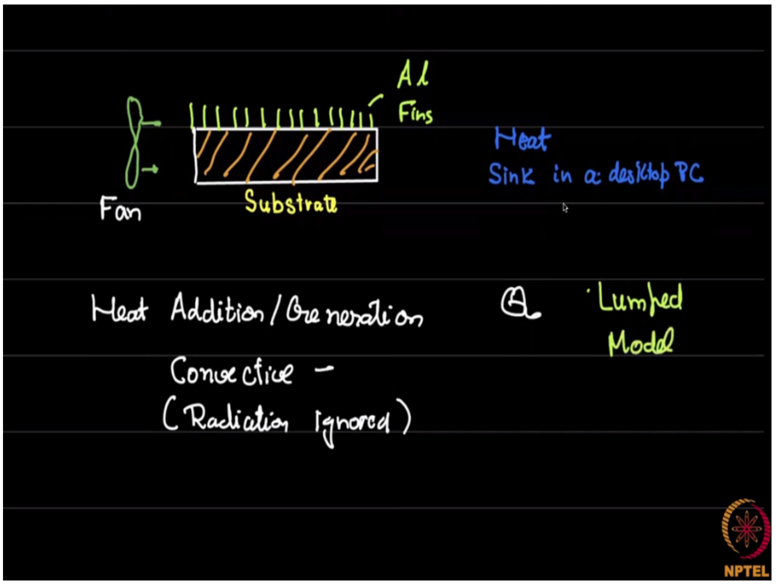


So, let us take a case let us say you have a desktop PC or you have your mobile phone. If you open up your PC, you will see these chips and on them you will see some heat sinks or basically something like a fin. This is supposed to remove the heat away from the chip. Now you might recall this incident a few years ago which had happened with Samsung phones where if you have too many apps running, it would actually spontaneously catch fire.

And that is the reason for that there was not enough thermal control. In a desktop PC you will in fact find a fan close by to the chip this will not always be active, but in case there are too many programs running you will see this sound which sort of starts and the fan switches on and it sort of meant to cool down the chip. So, you have a few cooling mechanisms you have of course the fins and then you have the fan.

This is generally supposed to remove both passive as well as active cooling takes place of the heated chip. Now, if you notice this problem and we will try to abstract it into a simpler problem, so that we can make a simple forward model. The full solution of course involves a full-scale thermal modeling of the system, but let us look at a simpler case. So, the simpler case is like this. you see the heat transfer modes which are acted here.

You have of course got heat addition or heat generation this happens when apps run on your chip and that generates some Q. On the other hand, you have cooling, you have convective cooling, we can ignore let us say something like radiation and we have this convective cooling which is taking place. So, these are the two things that are going on and we are going to assume that we are dealing with a lumped model. We had discussed this sometime in the first week. So, we will deal with a lump model.

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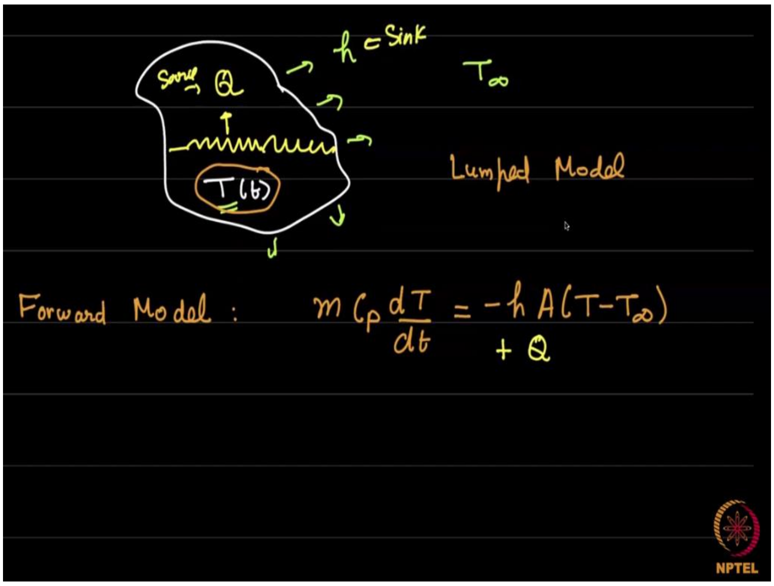


So, let me make up a simple lumped model of this, you can assume that this is somewhat like the chip and the fin system. It is kept at some constant temperature T so that is this temperature is especially constant wherever you keep a thermocouple roughly the same temperature is obtained. Of course, this depends on Biot number etcetera, which you would have seen in heat transfer since this is not a full course on heat transfer.

I am going to assume that you know these basics of heat transfer already. Now, we can abstract the heat addition within it, the heat addition that is being caused due to the apps running as some internal source of heat addition. so let us call them Q and of course outside you have these cooling effects both due to the fan and due to general convection or due to the fins.

So, let us say some convective average, convective coefficient is h, let us say outside temperature is $T_\infty$ which is cooler and inside temperature is T which is hotter. Now, if we want to make like as a lumped model of this and the lumped model is inherent in assuming this is constant in space then we can look at all the terms which are active and say here is the forward model.

As you saw in the last video in order to solve any inverse problem you actually need a forward model. So, the forward model is like this. you have some mass; this mass could be of the chip plus the fins plus the sink and $mC_p$; $C_p$ is the specific heat and this is changing in temperature.

So, you have the enthalpy $mC_p \frac{dT}{dt}$ and if you have just had no heat addition at all this would be losing its heat due to convection.

So, you will have $-hA(T - T_\infty)$, but of course you have heat addition also.

$$mC_p \frac{dT}{dt} = -hA(T - T_\infty) + Q$$

So, we can add a Q to it. So, this is a source term and this is the sink term. So, this is the source and, in some sense, you can think of this literally as the sink for which.

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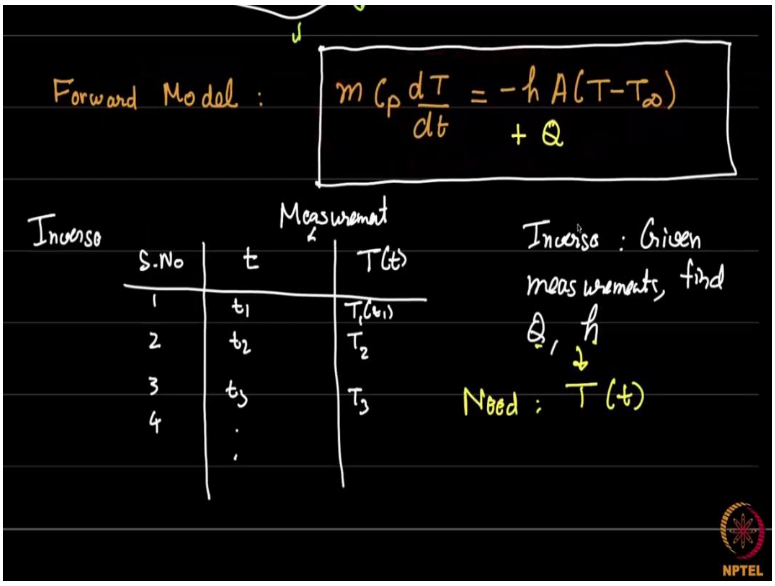


So, this model is a ODE as you can see, this is the temperature model. Now, what sort of inverse problem can we solve with this. So, the inverse problem could be something like this. Let us say we run an experiment and at various time points $t_1$, $t_2$, $t_3$ so on and so forth. We actually measure the temperature kept at the body. So, you could have one single thermocouple or multiple thermocouples and find out an average temperature since we are taking a lumped model.

And you can measure $T(t_1)$ let us call it $T_1$, $T_2$, $T_3$ etcetera. So, this is the actual measurement and what you want to find out are properties like what is Q, you could ask what is h or you could ask what is Q and what is $C_p$. So, these two will be the inverse problem. So, the inverse thing is given measurements, find some parameters. Let us say find out how much heat is being generated or what is the convective coefficient of this environment.

What is the effective convective coefficient of this environment. Now, of course to do this, we need not just the differential equation, but we need a forward model which is like given Q and h what is T. So, we need that model that is really how we did the slab problem also. So, unfortunately this happens to be in a differential equation. Now later on when we come to something called Physics involved neural network.

We will actually see how to go about solving this problem right from the differential equation end. When we do not even know the solution.

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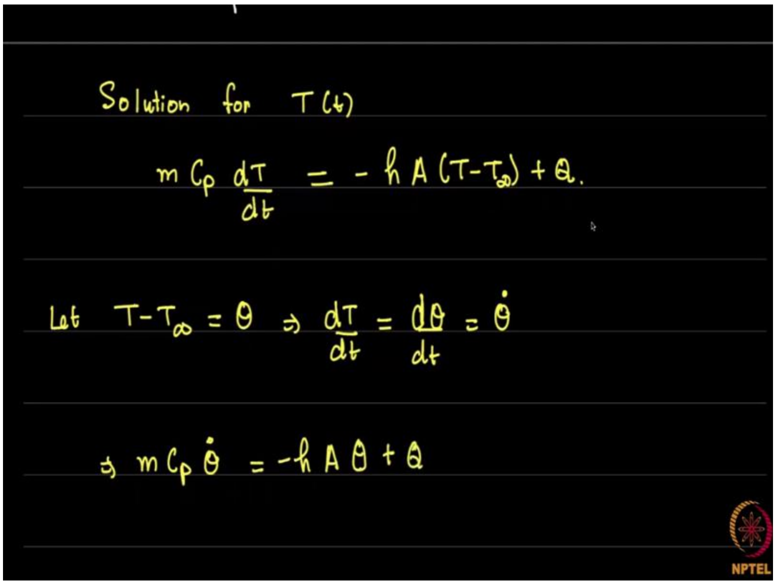


But right now, for conventional solvers and also to motivate why we need nonlinear methods we are going to look at a solution for this temperature. So, the equation once again was

$$mC_p\frac{dT}{dt} = -hA(T - T_\infty) + Q$$

Now, we make a substitution this is again a standard substitution with heat transfer. Let us say $\theta$ is,

$$\theta = T - T_\infty$$

we did something similar for the fin.

And this of course tells you that derivative of T with respect to time is the same as,

$$\frac{dT}{dt} = \frac{d\theta}{dt} = \dot{\theta}$$

I will call it $\dot{\theta}$ just for convenience. So, this gives us,

$$mC_p\dot{\theta} = -hA\theta + Q$$

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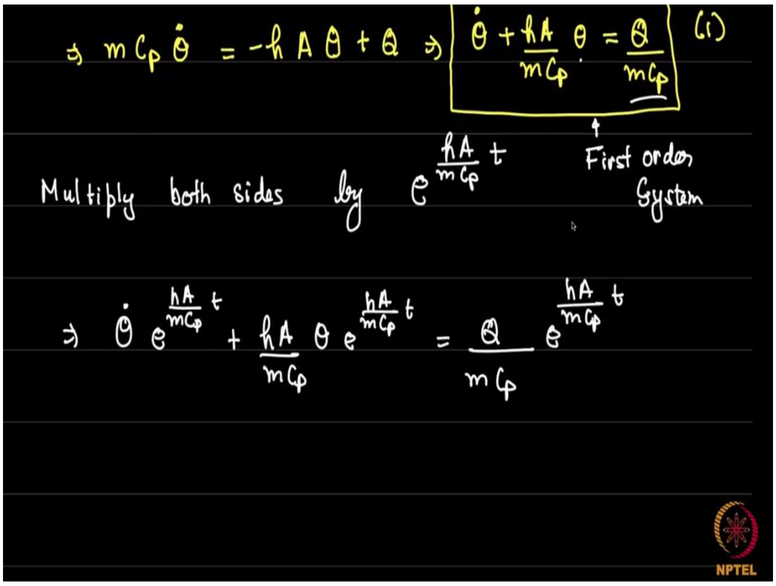


I am going to divide the whole equation by $mC_p$ so this is just for convenience,

$$\dot{\theta} + \frac{hA}{mC_p}\theta = \frac{\theta}{mC_p}$$

So, let us call this equation 1. So, we need a solution for this equation. There are multiple ways of solving this equation, what we will do is we will just multiply both sides by $e^{\frac{hA}{mC_p}t}$. So, this will give us,

$$\dot{\theta}e^{\frac{hA}{mC_p}t} + \frac{hA}{mC_p}\theta\, e^{\frac{hA}{mC_p}t} = \frac{Q}{mC_p}\, e^{\frac{hA}{mC_p}t}$$

So, this system typically is known as a first order system in case you have seen it, it has first order derivatives a constant term on the right-hand side and a term that depends on $\theta$. Of course, you might see spring mass, damped core systems that look like,

$$m\ddot{x} + c\dot{x} + kx = 0$$

So, all those systems are second order systems. You might have seen differential equation solutions for this. In case you have not I am just deriving it just in this case in order to motivate the nonlinear regression problems. In case of exams, in case you need some special thing or some special substitution like this it will be provided, in case you are writing the exam for this course.

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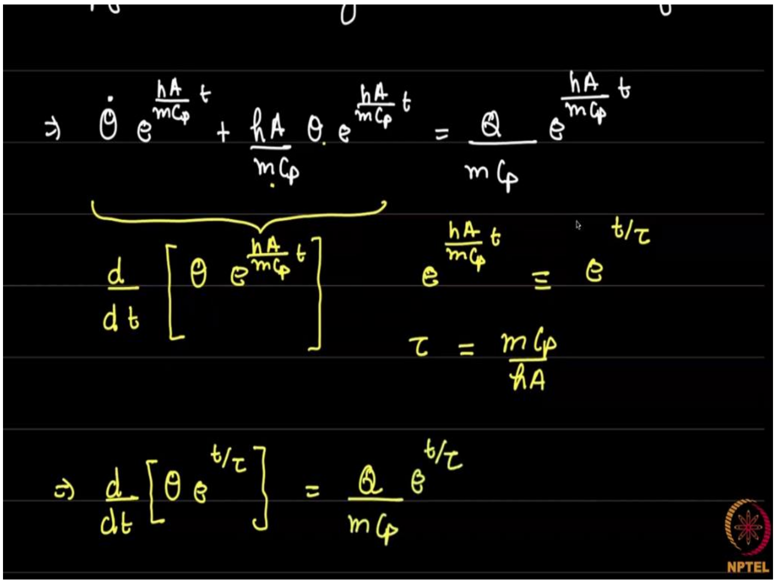


Now, if you notice carefully, the left-hand side here can be written as,

$$\frac{d}{dt}[\theta e^{\frac{hA}{mC_p}t}]$$

Why is that? Quickly consider this the derivative of one multiplied the other is derivative of the first term, so this will be, $\dot{\theta}$ times $e^{t/\tau}$ plus this term which is $\frac{hA}{mC_p}\theta\, e^{\frac{hA}{mC_p}t}$, which is the same as this.

So, this is a quick check that you can do and that is how we derive this term and that is in fact while you multiplied by this exponential function again in case you have done a differential equation course or in other courses you would have seen a sort of substitution. So, just for simplification let us call $e^{\frac{hA}{mC_p}t}$ because this term is going to repeat again and again as $e^{t/\tau}$.

So, where $\tau$ is what is called the time constant of the first product system which is $\frac{mC_p}{hA}$. So, using this we get,

$$\frac{d}{dt}\left[\theta e^{t/\tau}\right] = \frac{Q}{mC_p}\, e^{t/\tau}$$

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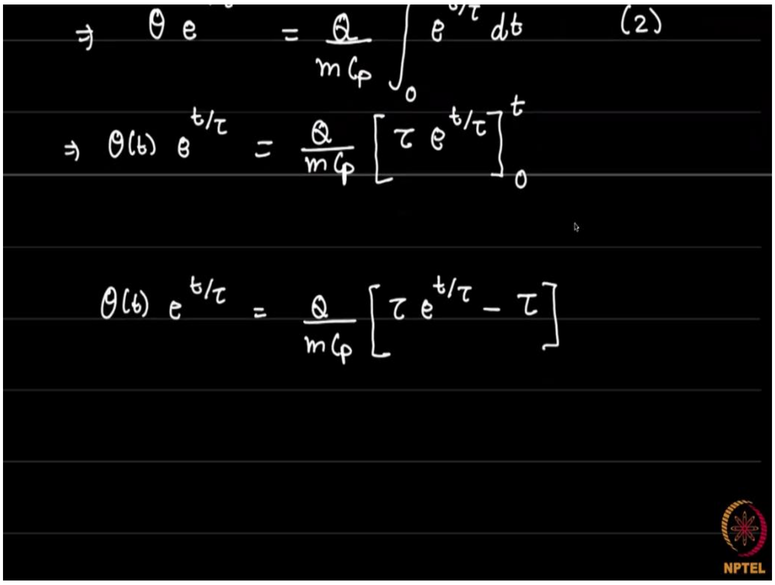

So, we can integrate both sides. So, this gives us,

$$\theta e^{t/\tau} = \frac{Q}{mC_p}\int e^{t/\tau}\ dt$$

So, let us call this equation 2. So, theta of course is theta as a function of time,

$$\theta(t)e^{t/\tau} = \frac{Q}{mC_p}[\tau e^{\frac{t}{\tau}}]_0^t$$

So, if you open up the right-hand side,

$$\theta(t)e^{t/\tau} = \frac{Q}{mC_p}[\tau e^{\frac{t}{\tau}} - \tau]$$

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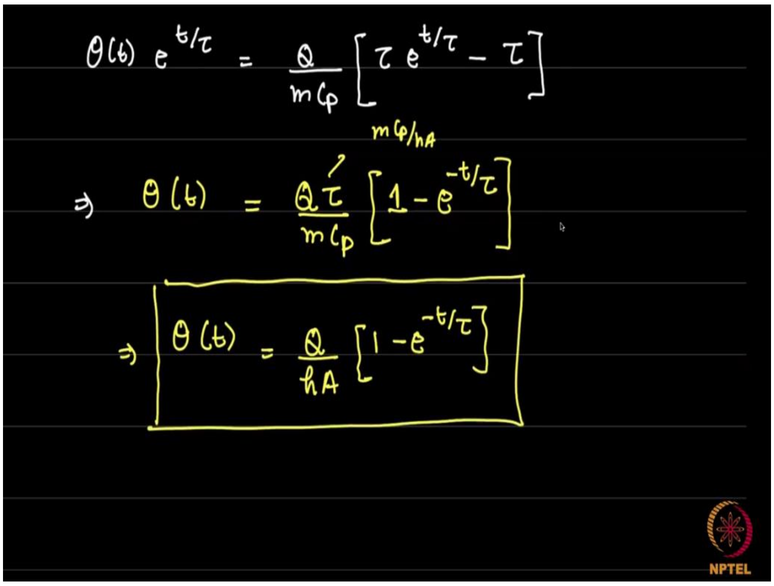

So, dividing both sides by $e^{t/\tau}$ to get,

$$\theta(t) = \frac{Q\tau}{mC_p}\,[1 - e^{-t/\tau}]$$

and this immediately tells us,

$$\theta(t) = \frac{Q}{hA}\,[1 - e^{-t/\tau}]$$

So, we basically get $\frac{Q}{hA}\,[1 - e^{-t/\tau}]$. So, this is our forward model.

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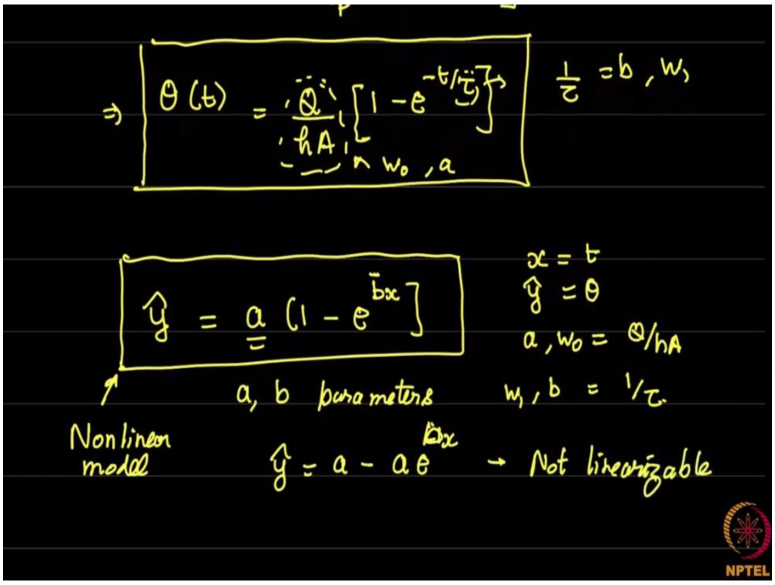


Now this forward model as if you can look at it basically has two parameters, it has this parameter $\frac{Q}{hA}$ we can call this parameter as $w_0$ or a. We have this other parameter $\tau$ we can call $\frac{1}{\tau}$ as b or $w_1$. So, in that case we can write this in the form,

$$\hat{y} = a(1 - e^{bx})$$

where x is time our predictor variable and $\hat{y}$ is basically our $\theta$, the predicted variable and a or $w_0$ depending on what we find convenient is $\frac{Q}{hA}$ and b or $w_1$ is $\frac{1}{\tau}$.

So, these are the parameters of the problems. So, a and b are parameters. This is what we try to find out in our inverse solution. Now the deal here is, that this model it is a nonlinear model. what is meant by nonlinear? Nonlinear means nonlinear in the parameters, that is this is not linear in a or b. what is that, it is not linear in either a or in b or you can treat it to be somewhat linear in a, in that you can write this as, $a - ae^{bx}$.

But this is not linearizable you cannot take log, you cannot do anything clever and somehow turn it into a problem which is linear in a and b.

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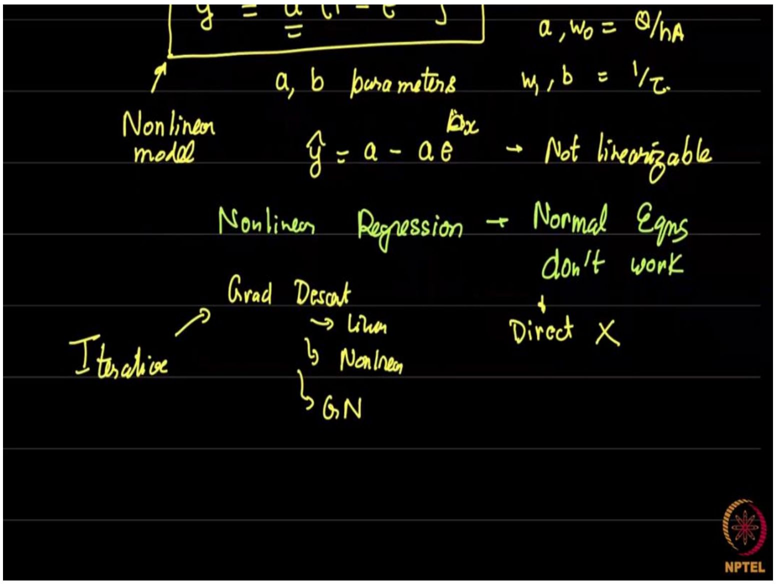


This is exactly why you need nonlinear regression. So, there is no simple solution to this just like we did linear regression, we will have to do nonlinear regression. unfortunately for nonlinear regression normal equations do not work, because normal equation work, because we had a system of linear equations. So, this is a system of nonlinear equations, what do I mean by system of nonlinear equations.

Now notice for each x or for each time you have a predicted temperature you will have multiple times at which we will have multiple temperatures I will show you an actual example later on in this week and you have to solve a regression problem for these two parameters a and b and you cannot do it in any of the methods that we have seen so far. So, what we will do in the upcoming videos is we will set up a systematic way to approach this.

So, the normal equations were direct approaches as I said earlier. This will not work. So, we have to use an iterative approach which we will start in the next video. So, the iterative approach we will start with will be called gradient descent. First, we will apply gradient descent to the linear case, then we will apply it to this nonlinear case and then we will see an even better method called the Gauss Newton algorithm which we will also develop.

So, that is the development which is planned for this week. So, I hope to see you in the next video. Thank you.