**Fundamentals of Combustion**

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**Lecture - 31**

**Characteristics of combustion flame and detonation Part 3**

**Rankine-Hugoniot relation**

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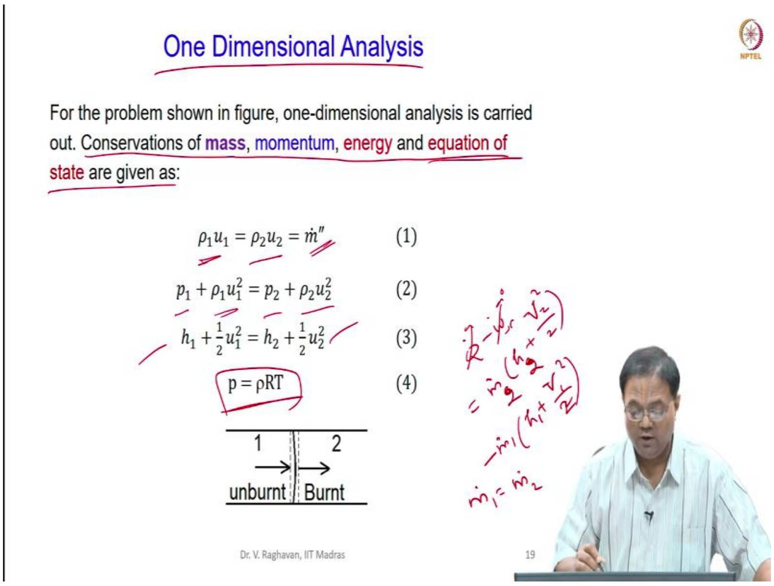


Now, is there any possibility to analyze this using some one-dimensional analysis? It is a predominant one-dimensional movement of the flame correct. So, easily we can try to do some one-dimensional analysis and understand what are the important factors here. So, that is what we will try to attempt now.

So, for the problem what we have shown here, you can see that the unburned gas approaches the flame and burnt gas leave the flame. So, it may be detonation anything. So, please understand we are trying to solve for both.

So, conservations of mass, momentum, energy and the equation of state are given here; conservation of mass, momentum, energy and equation of state. What is mass conservation? Per unit area of the duct, what is the mass flux, $\dot{m}''$, mass per unit area. This is the left hand side $\rho_1 u_1$; $\rho_1$ is the unburnt density, reactant density, and $u_1$ is the speed at which it approaches the flame.

$$\rho_1 u_1 = \rho_2 u_2 = \dot{m}''$$

And $\rho_2 u_2$ is the burnt side, so that mass should be preserved, so that is conserved. So, that is the first equation.

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

Momentum, $p_1$ pressure + $\rho u_1^2$, a dynamic pressure should be equal to $p_2$ + $\rho_2 u_2^2$, very simple. Energy conservation, so it is a steady flow as I told you steady propagation let us assume.

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

So, in this case, what happens? We will also assume there is no heat transfer, external heat transfer. So, both are 0, Q and W both are 0. You have m, steady flow, so what is mass coming in and going out are the same. So, $m_e h_e$ plus or maybe I will say 2, $m_2 h_2$ + $v_2^2/2$. So, potential energy I neglect, minus $m_1 h_1$ + $v_1^2/2$. Now, this is the equation.

Now, $\dot{m}_1 = \dot{m}_2$ and left hand side is 0. So, I can write $h_1$ + $u_1^2/2$, that is the velocity square will be equal to $h_2$ + $u_2^2/2$, so, that will be energy conservation in simple term. You may wonder there is combustion taking place here why we are not able to take into account of that. Because the h will take care of that basically we will see that later.

Then, as I told you equation of state is important for this to derive the property like density, so, based upon temperature and pressure, we can get the value of density.

So, I am writing the equation of state in this form p = ρRT. So, these are the fundamental governing equation. So, let us see how to extract some useful results from this.

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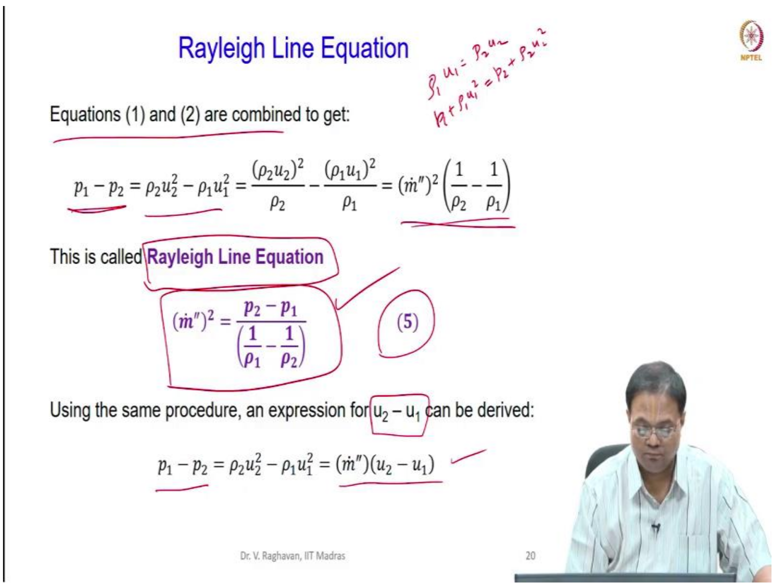

Now, first what we are going to see is, I will try to combine the continuity equation and the momentum equation that is $\rho_1 u_1 = \rho_2 u_2$, $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$. Now, I combine these two,

$$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2 = \frac{(\rho_2 u_2)^2}{\rho_2} - \frac{(\rho_1 u_1)^2}{\rho_1}$$
$$= (\dot{m}'')^2 \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)$$

I get $p_1 - p_2 = (\dot{m}'')^2 (1/\rho_2 - 1/\rho_1)$. So, what I do? I substitute here $\rho_1 u_1$ I know $\rho_1 u_1 = \rho_2 u_2$, I substitute here like this.

And I know that $\rho_2 u_2 - \rho_1 u_1$ that square, so I can write this, this is nothing but $(\dot{m}'')$. So, this equation when I say $(\dot{m}'')^2$ is nothing but $(p_2 - p_1)/(1/\rho_1 - 1/\rho_2)$. I am writing like this. This is called the equation for Rayleigh line.

$$(\dot{m}'')^2 = \frac{p_2 - p_1}{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)}$$

So, from the same procedure, we can evaluate the value of $u_2 - u_1$. So, $p_1 - p_2 = (\dot{m}'')(u_2 - u_1)$. So, from that, we can just. I am just skipping these equations which are handy in certain calculations. So, please understand the equation number 5 - this Rayleigh lines equation, it is very important for us.

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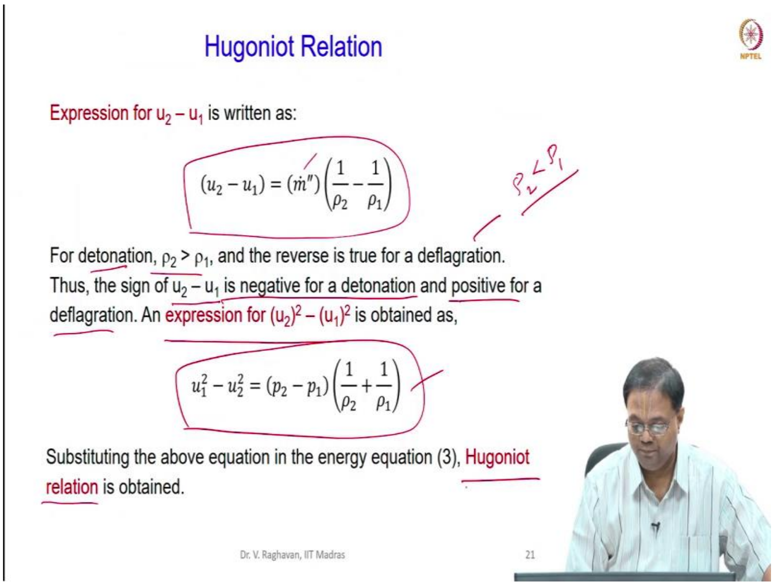
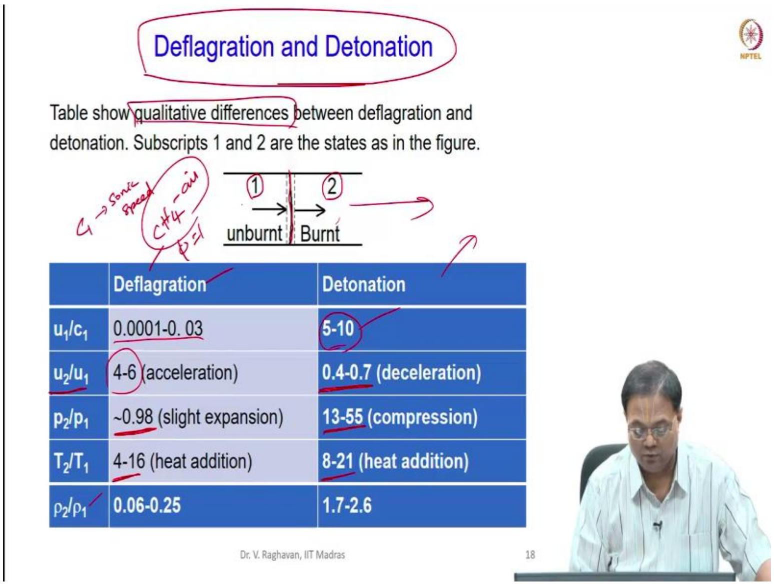



Now, the expression for $u_2$, $u_2 - u_1$ is derived like this. Here, you can see that $p_1 - p_2$ is this $(\dot{m}'')(u_2 - u_1)$. So, $u_2 - u_1$ will be equal to $p_1 - p_2/(\dot{m}'')$. Now, $p_1 - p_2$ is known to be this. So, using that, I can write $u_2 - u_1 = (\dot{m}''))(1/\rho_2 - 1/\rho_1)$.

Now, we can see why I am doing this I want to see the magnitude of this $u_2 - u_1$. So, for detonation, see the table again. (Refer Slide Time: 05:36)




| | Deflagration | Detonation |
|---|---|---|
| $u_1/c_1$ | 0.0001-0. 03 | 5-10 |
| $u_2/u_1$ | 4-6 (acceleration) | 0.4-0.7 (deceleration) |
| $p_2/p_1$ | ~0.98 (slight expansion) | 13-55 (compression) |
| $T_2/T_1$ | 4-16 (heat addition) | 8-21 (heat addition) |
| $\rho_2/\rho_1$ | 0.06-0.25 | 1.7-2.6 |

Go back to the table; detonation $\rho_2/\rho_1$ is 1.7 to 2.6 that is the range it varies.

For deflagration, it is very low, maximum is 0.25; for detonation it is 1.72 that means, $\rho_2 > \rho_1$ for detonation. For deflagration, for deflagration, $\rho_2 < \rho_1$.

Now, what happens due to this? So, when $\rho_2 > \rho_1$, $u_2 - u_1$ is negative for detonation. So, because you can see $u_2 - u_1$ is this. When $\rho_2 > \rho_1$, then this will be negative.

So, $u_2 - u_1$ will be negative. $(\dot{m}'')$ cannot be negative $(\dot{m}'')$ should be positive. So, this is negative. Now, for deflagration, since $\rho_2 < \rho_1$, this $u_2 - u_1$ is positive. So, that we have to understand.

Now, expression for $u_2^2 - u_1^2$ which is going to be useful later that is also derived using this same phenomena, What I do here is again use the momentum equation and combine with continuity. So, momentum and continuity are used to derive this. So, this is another important equation.

So, now, this equation and the energy equation if you use, energy equation is $h_1 + 1/2\, u_1^2 = h_2 + 1/2\, u_2^2$, this equation, and this equation $u_1^2 - u_2^2$ if I use this, then I get what is called Hugoniot equation or relation.

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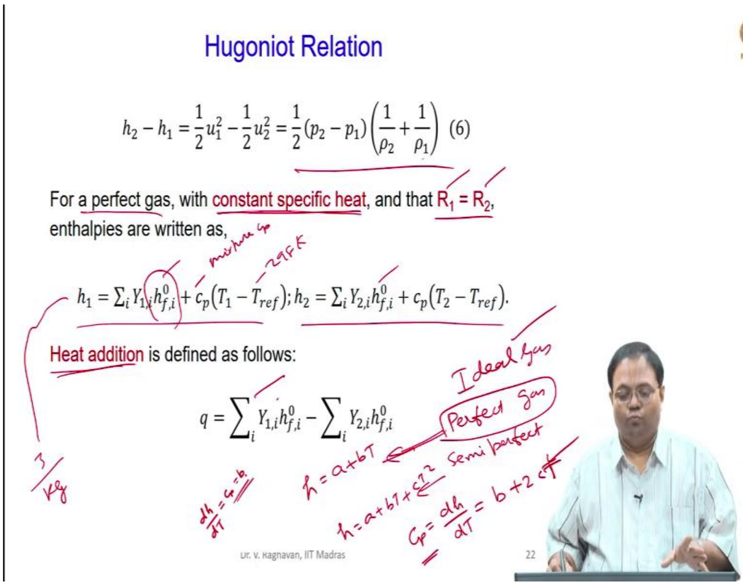


So, that is, is this $h_2$ - $h_1$. So, I am just writing the entire difference $h_2 - h_1 = 1/2 u_1^2 - 1/2 u_2^2$.

Now, I know the value of this $u_1^2 - u_2^2$ is $p_2 - p_1$ ($1/\rho_2 + 1/\rho_1$), so that I substitute here half times that, that will be this. Now, some assumptions have to be made. So, what we assume is a perfect gas, see ideal gas can be perfect or semi perfect.

When I say perfect U will linearly vary with temperature. So, for example, a + bT or say h, let us say h, h also vary like this, for perfect gas h will vary linearly. So, this means $dh/dT = c_p = b$ and its constant for the perfect gas. So, $c_p$ is constant for perfect gas.

But semi perfect gas, again there is no deviation, but h now is a function of temperature only, but it will vary like this; non-linearly it will vary. So, this implies $C_p = dh/dT = b + 2cT$, $C_p$ depends on temperature.

Now, when I say constant specific heat, it will not vary with temperature. So, I take this as a perfect gas. Ideal gas can be perfect gas or semi perfect gas. So, I take perfect gas. So, in the temperature range surely $C_p$ will vary, but this is an assumption made.

When I do that and additionally when I say molecular weight of the reactant and molecular weight of the products are almost the same, so $R_1$ the specific gas constant of reactant and that of the product are almost the same. If that is the case, then I will write enthalpies like this.

What is this? Enthalpy of the reactant $h_1$ or unburnt reactant is

$$q = \sum_i Y_{1,i} h^0_{f,i} - \sum_i Y_{2,i} h^0_{f,i}$$

$Y_{1i}$ means in the reactant stream what are the mass fractions associated with the reactant stream $Y_{1i}$ or the reactant region into the enthalpy of formation of that species added together plus this is the enthalpy formation net enthalpy formation.

Now, $C_p$, this is $C_p$ of the mixture, $C_p(T_1 - T_{ref})$ some reference temperature. So, the reactants are at the unburnt temperature of $T_1$, so that may be not equal to $T_{ref}$ always. If it is equal to $T_{ref}$, this is 298 K. If it is equal to this, then this term will go out to 0, only the enthalpy formation has to be taken into account, and basically that will be for fuel alone in the reactant side.

Because oxygen, nitrogen are basic elements they will not have any heat of formation. So, the product is same thing sigma $\sum Y_{2i}$ means whatever be the $Y_i$'s in the stream of the burnt gases that into enthalpy formation of this, so $CO_2$, $H_2O$, etcetera.

Similarly, $C_p(T_2 - T_{ref})$; so $T_2$ is the temperature. If it is same as the reference temperature, then that term will be 0. So, now, this is the expression for $h_2$ and $h_1$.

And heat addition now comes into play by standard heat addition, this is standard heat addition. When the reactants are at 298 K, what is the formation enthalpy for that entire reaction mixture ok? Please understand that this h etcetera has a unit of J/kg.

Similarly, so you have to convert. So, normally we get the $h_f$ value in J/kmol, now we have to convert into J/kg, and use it here, so that the mixing process we use the mass fractions.

Now, this is the enthalpy of formation of the reactants. And this is the enthalpy of formation of the products, the standard enthalpy of the products added together. Now, this difference will be the negative of standard heat of reaction.

So, what is standard heat of reaction? Enthalpy of the product, standard enthalpy of product minus the standard enthalpy of reactants. Now, here it is otherwise, that is heat of combustion, so that is heat addition. So, heat addition is this. So, this term can be absorbed in heat addition. So, q will be added to this.

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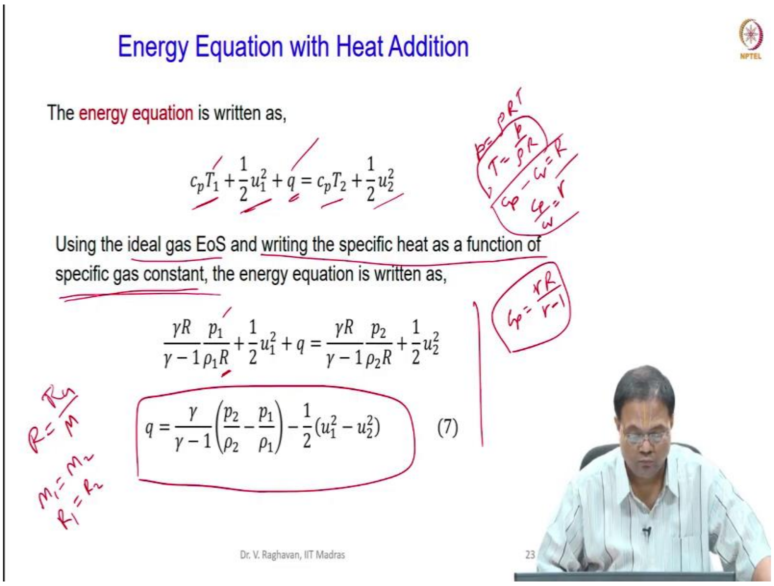


So, $c_pT_1 + q + 1/2u_1^2 = c_pT_2 + 1/2u_2^2$ that will be the energy equation. So, instead of writing in terms of enthalpy, when you write in terms of T, temperature and also recognizing that the enthalpy formation of the reactant and products can be combined to get the heat of combustion that is heat addition q. You can write the equation like this.

So, when there is a chemical reaction, the heat is released which is nothing but the difference between the enthalpies of the reactants and the products, so that is given out as a heat. Now, using the ideal gas equation of state and writing the specific heat as a function of specific gas constant, the energy equation can be written like this.

So, when you apply the ideal gas equation of state, you have $p = \rho RT$. So, now $T = p/\rho R$ that is what I will first use. So, instead of $T_1$, I will write $p_1/\rho_1 R$. Now, $c_p - c_v = R$ and the ratios $c_p/c_v = \gamma$

So, by combining this, I can get $c_p = \gamma R/(\gamma-1)$, now that is what I mean by saying writing specific heat as a function of specific gas constant and $\gamma$. So, this $\gamma$ is nothing but the ratio of specific heats at constant pressure and the specific heat at constant volume, so that this ratio.

So, using these two, I can write this. So, I will write temperature as $p/\rho R$, and $c_p$ as $\gamma R/(\gamma-1)$. So, now, as I already assumed $R_1 = R_2$ state 1 also it is R only because molecular weights are same.

So, R is nothing but $R_u$/molecular weight. Now, $m_1$ is approximately equal to $m_2$ let us assume. So, $R_1 = R_2$, molecular weight of the reactant and product are almost the same, so that is the assumption made.

So, when you do that, then $c_p$ is written as $\gamma R/(\gamma-1)$, $T_1 = p_1/\rho_1 R + 1/2\, u_1^2$ + q, same thing in the right hand side $\gamma R/(\gamma-1)$, $p_2/\rho_2 R + 1/2\, u_2^2$.

So, now combine the terms and write this equation like this,

$$q = \frac{\gamma}{\gamma - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) - \frac{1}{2}\left(u_1^2 - u_2^2\right)$$

So, that will be the equation I arrive at by combining this.

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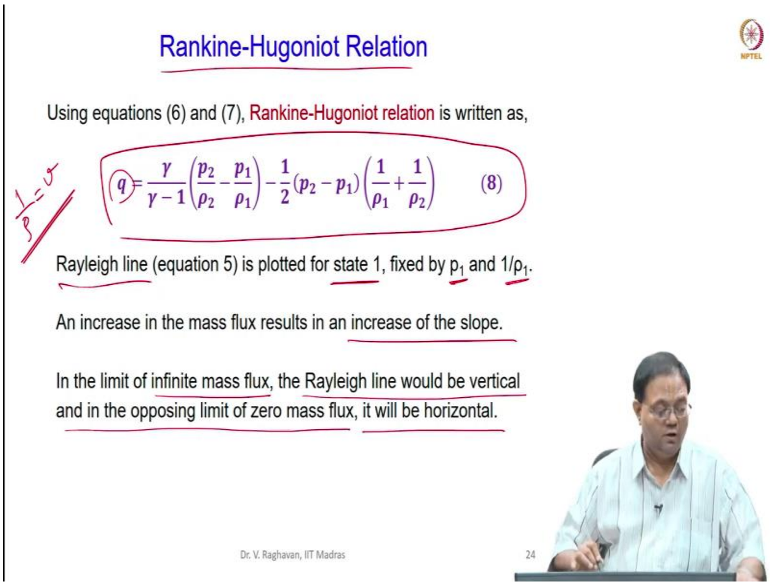


Now, this is called Rankine-Hugoniot equation; q, this is equation 8.

$$q = \frac{\gamma}{\gamma - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) - \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Rankine-Hugoniot relation $\gamma/(\gamma-1)$, $p_2$ you can also say $1/\rho$ is equal to specific volume. So, I say $p_2 v_2 - p_1 v_1$. So, specific volume also you can substitute here. So, $-1/2(p_2 - p_1)(v_1 + v_2)$, so, that will be the relationship.

So, please understand that this is energy conservation which is written in this form by combining the useful relation what we arrived by combining the mass and momentum. So, when you do any analysis, we have to take into account of both the Rankine-Hugoniot relation plus the Rayleigh line, both should be obeyed.

So, now Rayleigh line is this equation this mass flux square. This $(\dot{m}'')^2 = p_2 - p_1 / (v_1 - v_2)$ or $1/\rho_1 - 1/\rho_2$. So, a small $v_1 - v_2$.

So, this equation is the Rayleigh line equation and this is the Rankine-Hugoniot equation where you can see that q is expressed in terms of pressure and specific volumes or densities.

Now, when you fix the state 1, state 1 is any two properties are required. So, $p_1$ and $v_1$ small $v_1$ or $1/\rho_1$ let us take these two properties. I fix the state, then Rayleigh lines can be drawn. Rayleigh lines only need the initial condition. So, for several $p_2$ and $\rho_2$'s, I can draw the Rayleigh line.

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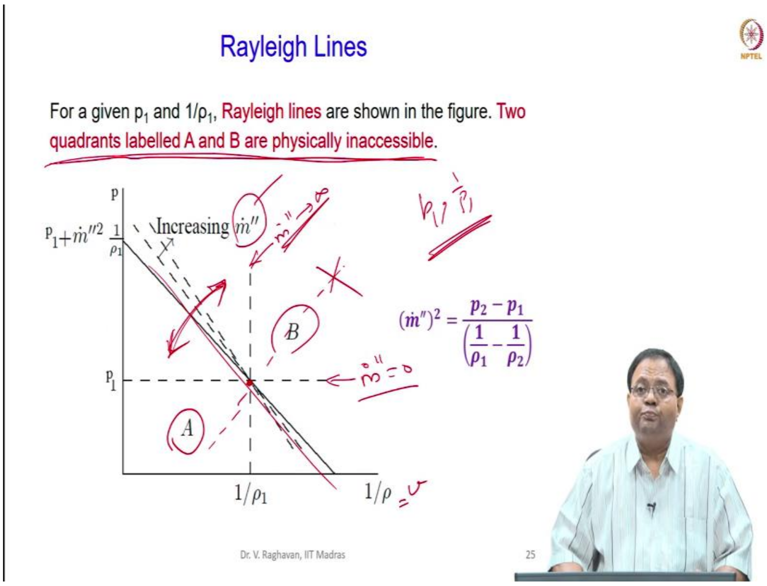


So, now, this is the Rayleigh line. So, p versus $1/\rho$ or v that is the Rayleigh line coordinates. In which you see that when I have different mass fluxes fix this state initial state $p_1$, $1/\rho_1$ - this is fixed, so that is this point $p_1$ and $1/\rho_1$. So, this point is fixed. Now, I can vary this based upon the mass fluxes, I can vary this.

So, what happens? If the mass flux is 0, then I get this line, this is corresponding to mass flux equal to 0. Where the mass flux is infinity very large, then I get this, So, in between, I can have this from horizontal to vertical anywhere I can move. This is not possible. So, I can approach the vertical line. Similarly, mass flux 0 does not make any sense.

So, I can approach the horizontal line. So, in between somewhere here this range I can operate this, that means, this coordinate, the line cannot go and cross this vertical line and becomes like this. There is no solution. So, this is not possible.

So, the solution in the A and B coordinate is not possible, because there would be negative flux. So, that is not possible for us. So, only it should be varying like this. The minimum is 0 and maximum is tending to infinity that is what we can have. And these two coordinates we can say that the quadrants labelled A and B are physically inaccessible.

So, Rayleigh line, the increase in mass flux there is an increase in the slope, obviously, in the limit of infinite mass flux Rayleigh line would be vertical; and in the opposing limit of zero mass flux, it will be horizontal. These are the two dashed lines which are drawn vertical and horizontal dashed lines.

And in between the mass flux can vary, so that we always have a slope line drawn only in this direction; the opposite direction line cannot be drawn, that means, the line cannot cross the vertical and go to the other side. And this type of line is not possible.

So, because of the negative flux, it will incur negative flux which is not correct. So, quadrants labelled A and B are physically inaccessible, this has to be obeyed. So, when you draw the Rayleigh line, you understand that these two coordinates are not going to be useful.

And possible Rayleigh lines I can draw in these two coordinates by varying the mass flux for a given initial condition. By varying the mass flux, I can draw several Rayleigh lines only in in this range, a typical range, not crossing the vertical and horizontal lines that is what I can do.

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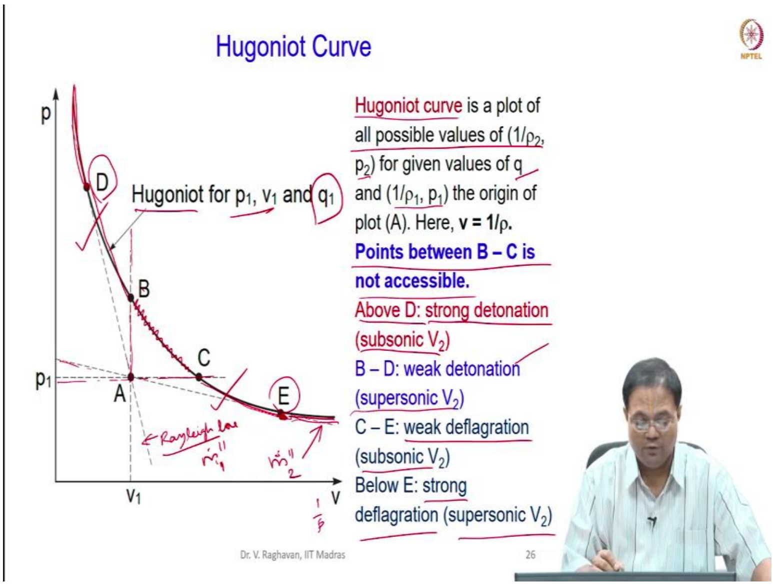


Now, Rankine-Hugoniot what you only draw is this. So, again we will say p versus v this is 1/ρ, p versus v, but please understand this Hugoniot curve you have to consider the Rayleigh line to draw this.

So, for example, the curve will look like this. This is the curve. But all these sectors of the curve will not be feasible for us to access. Obviously, you know this is the maximum mass flux this will be the minimum mass flux what we have as seen in the Rayleigh line.

Now, this means any portion in this curve here, where Rayleigh line cannot be drawn, I cannot access this; that is impossible. So, point between B and C is not accessible since

Rayleigh line cannot be drawn. So, continuity, Rayleigh line is basically the continuity equation and momentum equation obeyed you know that it should be obeyed.

Energy equation also should be obeyed. All these three are coupled. So, when Rayleigh line cannot operate in a particular portion, Hugoniot curve also cannot operate there. So, B and C is inaccessible, so that is ruled out. Then what you do is you take a flux now and now if you cross this you know, there is no meaning for us.

So, when you take a line, Rayleigh line, this is typical Rayleigh line which is feasible, possible Rayleigh line I take. And I draw this, and draw this line as a tangent to this Hugoniot. So, please understand when I draw this Hugoniot curve, I fix $p_1$ $v_1$ as I fix in the Rayleigh line and also I fix the q, q is fixed. Now, this curve will vary when $p_1$ $v_1$ or q varies.

So, let us fix three first - $p_1$, $v_1$ and $q_1$. So, I get a particular curve. Now, fix the mass flow rate, so that I get a Rayleigh line given like this. Now, that forms a tangent at the point D on the Hugoniot.

So, we can see that this particular Rayleigh line forms a tangent at this point of the Hugoniot curve. And this point is called D. So, this point is called D you understand this will again divide this into region. So, we have a region now B, D, and the region above D.

Similarly, I choose another Rayleigh line here, so that it becomes tangent here at D. So, this is another Rayleigh line. So, another difference, see this is m, $\dot{m}_1''$, this may be $\dot{m}_2''$. So, you get another Rayleigh line here. This Rayleigh line forms tangent to the Hugoniot curve at the point E. So, now you have accessible region C to E.

And I can also access the curve above D. So, the curve Hugoniot curve can be accessed above D; in between B and D, and between C, C and E, and below E - these are the four regions I can access. The region between B and C cannot be accessed, because there is no Rayleigh line possible there. So, Hugoniot curve is a plot of all possible values of $v_2$ and $p_2$ for given q and $p_1$ $v_1$.

Now, points between B and C is not accessible, because no Rayleigh line can be drawn there. Above point D, we say strong detonation occurs, but the velocity of the burnt gas will be subsonic, strongly detonation occurs. The incoming velocity will be supersonic, but after the wave propagates the burnt region, the velocity will be subsonic. B to D you would get weak detonation weak detonation and supersonic $v_2$. We will see about that.

Then C to E we will get weak deflagration, subsonic $v_2$ obviously. And strong deflagration and supersonic $v_2$ is possible here. Please understand that means, practically in lab scale, we can only get B to D here, and this. These are the two possible regimes. Above D it is not practically attainable.

Similarly, supersonic $v_2$ strong deflagration is not achievable. So, these are the things. But theoretically this below E region or above D region are possible, but practically we cannot get that. So, this is about Hugoniot. So, this curve, this equation is plotted with equation 8, and this Rayleigh line is plotted with the equation 5.