# Computational Fluid Dynamics Using Finite Volume Method


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**Lecture – 22**
## Finite Volume Method for Diffusion Equations: Truncation errors and stability analysis

Alright good morning, let us get started. So, today's lecture, we are going to look at evaluation of Truncation Error for Unsteady Diffusion ok.

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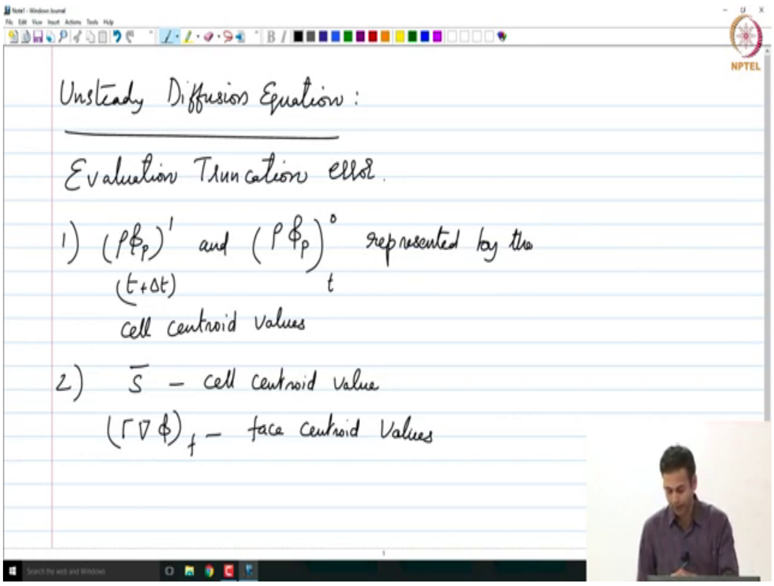


So, this is unsteady diffusion evaluation of truncation error. So, what are the assumptions we have made in the unsteady diffusion equation?

Student: (Refer Time: 00:45).

Well so, there is no convection term that is why we got the unsteady diffusion equation. In terms of how did we approximate some of the things for example, in the steady diffusion equation, we had the source term was evaluated as the cell centroid value right. Likewise, we have made several assumptions in the unsteady diffusion equation what are those?

So, if you look at the unsteady term, we had this $(\rho\phi_P)^1$ right and $(\rho\phi_P)^0$ right these corresponds to the $t + \Delta t$ and t values right. These values what did we say? We said that these are representative of the.

Student: Cell.

Cell centroid values right $\phi_P$ is what we have used right. So, these are again represented by the cell centroid values and then, if you move on to the other terms, we had the same spatial approximations right. For example, the source term that we will have we will again say that this would be represented by the again the cell centroid value and the face fluxes that is $\Gamma\nabla\phi$ on the faces are represented using the face centroid values right and what else? What are the assumptions we have made?

Student: Linear profile assumption.

Linear profile assumption for the diffusion fluxes as well as the source terms between $t_0$ or t and $t + \Delta t$ right that is what we have made. That means, and we said depending on the type of the scheme, either $t_0$ values prevail over the entire delta t or $t + \Delta t$ values prevail over the entire time step, that is what we said.

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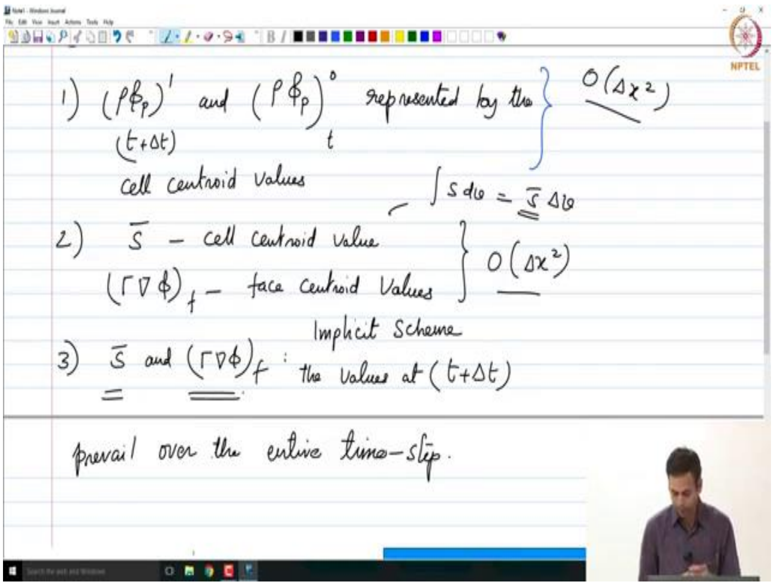


So, that means, the source terms $\bar{S}$ and $(\Gamma\nabla\phi)_f$ let us say if we are talking about an implicit scheme, this is a fully implicit scheme, then the values at $t + \Delta t$ prevail over the entire time step right, that is what we have said right. Anything else? We have any other approximations we have made in discretizing the equation? That is alright we had an unsteady term and the diffusion and the source terms right. So, both of them we said either it will prevail or t values or the $t + \Delta t$ values prevail over the entire time step and if you consider an implicit scheme, it is the $t_1$ values right ok

So, then let us evaluate each of these approximations. So, what is the first approximation? $(\rho\phi_P)^1$ and $(\rho\phi_P)^0$ these are represented between cell centroid value. So, what will be the order of accuracy for these terms? This is basically a mean value approximation right. So, we have already seen that this would lead to.

Student: A mean value.

A mean value approximation lead to what?

Student: Second order.

Second order in space or time?

Student: Time.

Student: Space.

Space right this will lead to second order in space because $\phi_P$ is what we are looking at right. So, this is order $\Delta x^2$ in space. What about the second assumption $\bar{S}$ and $\Gamma\nabla\phi$? This is also mean value approximation right. We are calculating the integral S dv with $\bar{S}$ times $\Delta V$ right. So, this is basically integral S dV is represented as $\bar{S}$ times $\Delta V$ saying that this is the centroid value. This is also both these assumptions lead to a spatial order of accuracy of what? Which order? Second order.

What is the third one? Third one is about time right. So, this is basically talks about if we have a source term or if we have the diffusion fluxes, these are will kind of incur some temporal order of accuracy, there will be some temporal error that is what we have t evaluate ok. So, that is what we are going to see next.

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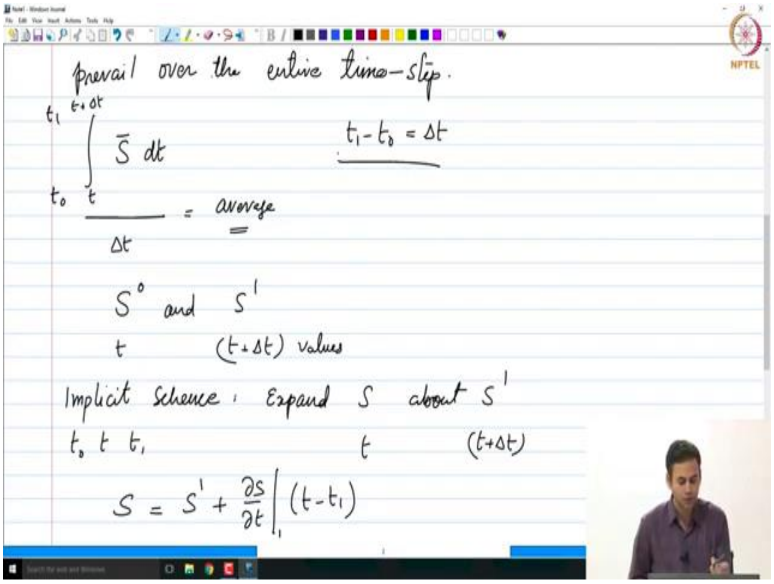


That means, let us say if we consider a source term that is S dV dt ok. So, this is already let us say some $\bar{S}$ that is on the volume $\bar{S}$ dt integral from t to $t + \Delta t$ right this is what we have to evaluate upon $\Delta t$ would give us some kind of an average value right for the $\bar{S}$ or the entire time step ok.

So, let me consider first a source term that is S which is expanded about S at $t + \Delta t$ ok. So, we have now $S^0$ and $S^1$ which represent t and $t + \Delta t$ values. We are interested to calculate what is integral S dt over the range of $\Delta t$ right. So, then, let me because I am considering an implicit scheme, let me expand S about $S^1$ ok. So, that is we are expanding S that is at any time instant t about $S^1$ that is at $t + \Delta t$ or basically we have $S^0$ and $S^1$ fine.

Then, if I use Taylor series expansion, what would be S? S would be if I were to expand about $S^1$ let me call it as this is basically $S^1$ plus $\frac{\partial S}{\partial t}\Big|_1$ right times t minus $t_1$. Is that correct? Just like what we have done for the spatial expansion, I am expanding about $S^1$ in time right. Yes question.

Student: Sir are we approximated for $S^0$ in terms of $S^1$?

Not $S^0$, I have S at any time level t ok. So, I have here basically $t_0$ and $t_1$ and t that is intermitted between these two ok. So, $S^0$ corresponds to $t_0$, $S^1$ corresponds to $t_1$ and S corresponds to any time t ok. Now, we are interested in calculating what is the integral of

S dt or the range t to $t + \Delta t$ or I can write this as $t_0$ to $t_1$ that is what we are interested in. $t_1$ minus $t_0$ would be our $\Delta t$ right.

Student: So, the t values are?

t values are at any time; at any time.

Student: So $t_1$ is the final value after.

$t_1$ is the final value after the delta t, $t_0$ is the previous value between the time steps.

Student: So, in the second term should it be $t_1$.

This one? You mean you mean this guy?

Student: Yeah.

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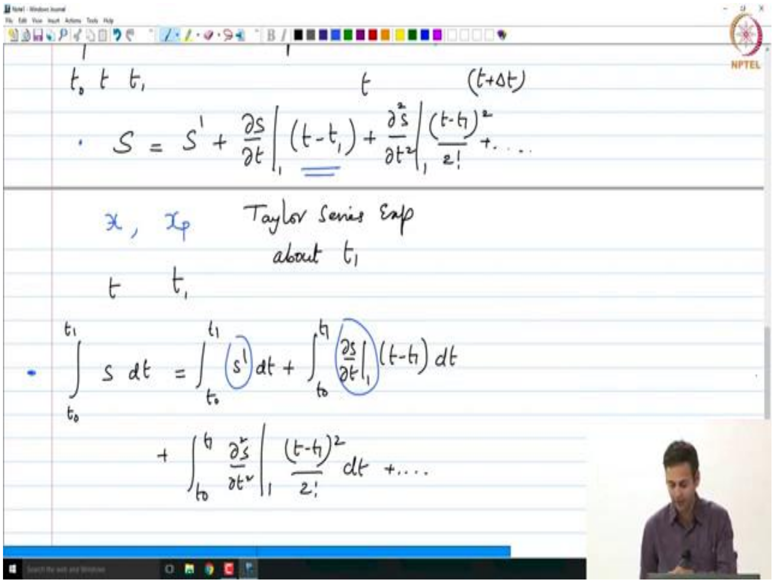

So, essentially you have let us say if you go back to the spatial expansion, we had some x right and then, we had expanded about $x_P$ right similarly, I have some time t which I am expanding about $t_1$ right or t sub 1 right basically that corresponds to be next time step $t + \Delta t$ right next time yes.

Student: Sir it is $\frac{\partial S^1}{\partial t}$.

Essentially it is $\left.\frac{\partial S}{\partial t}\right|_1$ evaluate at 1 right that is not $\frac{\partial S^1}{\partial t}$. It is essentially everything is evaluate at time level 1. So, that carries a time step of that of 1. Other questions? Should this be minus or plus?

Student: Plus plus.

Plus right fine. Any other questions? No then shall we complete this. So, what will be the next term? $\left.\frac{\partial^2 S}{\partial t^2}\right|_1$ evaluated 1, times $\frac{(t-t_1)^2}{2!}$ and so on right will have the third order terms and so on ok.

Now, of course, what we are interested in is basically integration of S over the time step that is going from $t_0$ to $t_1$ is what we are interested in. So, let us calculate this. So, basically integrate on both sides of this equation over the time step $t_0$ to $t_1$ ok. So, this will be integral $t_0$ to $t_1$, $S^1 dt$ plus integral $t_0$ to $t_1$, $\left.\frac{\partial S}{\partial t}\right|_1$ times t minus $t_1$ plus integral $t_0$ to $t_1$ I would write it down here plus integral $t_0$ to $t_1$, $\left.\frac{\partial^2 S}{\partial t^2}\right|_1$ evaluate 1, $\frac{(t-t_1)^2}{2!}dt$ and there will be a dt here and so on fine.

So, whatever the Taylor series expansion we have here ok, we have integrated over the time step right. This is essentially what we have in the unsteady diffusion equation right. We have approximated this using $S^1$ right $S^1$ times $\Delta t$ that is what we have approximated or $S^0$ times $\Delta t$ if it were an explicit scheme. Questions yes?

Student: First term (Refer Time: 10:59) minus t.

First term which one, which first term you mean you mean this equation this equation?

Student: The one above.

The one above here?

Student: $t_1$ minus t.

This should be $t_1$ minus t. Why it should be $t_1$ minus t?

Student: (Refer Time: 11:19).

So, we have time step t that is between $t_0$ and $t_1$ and we are expanding about $t_1$ right. So, this is basically t minus $t_1$ right. So, $t_1$ is the value set 1 are known or we are actually expanding about the next time level values ok. So, this is kind of a forward values we are using fine. Other questions? No, this is good.

Then, what will be; what will be the values of $S^1$ and $\left.\frac{\partial S}{\partial t}\right|_1$? What about these values?

Student: Constant.

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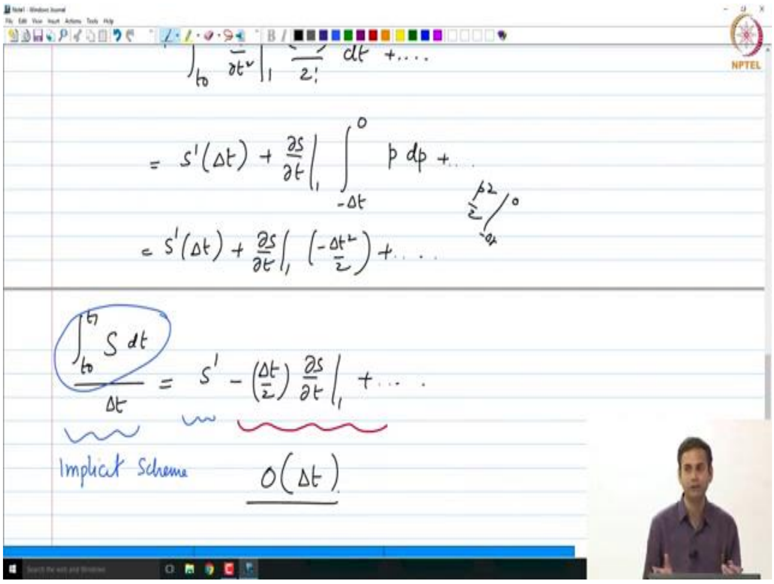

These are all constants right. So, can we integrate this equation on the right-hand side and tell me what would be the right-hand side. So, this would be $S^1$ times integral dt would be how much? t, $t_1$ minus $t_0$ would be $\Delta t$ and then plus $\left.\frac{\partial S}{\partial t}\right|_1$ times what would be t minus $t_1$ dt? Again, we have to use some other variable some p or something right and change of variables right. So, if I use a change of variables, what will be the limits?

Student: $-\Delta t$ to.

Minus $\Delta t$ to.

Studnt: 0.

$-\Delta t$ to 0 and this would be some p dp right if I use some other variable this would be plus and so on. We will have some more terms here.

Now, what will be the integration here? So, this will be $S^1$ times $\Delta t$ plus $\frac{\partial S}{\partial t}\Big|_1$ times what would be this value? Integral p dp would be $\frac{p^2}{2}$ right that would be?

Student: Delta.

$\frac{\Delta t^2}{2}$ with a plus or minus?

Student: Minus.

Minus. So, this will be a $-\frac{\Delta t^2}{2}$ plus something fine.

So, essentially if you go back, what we are interested in actually is there an average value right that is nothing but, integral $t_0$ to $t_1$ Sdt upon $\Delta t$ right that would be the average value that is what we are interested in the in the equation that we have already derived. So, what would this lead to? Essentially, we are dividing the entire equation with delta t right, this would leave us with $S^1$ minus $\Delta t/2$ times $\frac{\partial S}{\partial t}\Big|_1$ plus and so on. Is it correct? Yes.

Student: What is the minus $\Delta t^2$ (Refer Time: 14:04).

This is the lower limit so; you will get a minus.

Student: (Refer Time: 14:07) in a.

There is a square so, upper limit is 0 minus lower limit, lower limit has a square, this will lead to minus $\frac{\Delta t^2}{2}$. Is not it? Correct or no?

Student: Correct.

Correct essentially you get $\frac{p^2}{2}$ going from $-\frac{\Delta t}{2}$ to 0, right. So, this will be 0 minus you will get minus $\Delta t^2$ it will be a minus fine so, this is what we have.

Now, what is on the left-hand side? Left-hand side is what we have used in the equation right. This is what we have used in the implicit scheme right, we have replaced this term with what?

Student: S.

S times S of 1 times $\Delta t$ or it would be this value would be $S^1$ is what we have used right. We said the source term and the diffusion fluxes are assumed to be the $t + \Delta t$ values over the entire time step that is what we have used. So, that is what we said we approximated integral $t_0$ to $t_1$, S$\Delta t$dt right this guy with $S^1$ times $\Delta t$ that is what we have used and what is the error we are committing in doing so? Whatever is left over is what we are committing as an error right. Now, what is the order of accuracy of this error?

Student: First order.

First order so, this will be only order $\Delta t$ ok. So, this implicit scheme that we are working with is only first order in time and all other spatial assumptions are second order in space if we have a uniform mesh.

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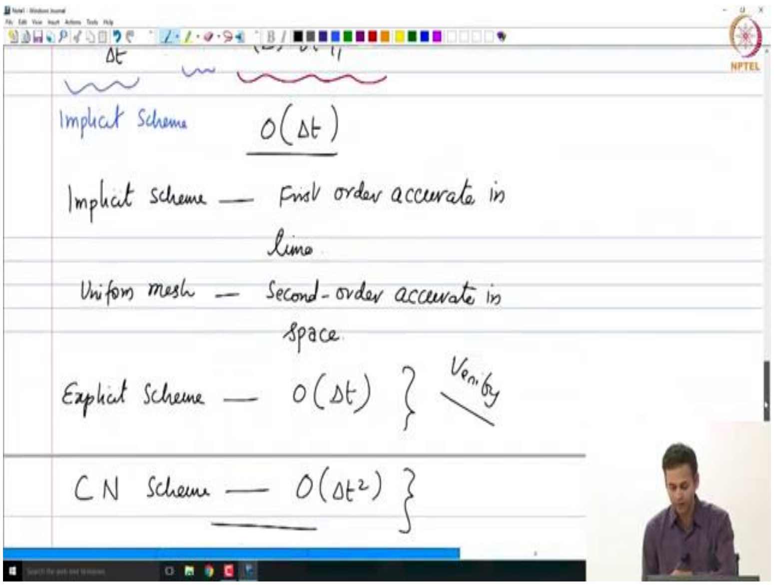


So, the implicit scheme is first order accurate in time and if you have a uniform mesh, then this is second order accurate in space fine. Now of course, we are not going to derive the order of accuracy for the other two schemes which are basically the explicit scheme and the Crank-Nicolson scheme which is what you will do and verify them later ok.

So, without proof we are stating here that the explicit scheme would turn out to be also order $\Delta t$ and the Crank-Nicolson scheme would turn out to be what?

Student: $\Delta t$.

Order $\Delta t^2$ ok. So, this is what I am not proving, but you have to verify these ok. Now, how do you go about calculating the temporal error for explicit scheme? What is that that you would change in the beginning?

Student: (Refer Time: 17:06) $S^0$.

You will expand about?

Student: $S^0$.

$S^0$ and so on ok. What about Crank-Nicolson scheme?

Student: (Refer Time: 17:11).

Where would you expand about?

Student: S half (Refer Time: 17:14).

S half ok. So, that is what you have to kind of try and see whether you are getting the $\Delta t^2$ for the Crank-Nicolson scheme or not ok. So, that is for you to verify. Questions on this till now? No questions clear ok. So, now, we have established the truncation errors both for space and time for the steady diffusion as well as for the unsteady diffusion equations all right.

So, we have one more topic that is left over which is basically the stability analysis right. We said in looking at these temporal equations right these temporal schemes, we said the explicit scheme is only conditionally stable whereas, we also stated that the implicit scheme and the Crank-Nicholson are unconditionally stable right.

We have said this, but we have never proved them through any analysis, but for explicit schemes, we have used something known as a heuristic method right where we said if the coefficients of $\phi_P^0$ let us say have to be positive, then you ended up getting a condition right that is what we have done which is a heuristic method.

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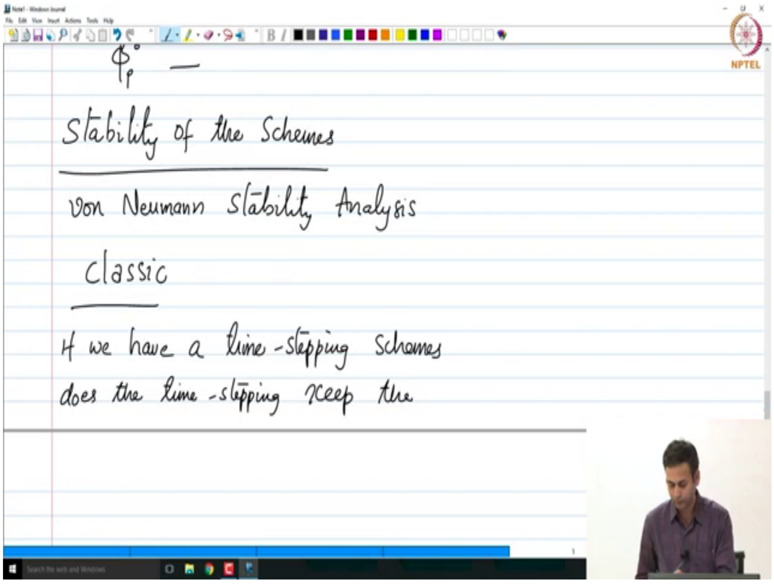


So, now we are going to look at a proper analysis that is known as Von Neumann stability analysis. So, we are looking at stability of the time stepping schemes ok.

So, we will use one method known as the Von Neumann stability analysis. So, this is a classic method that can be used to check the stability of the time stepping schemes we have ok. So, this can be applied for if you have simple 1D problems with 0 source term and without any non-linear descent things like that ok. So, that is what we are going to do.

So, essentially what this tells us through this stability analysis is if we have a time stepping scheme are the time steps that we are choosing going to keep the round off error stable or not that is what the stability analysis tells you.

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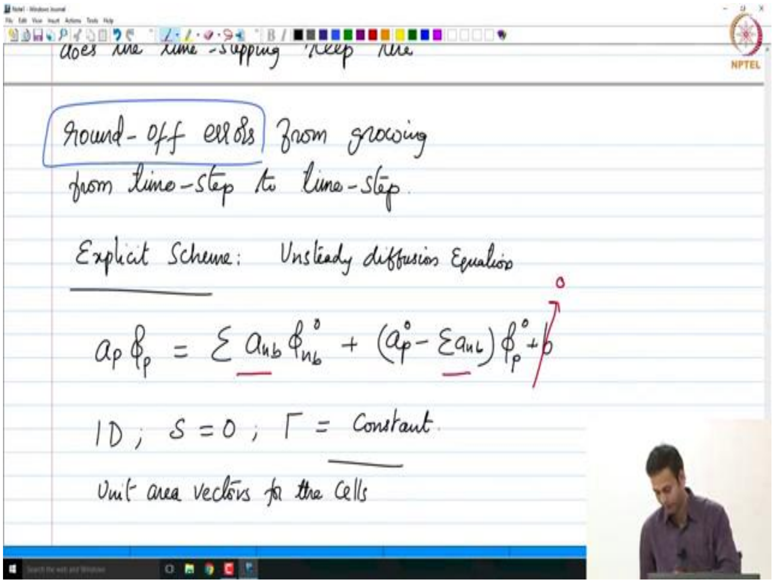


So, essentially if we have a particular time stepping scheme, do the or does the time stepping keep the round off errors from growing from iteration to iteration or from time step to time step ok.

So, here we have to kind of pay attention to this word which is basically the round-off errors. What are these round-off errors? Why do we get round off errors or where do we get them?

Student: (Refer Time: 20:29).

Computationally essentially numerically right. If you want to. So, truncation error is different from the round-off error right. Up till now, we have looked at truncation error which is order $\Delta t$, order $\Delta t^2$, $\Delta x^2$ and so on.

What is round-off error? Round-off error is basically the inability to represent a floating-point number exactly in a system right in a computer. So, essentially, we have round-off errors right, does not matter whatever is the accuracy you would use you still have round-off errors. Now, that is what we are using to kind of base the entire analysis on ok.

So, these round-off errors if they are bounded if they do not grow from time step to time step, then we end up calling the scheme as a stable scheme. If the round-off errors grow from time step to time step, then we call it as a unstable scheme which will eventually lead to the solution diverging ok. So, the entire analysis is based on existence of round-off errors in the computer ok. So, that is what we will do. So, the error we encounter at any time step is the round off error is what we would call in this stability analysis fine.

So, again the stability analysis can be performed for each of the three schemes that we have looked at which is the explicit, implicit and the Crank-Nicolson ok. For the purpose of the demonstration, I am going to use only the explicit scheme and see if we can come up with a stability analysis for this particular scheme ok. If we can come up with the condition that we have already realized in the previous lectures ok.

So, let us look at the explicit scheme and if you go back to your notes and see what would be the discretized equation for the explicit scheme for an unsteady diffusion equation. So, that is basically $a_P \phi_P$ equals.

Student: $\sum \mathrm{a_{nb}}$.

$\sum \mathrm{a_{nb}} \phi_{nb}^0$ plus.

Student: Plus $a_P^0$.

$a_P^0$ minus $\sum \mathrm{a_{nb}}$ times $\phi_P^0$ plus some b right that is what we have here $\phi_P$ without any superscript indicates the current time level values at time level 1 right or $t + \Delta t$ and $\phi_P^0$ or $\phi_{nb}^0$ they all indicate values at time level t right that is what we have.

Now, let for the sake of simplicity, let me assume we are looking at a one-dimensional problem and the source term is also 0 and I will also assume that the diffusion coefficient gamma ($\Gamma$) is a constant. This will make our life simple in the analysis. So, I am assuming a 1D problem with no source term and gamma equals constant ok. So, 1D problem meaning the area is also 1 right. So, essentially the unit area vectors for the cells. So, you do not get the $\Delta y$ or a for east and west they are all 1's ok.

So, if the source term is 0, which terms in this equation go to 0? b is completely composed of $S_C$ and $S_P$. So, this would be 0 this term would not be there and what would be the neighbors if you are looking at a 1D problem what are the neighbors? Only east and west nothing else fine.

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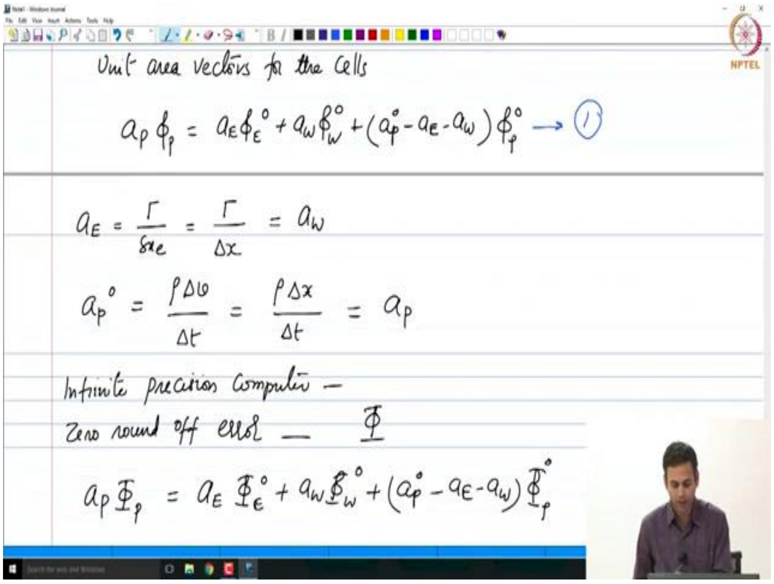


So, let me write down the corresponding equation kind of a simplified equation that would be $a_P \Phi_P$ equals $a_E \Phi_E^0$ plus $a_W \Phi_W^0$ minus sorry plus $(a_P^0 - a_E - a_W)$ times $\Phi_P^0$ right that is all we get for a 1D no source constant gamma.

Now, what is $a_E$? What will be the expression for $a_E$? $\frac{\Gamma \Delta y}{\delta x_e}$. So, because it is uniform, I would represent it with using $\frac{\Gamma}{\Delta x}$ we are using a uniform mesh as well. So, that is another assumption we are looking at a uniform mesh fine. Then, what will be the value of a west?

Student: Same.

Same as this right because it is uniform and it also this will also equal to a west. What will be the value of $a_P^0$?

Student: $\rho \Delta V$.

$\rho \Delta V$ by.

Student: $\Delta t$.

$\Delta t$. So, for the context of 1D, what will this p?

Student: $\rho \Delta x$.

$\rho \Delta x$ upon.

Student: $\Delta t$.

$\Delta t$ and what will be the value of $a_p$ for an explicit scheme?

Student: (Refer Time: 25:28) $\Delta t$ minus (Refer Time: 25:30).

What will be value of $a_p$? Can you go back and see? $a_p$ would be equal to $a_P^0$ for explicit scheme isn't it right for explicit it is the same for implicit you will have extra terms. So, this will be same as

Student: $a_p$.

$a_p$, $a_p$ equals $a_P^0$ for an explicit scheme right we have derived this before ok. So, far so good. So, we have now made several assumptions and got these equations.

Now, what I would say is that let us say we have an access to a an infinite precision computer ok. So, we have a some computer which has infinite precision right so; that means, the solution you would get out of that computer would have 0 round-off errors right. So, we have an access to infinite precision computer ok; that means, it would give you 0 round-off error.

As a result, the solution you would get from this infinite precision computer we would like to denote it using some capital phi ($\Phi$) whereas what we would get from this above equation we would call it as let us say 1 that is basically solution coming out of a finite precision computer like what we have with us right. And the second one which we are talking about is an infinite precision computer whose solution will be $\Phi$, but what equation do we solve on this computer? Same discretized equation right only thing is that the solution a little phi p($\phi_P$) and the capital phi p ($\Phi_P$) would differ by what?

Student: (Refer Time: 27:16).

By the round off error at that cell centroid right that is the only difference ok.That means are the accumulated round off errors at that cell centroid ok; that means, the solution $\Phi_P$ on a infinite precision computer would come out of solving this equation

$$a_P\Phi_P = a_E\Phi_E^0 + a_W\Phi_W^0 + \left(a_P^0 - a_E - a_W\right)\Phi_P^0$$

fine. So, this is basically a solution that is coming out of 0 round off error calculation ok

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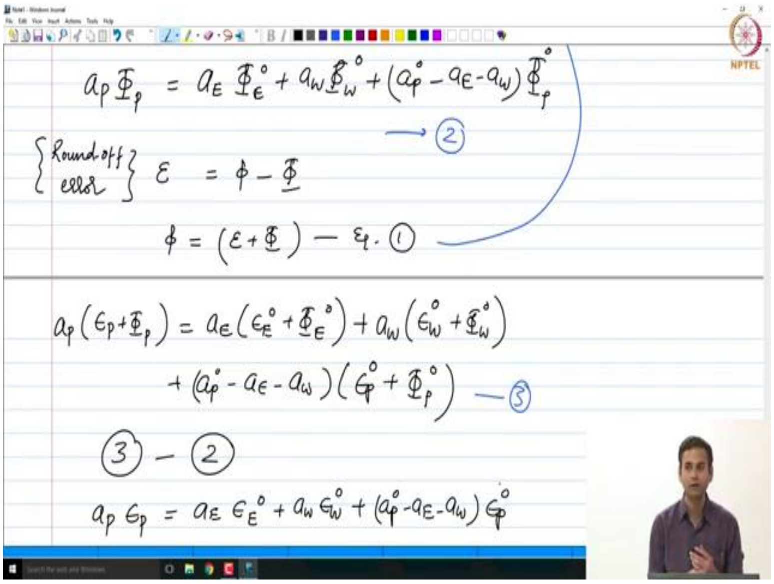


And now, we would like to call the difference in solution between ϕ and this Φ as the what?

Student: Round off.

As the round off error that is basically the all the accumulated round-off errors coming into play. That means I can calculate what is my ϕ from here as epsilon ($\epsilon$) plus Φ right. The solution I have is basically add the round-off error to the exact solution and that is what my actual solution that I got ok.

Now, can I substitute this ϕ back into equation 1 right essentially take this guy and substitute into equation 1 can I do that right essentially we have now ϕ which is the actual solution so, we will plug it in here. So, what will that be? That will be, $a_P(\epsilon_P + \Phi_P) = a_E(\epsilon_E^0 + \Phi_E^0) + a_W(\epsilon_W^0\Phi_W^0) + (a_P^0 - a_E - a_W)(\epsilon_P^0\Phi_P^0)$ right we got another equation which is in terms of $\epsilon$ and Φ the round-off error and the exact solution. Let us call this as equation number 3 fine.

Then, let me do a some algebra here. Let me subtract equation 2 from equation 3 so that means, 3 minus 2 what would this give rise to? By the way are these equations linear? They are all linear right because we have its basically a linear algebraic equation for one cell is what we are looking at.

So, can we subtract 3 minus 2 right if we do 3 minus 2 what would you get? You would get an equation in terms of $\epsilon$ that will be what $a_P\epsilon_P$ equals what will be on the right hand side? $a_E\epsilon_E^0$ right and $\Phi_E^0$ term goes away plus $a_W\epsilon_W^0$ plus $\left(a_P^0 - a_E - a_W\right)$ times $\epsilon_P^0$ is that correct? For every term, there are two terms in the corresponding term in the other equation. So, essentially the $\Phi$ terms all go away and then only the $\epsilon$ terms remain.

So, what have we got? We got an equation that governs the round-off error and the round-off error apparently satisfies the same discrete equation as the original equation that we have ok.

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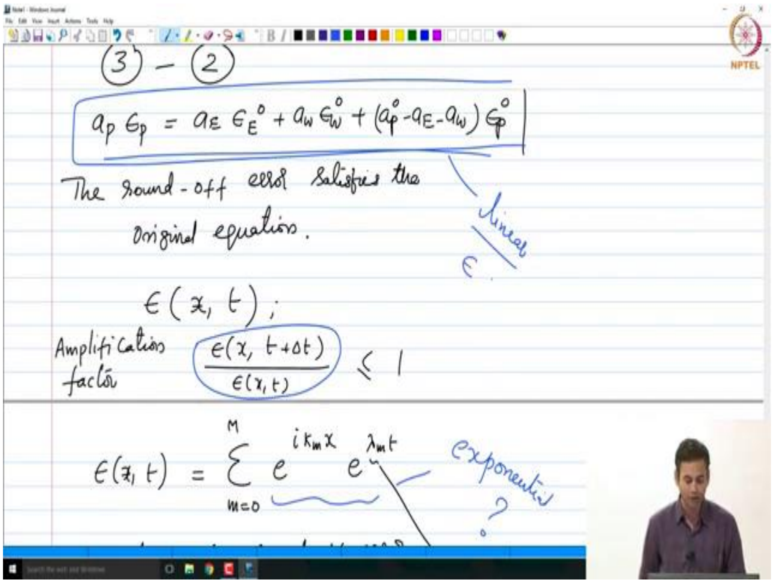


So, that is what we got; that means, the round off error satisfies the original equation as the solution right as the dependent variable, the round off error also satisfies the same equation ok. So, far so good.

Now, of course, we talked about this round off error $\epsilon$ ok. Now, we need to introduce a model for this round-off error right. So, we got some round-off error which is basically a function of what both space as well as time right. It is basically a function of both space and time. Space we are for the moment considering it as only x direction otherwise it will be x, y, z as well if we have a 3D system. Now, what is that we are actually looking for in terms of stability? We said a scheme would be stable if the round-off errors.

Student: Decrease.

Decrease or do not decrease?

Student: Decrease.

They should not increase right that is only condition; that means, the round-off error should not grow from time step to time step right. So, if we consider a particular x and between two time steps that is t and $t + \Delta t$, what will be this ratio between the round-off errors for these two time steps?

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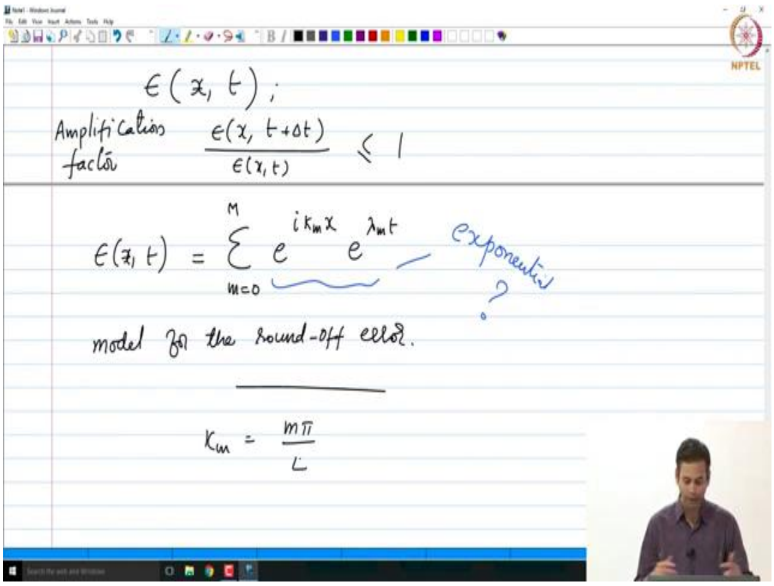


So, we are looking at the ratio of round-off error between in a time step alright. So, that will be $\epsilon(x, t + \Delta t)$would be the round-off error at time $t_1$ right divided by $\epsilon(x, t)$ would be round-off error at time level $t_0$ right. Now, what do we want this term to be?

Student: Less than or equal to.

Less than or equal to?

Student: 1.

1 so, let us call this as amplification factor which is basically less than or equal to 1 such that the round-off errors do not grow from time step to time step ok. So, that is what we are saying fine.

But as of now, we do not know any form for this round-off error ok. So, we can assume that the round-off error is composed of several modes right as in a Fourier series right so; that means, I am introducing a model for my $\epsilon(x, t)$ it is being composed of several waves that are basically as a function of space and time that will basically be and I am also assuming that it is in the form of a an exponential function or is made up of cosines and sines. So, basically is a function of your trigonometric function so that means, $e^{ik_m x}$ times $e^{\lambda_m t}$ sigma m goes from 0 to some capital M or infinite ok.

So, essentially I have introduced a model here for the round-off error that we get in a computation. Now, I assume that there is round-off error is composed of several waves in space and in time and there are also functions of exponentials. Now, why should this functions be exponentials? That is a question to you. Why cannot it be some so, polynomial some x square, x cube or something like that right that is a question for you to ponder and come back to me fine.

And this $k_m$ is a wave number that is basically $m\,\pi/L,$ the length of the domain right if we take a length as L for the problem, this would be the number of waves that you can get right. So, what we are saying is that these round-off errors are composed of several waves right, each of these waves are defined by m right if you set m equal to 1, you get 1 round-off error and you get another round-off error and so on.

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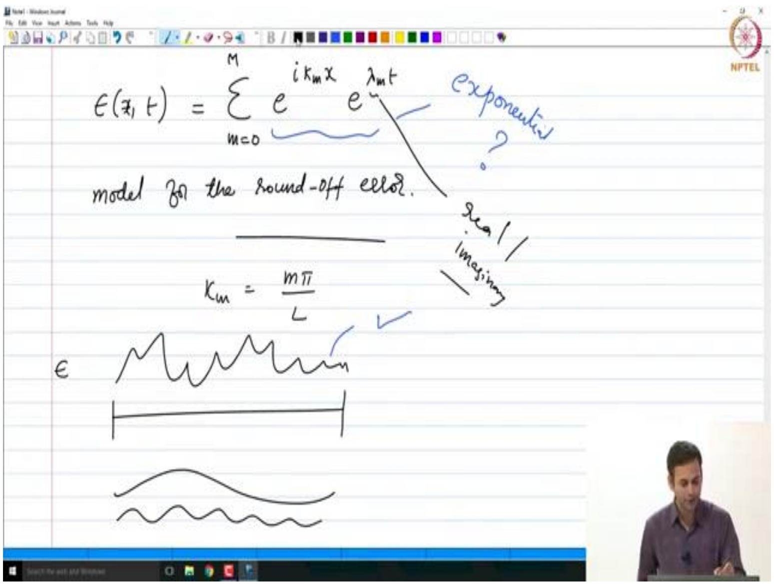

That means what we are essentially trying to say is if I have a let us say this is my domain L right. If I have some epsilon this is a round-error, I would say my round-off error is something like this in the computer I am using or what we are saying is we are modeling this round-off error using a set of cosines and sines right essentially using a Fourier expansion is that correct right.

We have several waves and then, we are modeling as a linear combination of all these several waves and then, we say we can somehow represent this using a linear combination of all of these things right and the round-off errors also may grow in time or may decay in time we do not know. For that matter, we have introduced this thing called $e^{\lambda_m t}$ where$\lambda_m$ can again be either a real or a an imaginary constant fine. Questions is it ok, everybody with this?

We have some random distribution of the round-off error in a computer or the entire domain and that distribution is what we are trying to approximate using several waves right, with several modes right, several frequencies that we have using the wave number $k_m$ fine everybody with that.

Of course, the question remains is why it should be exponential that is one thing which you have to kind of look at and come back ok. So far so good; that means, if I would substitute this model for the round-off error, I can go back and look at what would be my values for this amplification factor right. Essentially, I want to see the amplification factor for what? For this particular scheme right. We have considered an explicit scheme for this scheme, what will be the amplification factor is what we want to look at. In order to do that, because we want to solve for this, we have introduced a model for the round-off error that is all we have done.

Now, what about the linearity of this equation. Is this equation linear or non-linear, the original equation this one, discretized equation? This is a?

Student: Linear equation.

Linear equation and even if you have source terms what would this equation be?

Student: Linear (Refer Time: 37:58).

This will always be linear right because you would somehow circumvent this so, non-linearity of the source terms or diffusion by introducing additional iterations, but the resulting [vocalized-noise] equation we would solve would always be linear. So, this is linear. Now, what is the solution variable here, what are we solving for?

Student: Epsilon.

Epsilon we are solving for epsilon.

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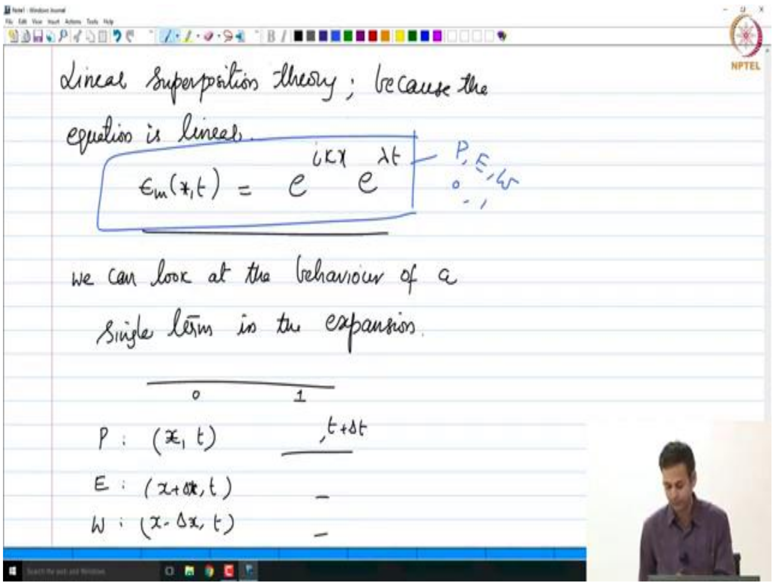


Now, we said epsilon is composed of several modes' sigma m equals 0 to capital M. Now because this equation is linear, we can use theory of linear superposition right because the solution is linear, it will be summation of all these solutions right.

I can use linear superposition theory because the equation is linear so, the simplification we are trying to get out of this is basically we do not have to look for each and every term of this; of this error right we do not have to look for each and every little m rather we can just look for one particular m right and if that is unstable, it will make the entire system unstable right. As a result, I do not have to look for each and every wave here rather I would look for just one component of the solution and that component happens to be just some arbitrary value ok.

So, we are looking at some let us say some epsilon m of x comma t this would be one particular mode. This is basically $e^{ikx}$ $e^{\lambda t}$. So, we say if this mode is unstable if this particular solution is unstable, then the original equation be unstable because it is all linear, there is no non-linearity. If you have non-linearity, then things could probably get damped and there will be some interaction right.

As a result, because of the linearity of the original equation, we can look at the behavior of a single term in the expansion only one term is what we can look at fine good. Then, what will be

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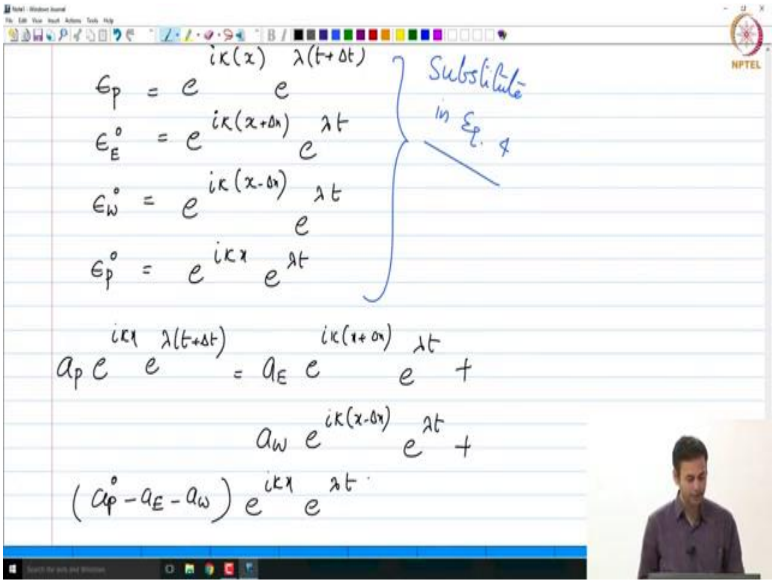

Now, what are the grid points we have? We have east, west and P right. Let us say P is characterized by $(x, t)$. What would be the east cell will be?

Student: (Refer Time: 40:40).

$x + \Delta x$ comma.

Student: t.

t and west would be?

Student: x minus.

x minus sorry it is be minus $\Delta x$ comma t and if it were at time level 1, you would also get in addition $t + \Delta t$ right. These are all at values 0 if it were at 1, you would get it corresponding $t + \Delta t$ right all these things. Correct? That is all we have, we have looked at the solution.

Then, if I go back and use the same equation, basically this equation right and substitute for the error in terms of the single mode that we have right. So, what are the terms we have to now calculate? $\epsilon_P$, $\epsilon_E, \epsilon_W$ and $\epsilon_P$ all at 0 right all these epsilon 0.

So, what will be $\epsilon_P$ at time level 1 what will this be? $e^{ikx}$ what will be the x value? x what will be e to the power lambda $t + \Delta t$ right. We are plugging in what is the p values for epsilon p. What will be $\epsilon_W^0$ or $\epsilon_E^0$? What will this be? $e^i$.

Student: k

k.

Student: $\Delta x$.

$x - \Delta x$ times $e^{\lambda t}$?

Student: (Refer Time: 42:06) $x + \Delta x$.

Sorry this is x plus.

Student: (Refer Time: 42:10).

And then, what about $\lambda t$ or t minus $\Delta t$ or?

Student: t.

t ok; t; ok. What about $\epsilon_W^0$? $e^{ik(x-\Delta x)}$ $e^{\lambda t}$ right. So, essentially what we are doing is we have this particular mode, we are just plugging in what is this for each of the p, east, west and at time level 0 and 1 right that is what we are doing.

Student: Sir.

Yeah.

Student: So all these modes should satisfy the equation.

All of these modes will satisfy. So, essentially your equation would become a equation in terms of a sigma right.

Student: Cannot you directly calculate x (Refer Time: 42:53) $\Delta t$ from this (Refer Time: 42:55).

Cannot you directly calculate what?

Student: (Refer Time: 43:00) x at the epilson t plus delta x comma t as delta from the equation.

No, but x is what varies over 0 to L I think I did not get your question. So, we have a model for the error, in terms of some exponential functions and we are considering one particular mode of that right and with a hope, we want to substitute this model that we have introduced into the original equation right and see if that error grows from time step to time step that is what we are trying to do.

Student: By [vocalized-noise] substituting we will get the value at the next time step.

We will get a value of at the next time step yes.

Student: We can get directly from the (Refer Time: 43:42).

Sure, essentially that is nothing, but you will get again in terms of $e^{\lambda \Delta t}$ right. So, we do not know what is that λ $\Delta t$. Now, that λ is what we are interested in right. So, we do not know lambda, now is there a λ that will make it grow from time step to time step that is where we have to bring in the governing equation ok.

So, essentially the question is we said we have a we said we have modeled the error using something right. I said I gave you a problem or you give me a problem and then, we said the error is basically some $sin(2x)$ is what we said. Now, $sin(2x)$ times $e^{t}$ I said. Of course, this will grow because you have $e^{t}$. If it were $e^{-t}$, it would actually reduce.

Now, the idea is cannot we just substitute this back or cannot we just calculate what is e at $\Delta t$ divided by e at t? We cannot because that is where this constantly $e^{\lambda t}$ still remains right.

For that, we have to put back this model that we have introduced for the solution back into the discrete equation for error right that is where the equation comes into play otherwise your equation is disconnected from the model we have introduced ok. But we have to make sure that this model we have introduced would not over ride all the things that we have in the equation right fine. Is that clear? Ok fine. We will come back to actually we will use what you have suggested, it will it will come as a part of it ok.

Now, so, these are other values. Do we need to calculate anything else? Do we need anything else in the equation? $\epsilon_P$is done, $\epsilon_E^0$ is done, $\epsilon_W^0$ is done, what else? $\epsilon_P^0$ needs to be calculated. What is $\epsilon_P^0$? $e^{ikx}$ times $e^{\lambda t}$ ok.

Now, can we substitute all of these back into in equation what equation number was that

Student: 4.

Let us call it 4 this is 4. So, substitute into equation 4 so, we have $a_P \epsilon_P$ equals $a_E \epsilon_E^0$ plus $a_W \epsilon_W^0$ plus this coefficient times $\epsilon_P^0$ ok. So, substitute into equation 4 what will be what is that we get? $a_P \epsilon_P$ is $a_P e^{ikx} e^{\lambda(t+\Delta t)}$ equals on the right-hand side what do we have? $a_E \epsilon_E^0$ so, this is $a_E e^{ik(x+\Delta x)} e^{\lambda t}$ plus $a_W e^{ik(x-\Delta x)} e^{\lambda t}$ plus what else? $(a_P^0 - a_E - a_W)$ times $e^{ikx}$e to the power.

Student: λt.

λt ok. Now, what do we see? We see that $e^{ikx}$ $e^{\lambda t}$ is common.

Student: Cancel.

In all the terms so, we can cancel it and also divide everything with $a_P$ on both sides ok.

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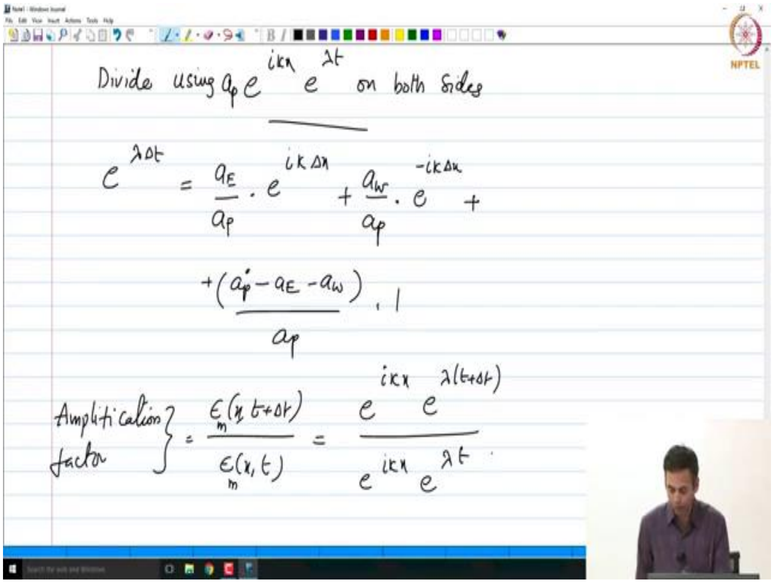


So, essentially divide using $e^{ikx}e^{\lambda t}a_P$ on both sides ok. So, if I do this what will remain on the left-hand side?

Student: (Refer Time: 47:37).

e to the power.

Student: λ.

λ $\Delta t$ equals on the right-hand side? $a_E$ upon $a_P$ times $e^{ik\Delta x}$

Student: $\Delta x$.

$\Delta x$ the next term would be $a_E$ upon $a_P$ times.

Student: e to the power.

$e^{-ik\Delta x}$ plus plus $(a_P^0 - a_E - a_W)$ times

Student: Upon $a_P$.

Upon $a_P$ times.

Student: 1.

1 that is all right. So, that is the equation we get alright. So, if we go back to our definition of what is the amplification factor $\epsilon(x, t + \Delta t)$ by $\epsilon(x, t)$ right this is for a particular mode let us say sub m.

So, if you substitute back into this equation, $e^{ikx}e^{\lambda t}$ what will this be? This will $e^{ikx}e^{\lambda(t+\Delta t)}$ upon $e^{ikx}e^{\lambda t}$.

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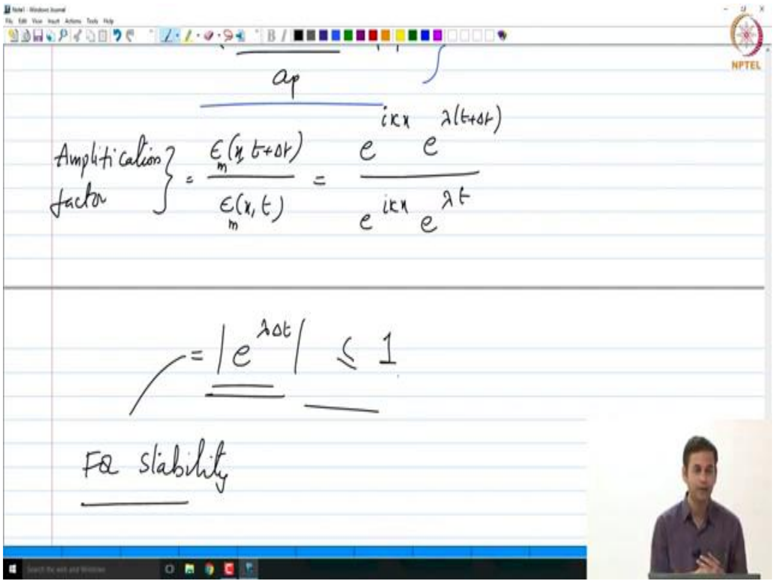


So, how much is this value?

Student: (Refer Time: 49:17).

e to the power.

Student: Lambda.

Lambda $\Delta t$.

Now, what do we want? We want this guy to be.

Student: (Refer Time: 49:24).

In a magnitude sense, we want this guy to be less than or equal to.

Student: 1.

1 right. Now, what have we obtained for this from the scheme? From the scheme, this value is nothing but everything on the right-hand side right. So that means, everything on the right-hand side here, we want it to be less than or equal to 1 if we do not want our errors to grow from time step to time step right. So essentially, we want this condition for what? Stability ok.

So, the derivation is not finished yet. We will introduce the coefficients which is $\frac{\Gamma}{\Delta x}$ for a east, a west and so on. We will plug this back into the $e^{\lambda t}$ expression and then simplify and see if we get a condition for the explicit scheme ok.

So, we will do that in the next class and you would verify the same thing for implicit and Crank-Nicholson schemes and you would see that you will not have any condition on the stability that is what you would do later on fine. So, we will pick it up in the next class and thereafter we will move on to diffusion on unstructured meshes and so on fine. Ok I am going to stop here

Thank you. See you in the next class.