Type equation here.**Computational Fluid Dynamics Using Finite Volume Method**

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Lecture – 14

Finite Volume Method for Diffusion Equation: Tri-Diagonal Matrix Algorithm

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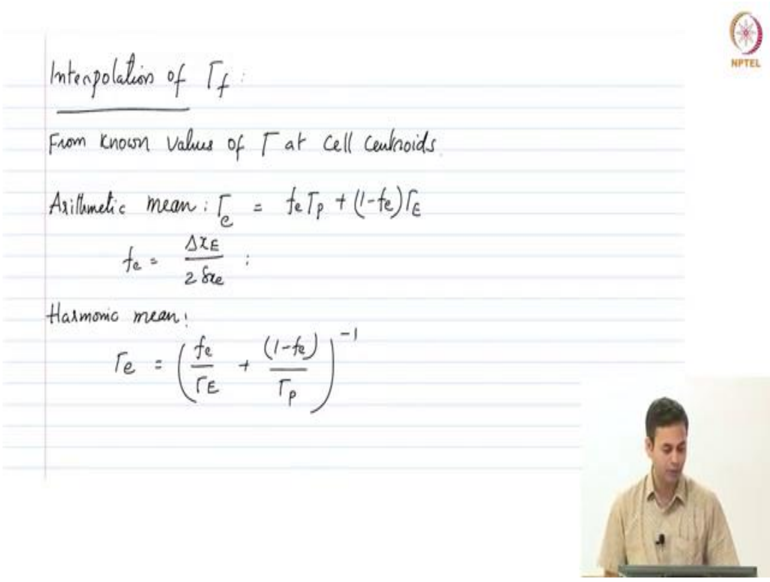

Good morning, let us get started. So, we were discussing interpolation of the diffusion coefficient on the faces right, this is interpolation of a gamma on the faces right, from known values of gamma at cell centroids ok. So, we have found there are two different ways we can calculate the diffusion coefficient on the face right. One was the arithmetic mean, which was given by what? $\Gamma_e = f_e\Gamma_p + (1 - f_e)\Gamma_E$ equals right.

What was the formula we had? , that was the formula we had. And, where$f_e = \frac{\Delta xE}{2\delta xe}$, that was the length of the you know arm length fine. And, then we also had this harmonic mean which was given by $\Gamma_e = \left(\frac{f_e}{\Gamma_e} + \frac{1-f_e}{\Gamma_p}\right)^{-1}$ equals what? is what we had right, gamma e inverse equals so and so or I could even write gamma e equals so and so to the minus 1 right. So, this is what we have for the harmonic mean.

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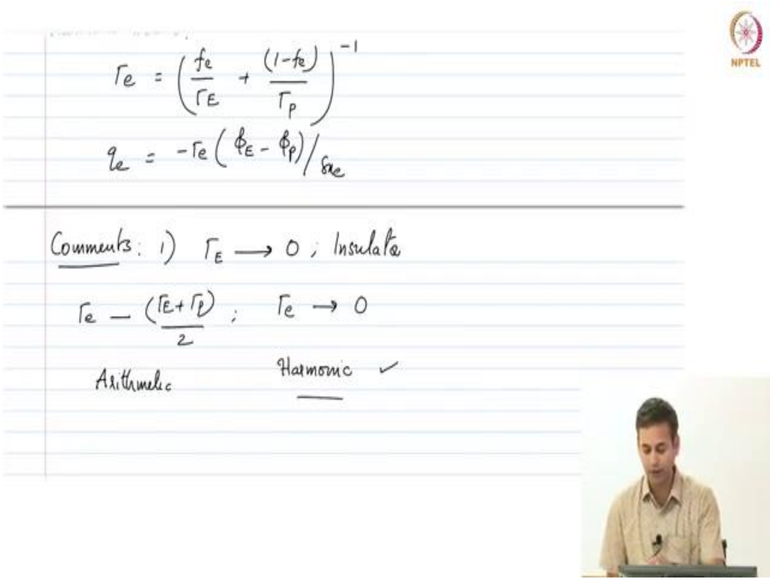

We also had a couple of different formulae in through which we have kind of derived this; that was what? That was the q e formula right which was given as $q_e = -\Gamma_e(\phi_E - \phi_P)\delta xe$ t. This is one formula we have used in order to arrive at the gamma on the faces right, by using the correctly modeling the heat flux alright.

So, that is what we have, then we have also seen that if we have a condition where. So, in the comment section what we have seen is if we have a thermal conductivity for $\Gamma_e \rightarrow 0$, essentially we have an insulator. Then what we saw was the arithmetic mean kind of retains the effect of both gamma E and gamma P right. This is the arithmetic mean part whereas, for the harmonic mean what happens?

Student: (Refer Time: 03:19).

A gamma e tends to 0 which is what it should be, because we have an insulator then the heat flux on the face of the insulator should be 0 right. So, that is what kind of recovered by the phase flux ok. So, this is recovered correctly by the harmonic mean, but not by the arithmetic mean ok.

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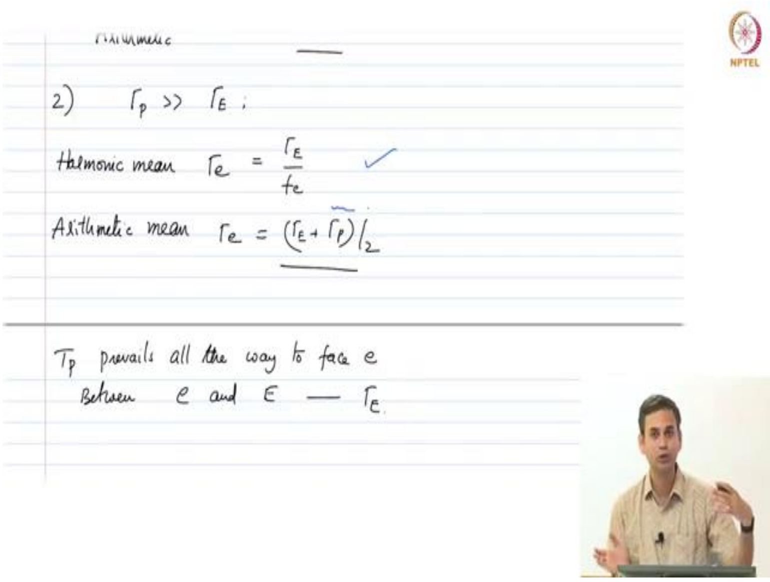

So, let us consider another case where let us say the thermal conductivity of the one of the blocks that is gamma E or gamma P is much higher than the east ok. So, we have a very large thermal conductivity for the P cell when compared to the gamma E ok. If we have a case like this then from the harmonic mean formula if we go back; so, gamma P is much larger than gamma E right. So, what will happen to this term, 1 minus f e by gamma P? This would tend to.

Student: 0.

This would tend to 0, as a result what will be gamma little e? Gamma capital E upon f e right; so, that is what we have. So, this is gamma capital E upon f e. So, essentially there is no dependence of gamma P in this formula right, it is essentially only depends on gamma capital E.

Whereas, what would have happened for the arithmetic mean? Gamma e would be how much? It would still retain the value of gamma P as well as gamma E right. So, this will still be gamma E plus gamma P upon 2 or the corresponding one. Now which of these kind of make sense? So, if we have a thermal conductive for the P cell very large compared to the east cell right, what will happen to the temperature drop within the P cell? The thermal conductivity is very large, much larger than E cell.

So, we will kind of think that the temperature should kind of remain the same all the way you reach the interface right. The T p that we have for the P cell should be the same all the way to the interface. Then you should have a decrease in temperature as you go along the E cell right; which is what the harmonic mean kind of gives you because it says that, because the interface is calculated only from gamma E right; the temperature T p that we have.

So, the temperature T p prevails all the way to face e right to the east face and from there you have a drop in temperature between.

Student: (Refer Time: 06:01).

Between e and capital E right and this should be dictated by which thermal conductivity gamma E or gamma P?

Student: Gamma E?

Gamma E, this should be dictated by gamma E right which is the case if you have a harmonic mean ok. If you have an arithmetic mean then it again retains the value of gamma P which would not kind of give you correct result, because you get a different slope right. Wherein, the temperature will start decreasing from the P cell to the E cell all the way within the P cell as well ok. Now that means, what we are trying to say is if I have let us say the cells like this, we have this is my P cell and then this is my east cell.

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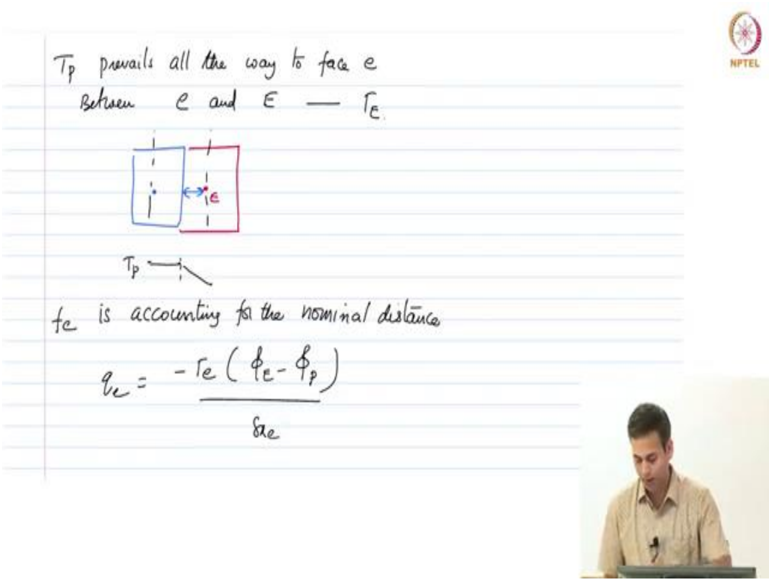

So, this is essentially we are talking about between these two and then we also have temperature as T p and then it would, it would pretty much remain the same or decrease very little as he reach the interface. And, then there will be a drop in temperature as you move within the E cell right. So, that is what would be captured correctly, if you have the harmonic mean ok.

Now, what about the other consequence? We say that of course, gamma e is not actually equal to gamma on the face is not equal to gamma capital E, but rather it is divided by little f e right. So, we have this factor in here. Now what is this doing? This is basically accounting for the nominal distance we have to use between the face and the capital E ok.

So, kind of the f e term; if you look at it is accounting for the nominal distance nominal distance ok. Now, how is that? How is that the nominal distance? Essentially, if you go back what is the formula for q e? $q_e = -\frac{\Gamma_e(\phi_e - \phi_p)}{\delta xe}$

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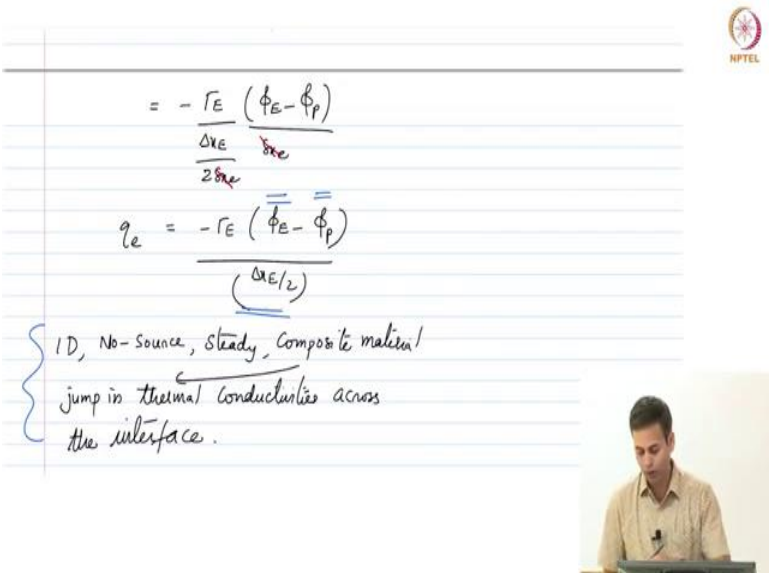

Ah Now, that is what we have; that means, if I were to plug in what is what is my gamma e now I have? Gamma e is gamma capital E upon f e. So, this will be$-\frac{\Gamma_E}{\frac{\Delta xe}{2\delta xe}}\frac{\phi_E - \phi_p}{\delta xe}$ gamma capital E upon, f e is how much?

Student: Delta x E.

.. So, what would this be? You have your del x e gets canceled as a result what you have is, you have $-\Gamma_E \frac{\phi_E - \phi_p}{\Delta xe/2}$. Now, do you see that the harmonic average recovers the correct heat flux; because when we talk about phi east, this is the temperature of the east cell.

When we talk about phi P now this is what? This is the of course, the temperature of the P cell as well as the temperature of the east face right. We are now talking about the temperature drop within the east cell only right. And what would be the distance between these two? Half of the cell distance right, that is delta x E by 2 and then we have the minus gamma E. So, essentially that is nothing, but whatever we have drawn here right this is T p and what is this distance? Delta x E by 2 ok.

So, essentially you recover the correct heat flux from the harmonic mean ok. So, these are kind of the implications and we have just demonstrated that the harmonic mean gives you physically correct results. Because, we started off with correct value for the heat flux Q rather than for the looking at what would be the interpolation I have to use for the diffusion coefficient ok.

So, that is the thing of course, we have made several simple give simplifications in deriving this thing; those are this is a 1D situation as well as there is no source right. So, this is a 1D situation, there is no source. What other simplifications we have made?

Student: Steady.

Steady ok, this was a steady and I have also made another simplification where there is a jump in the conductivities right. So, essentially we have a composite material right. So, essentially this is a composite material which is giving rise to a jump in thermal conductivities across the interface ok. Now of course, these are the assumptions we have made for which the harmonic mean is a kind of shown to be performing better than the arithmetic mean ok.

But, in situations where you do not have these things for example, even if you have a source term even if your problem is not 1D, even if it is not steady; you have transient problems.

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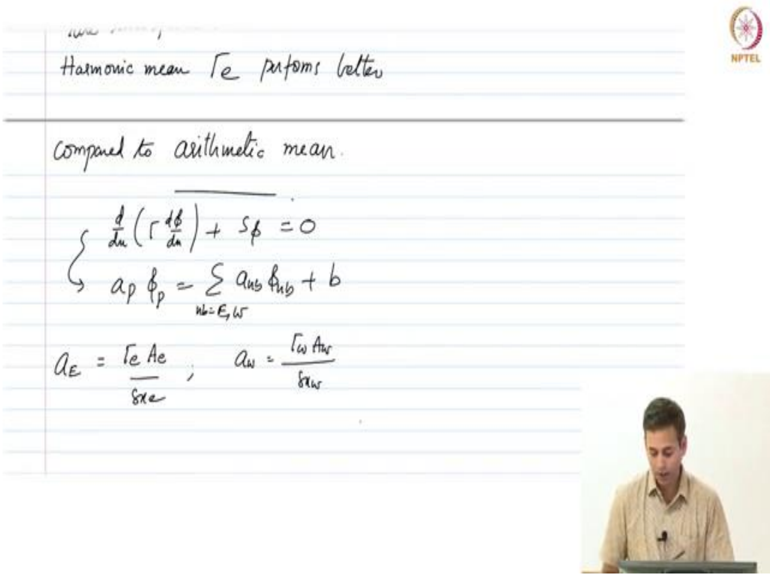

And, if it is a not a composite material rather you have the thermal conductivity varying continuously within the domain for all these cases as well; the harmonic mean formula for gamma e outperforms or performs better compared to the arithmetic mean. Or, essentially you get the correct result from gamma e rather than from the from the gamma e from the harmonic mean rather than from the arithmetic mean.

So, you can use in general if you want to obtain the diffusion coefficient on the faces, the interpolation by default would be the one is the harmonic average ok; that is what you would use unless otherwise stated somewhere fine. Now, let me kind of complete this discussion by just writing down the 1D equations ok.

So, those are we have already seen, essentially we are trying to solve a steady diffusion equation that is $\frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + s_{\phi} = 0$. And, after applying finite volume method you got $a_p \phi_p = \Sigma_{anb}\, \phi_{nb} + b$. So, where nb is capital E and capital W and we have these coefficients that is a east a west and so on. So, $a_E = \frac{\Gamma_e A_e}{\delta xe}$ an$a_w = \frac{\Gamma_w A_w}{\delta xw}$ a

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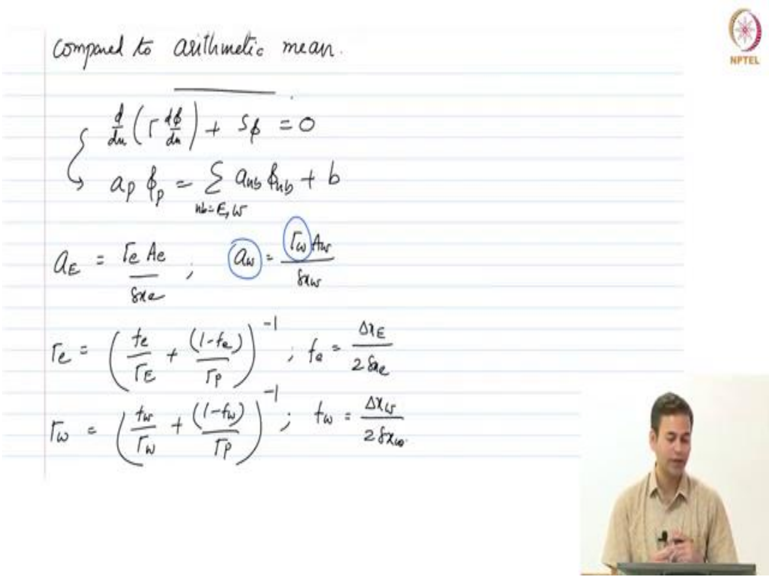

Ah Then how do you now model gamma little e, the diffusion coefficient on the east face as what?

Student: (Refer Time: 12:58).

Harmonic mean this is $\Gamma_e = \left(\frac{f_e}{\Gamma_e} + \frac{1-f_e}{\Gamma_p}\right)^{-1}$, $f_e = \frac{\Delta x_E}{2\delta_{xe}}$ f e upon gamma capital E plus 1. Similarly what would be gamma w that we get here in the coefficient for a w? This would be of course, $\Gamma_w = \left(\frac{f_w}{\Gamma_w} + \frac{1-f_w}{\Gamma_p}\right)^{-1}$, $f_w = \frac{\Delta x_W}{2\delta_{xw}}$. And what would be f little w? This would be

Student: x.

; of course, you can extend it to 2 dimensions and 3 dimensions and so on ok; questions till now, no questions fine. So, essentially you would use the harmonic mean by default ok. I may not say that in let us say an assignments or in exams or whatever, you would always use the harmonic mean alright.

So, we kind of take a we are trying to now fix up some things which we have not discussed in detail before right; interpolation of gamma is one of them. And, then a discussion on the linearization of the source term is one more thing right. If I just said that we are going to linearize, but we did not say how do I linearize it, why do I linearize it all these things.

So, we will come back to that discussion, before that I want to kind of discuss on the solution of the linear algebraic equations ok. So, we look at a direct method, but solving the equations and after that we will come back to the discussion on source term linearization, fine depending on the time either this class or in the on the next class fine.

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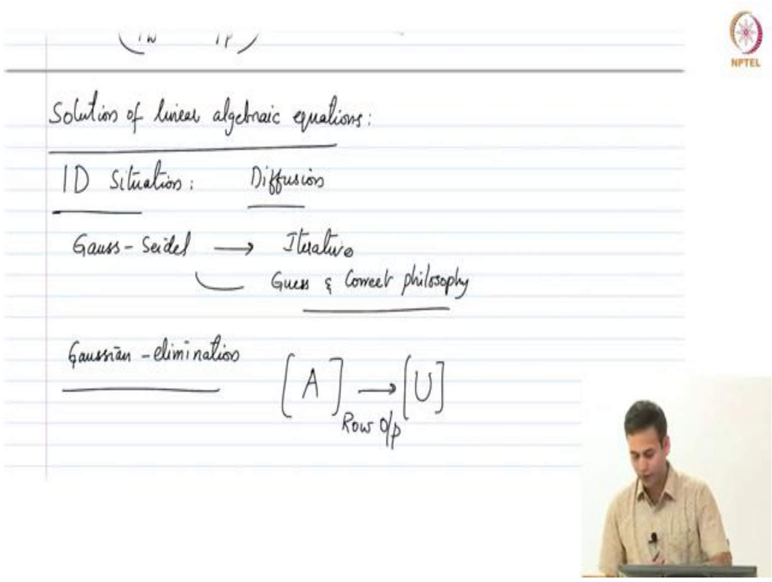

So, let us look at solution of linear algebraic equations, we are again looking at a 1D situation for now. So, we are looking at a 1D situation, let us say if we have a diffusion equation in 1D. We have already seen how to solve this the resulting linear algebraic equations right. How do we solve it? We have already used one method right, what was that? That was the.

Student: Gauss-Seidel.

Gauss-Seidel method right. So, we have Gauss-Seidel which is what kind of a method; is it a direct method or as an iterative method?

Student: Iterative method.

It is an iterative method ok, essentially works on the guess and correct philosophy, that is fine. Now, there is another method which is basically a direct method ok. We will kind of try to learn that, because we would use that in solving the some more problems that are coming up. Because, Gauss-Seidel is kind of slow, if you want to go for 2 dimensions and 3 dimensions the idea is to use a direct solver that works only in 1 dimensions ok.

And, then interlay this direct solver with Gauss-Seidel in the other dimensions such that you can still maintain the simplicity of a direct solver in 1 dimension and you can try to solve 2D and 3D problems relatively quickly ok. So, that is the idea. So, that is why we kind of learn a direct method ah.

This method is based on Gaussian elimination ok. Now, what is the difference between Gauss-Seidel and Gaussian elimination? Have you heard of the Gaussian elimination process? I think you have done this in linear algebra. What is Gaussian elimination? What is the philosophy? You have a matrix A right, I give you a matrix A; what do you do with it?

Student: (Refer Time: 17:02) row operation.

You essentially do row operations right and then convert this into you perform row operations and then convert this into an.

Student: Upper triangular.

Upper triangular matrix right and then you do the row operations on the right hand side also or no?

Student: Yes.

Do otherwise you are solving a different problem right. So, the solution you get will be something else which is not what ideally correct ok.

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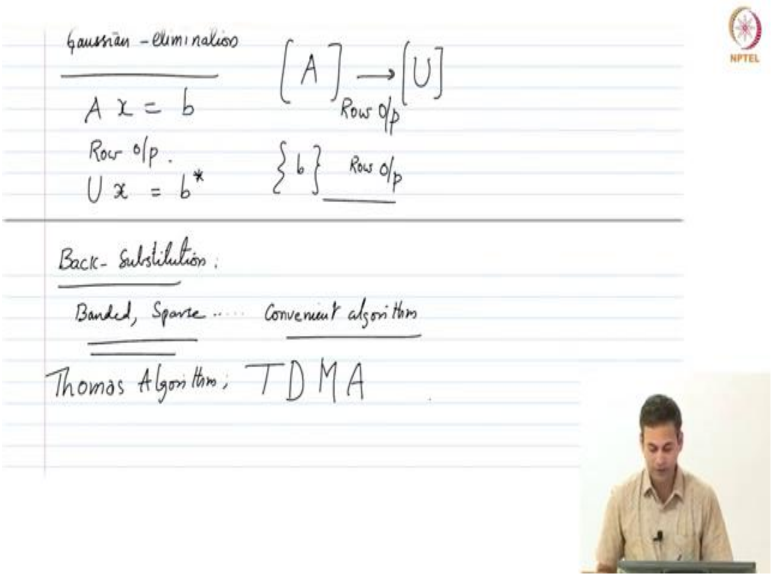

So, then you do the same operations on the b vector as well, on the right hand side as well you perform the same row operations that we have and then now we obtained some upper triangular matrix multiplying the. So, you started off with A x equals b, you did some row operations right. And then you ended up with what? You ended up with U times x also gets modified?

Student: No.

No, right because it the same, if it gets modified then again we are in trouble; x is the same equals b gets modified to some b star right; something like that. So, essentially you ended up with U x equal to b. Now why did we do this?

Student: (Refer Time: 18:03).

You can do essentially U x equal to b has what? Essentially has only one unknown per row right. So, essentially if you start off with the last row you have only one unknown right, you have some equation and you can solve for the unknown. And, then you go and come back to the last, but one equation; there you have only one unknown again because the last one was just solved and so on right.

So, and then you come back you essentially do a back substitution and then calculate all the unknowns that you have. So, this is a direct method. Now, does it sound like a direct method with the application or does it sound like an iterative method?

Student: Direct method.

It is a direct method, we are doing operations, but we are not iterating right; you are just doing one operation on each of them, you are converting the system. And, then you are getting the resulting like one go, one shot right. You perform this on all the rows ok. Now, this is the basis and we have two operations: one is forward elimination or forward substitution and the other one is back substitution ok; converting A to U and the other one is obtaining x from all these equations ok.

So, these are the two operations that we have. Now, for the equations that we get from the finite volume method ok, at least for the diffusion equation; all the equations are kind of have a nice pattern right. What was the pattern? These coefficients aligned.

Student: Diagonal.

Along the diagonal; so, there is a these are kind of banded.

Student: Yes.

And, also it is a sparse matrix right; all these things are there. So, essentially we can apply Gaussian elimination and Gaussian elimination kind of gets simplified into a very convenient algorithm ok; known as Thomas algorithm or also known as TDMA which stands for what?

Student: Tri Diagonal Matrix.

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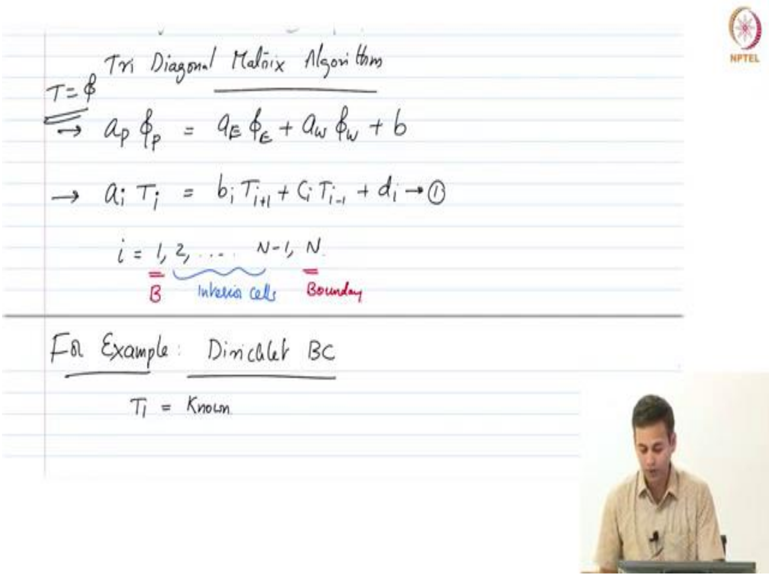

Tri Diagonal Matrix Algorithm which you know already ok, tri diagonal matrix algorithm; why is it tri diagonal? Because, on these trying on these diagonals the coefficients are non-zero; that is all ok. So now, we are going to look at tri diagonal matrix algorithm which is a direct method to solve a 1D set of equations right; essentially we have these equations.

Now, in order to kind of conveniently present these this algorithm we are going to do some modification to the set of equations we get ok. So, what was the equation set we always intend to solve? We always intend to solve or get it in terms of $a_p\phi_p = a_E\phi_E + a_w\phi_w + b$. This is the equation we usually work with in in this particular finite volume method.

Whereas, to present a direct method; I am going to kind of rewrite this equation which is one equation or one cell into slightly differently using the index notation ok; not actually index notation, using an index i ok. So, this is basically a i, let me use the term T i or essentially phi P stands for let me put back yeah. I mean so, essentially this is phi is T ok, T equals phi. We are solving for temperature or some phi. $a_iT_i = b_iT_{i+1} + c_iT_{i-1} + d_i$ these two equations are same?

Student: Same.

They are the same because east is nothing, but i plus 1, west is nothing, but i minus 1 fine. So, we have a i T i, i is nothing, but your P cell right; a i T i equals b i T i plus 1 plus c i T i minus 1 plus d i. Now, we have to also decide on what are these i notation ok; essentially what would be the boundary, what would be the interior cells? So, for that let me say that i equals goes from 1, 2 all the way to N minus 1 to N ok, where 2 to N minus 1 constitute the interior cells and 1 and N constitute the.

Student: Boundary cells.

The boundary cells ok, boundary cells I mean the boundary condition itself ok. So; that means, 2 is itself is a boundary cell because, it is sharing a bound a one face with a boundary fine. So, essentially 1 and N is where we apply the boundary condition, 2 to N minus 1 is the unknowns that we have to solve for alright. So, then then what we do is let us say if I have a Dirichlet boundary condition.

So, for example, if I have a Dirichlet boundary condition for temperature, I would say T 1 is known right. I would say T 1 is some constant. Now, if T 1 is constant what would, how can I rewrite this equation in a matrix form to; so, what would be the coefficients be if I have to say T 1 is known?

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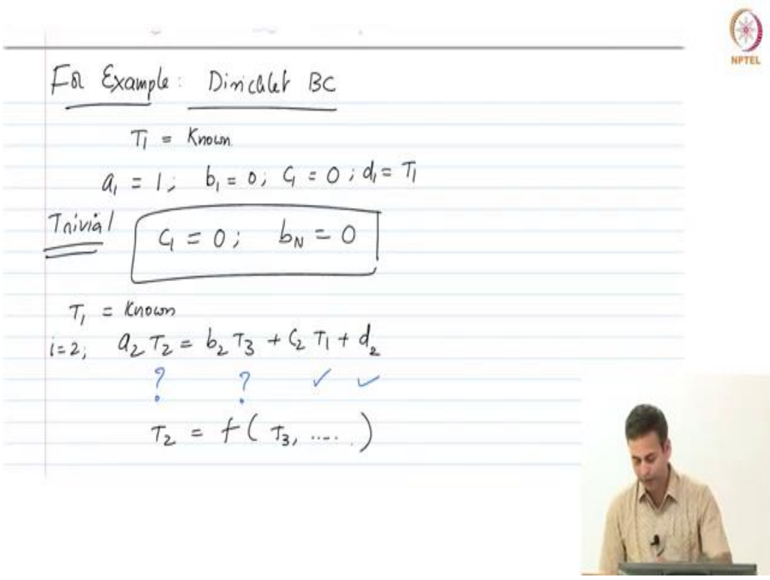

This would reduce to a 1 equals 1 right. What would be b 1? b 1 should be 0, because T 1 is already known, b 1 is 0. What would be c 1?

Student: 0.

0, because we do not have a cell where T 0 is there right. What would be d 1?

Student: (Refer Time: 23:49).

d 1 should be.

Student: T 1.

T 1, right essentially your matrix is like 1 times the unknown equals T 1. So, unknown gets equal to T 1 ok, that is kind of very trivial. So, this is trivial. So, the coefficients get modified like this if you have a Dirichlet boundary condition and it gets modified differently if we have a Neumann boundary condition ok. Now, again as we have seen that 1 and N are the boundaries itself, we do not have 0 and N plus 1 cells right.

So, we do not have. So, any variable that is referring to 0 and N plus 1 does not physically exist right. So, essentially what does that mean? That means, that I can set; so, we because we do not have T i minus 1, if I am writing an equation for i equals 1 right because this will be 0; that means, what would be the coefficient? The coefficient would be c 1, c 1 would be always 0 because it tries to connect to the T 0 which is physically not there. So, this is irrespective of whether I have a Dirichlet boundary condition or a Neumann boundary condition ok; c 1 is always 0. What is the other one that will be 0, if you go to the other end? b i, i equals?

Student: N.

N, right you are trying to write an equation for Nth cell and you do not have a connection to N plus 1, right because you do not have anything outside the boundary for that. So, this essentially b N is also equal to 0 ok. So, these are set to 0; so, that the connections to the outside the boundary are disconnected fine. These are always 0 irrespective of the boundary condition we apply ok. Now, if this is the case what will be an equation for.

So, we say T 1 equal T 1 is known. So, T 1 is known what would be an equation for i equals 2? I want you to just plug in i equals 2 in equation 1, that is all. What will that be? a 2 T 2 equals a 2 T 2 equals what on the right hand side?

Student: b 2.

b 2.

Student: T 3.

T 3 ok, b 2 T 3.

Student: Plus c 2 T 1.

Plus c 2 T 1 plus.

Student: (Refer Time: 26:05).

Some d 1 out of this, what are unknowns?

Student: T 2.

T 2 an unknown ok, what else? T 3.

Student: (Refer Time: 26:13).

T 3 is an unknown, oh this should be d 2 I am sorry thank you. This should be d 2 ok, should be with what? Ok. What are the unknowns? T 2 is an unknown.

Student: T 3.

T 3 is an unknown as well.

Student: T 1 (Refer Time: 26:31).

What about T 1?

Student: Known.

Known. So, where will we send this term?

Student: (Refer Time: 26:34).

To the right hand side, right its known. So, this is known, d 2 is known or unknown?

Student: Unknown.

d 2will have what terms?

Student: Source terms.

Source terms right.

Student: Depending upon source.

Depending upon the source, this will actually all the known known terms. This is actually the b term that we got in our original equation right. So, this is known or not known?

Student: Known.

Known, this is known of course, ok. So, all these things are known. So, I can write an equation for cell 2 as. So, essentially T 2 is now a function of other than the constants, what is it a function of the unknowns?

Student: T 3.

T 3 and plus some constants right, we have this.

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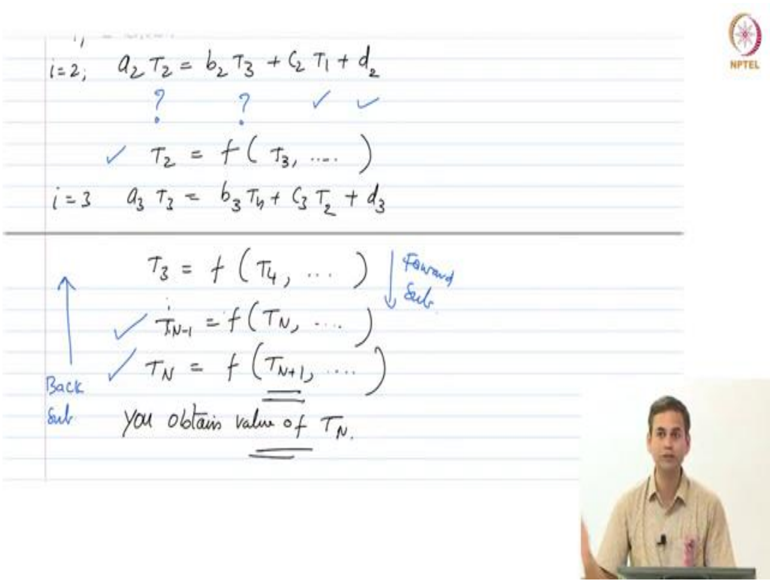

Now, can we repeat the same process for cell T 3? If you write i equals 3, you will get something here a 3 T 3 equals b 3 T 4 plus c 3 T 3 plus d 3; do you get this?

Student: T 2.

T 2 ok. So, this should be T 2, is this correct? i equals 3.

Student: Yes, sir.

Same thing ok, in this what will be T 3 a function of?

Student: T 4.

T 4 and.

Student: T 2.

T 2, T 2 is now what?

Student: Variable constant.

It is a function of T 3 again right. So, essentially that T 3 can be sent to the left hand side. So, then T 3 will be a function of what?

Student: T.

Only T 4.

Student: T 4.

Student: T 4 and constants.

T 4 and constants right, is that correct what I am saying do you do you see?

Student: Yes, sir.

For cell 2 T 2 is a function of T 3, for cell 3 T 3 is a function of T 4 right and so on; that means, we have some kind of a forward dependency for each of the cells ok. Then if I continue this what would be T N a function of?

Student: T N minus 1.

T N plus 1 and some constants, but do we have T N plus 1?

Student: No.

Because, do we have already set b N equal to.

Student: 0.

0. So, then what will be T N? You actually get the value of T N at this stage right. So, this this step gives you what? Value of T N right. So, essentially you obtain you obtain value of T N from this equation right, T N is obtained. Now, once you have T N can you use the previous equation would read T N minus 1 is a function of T N and constants right. Now, can I use the this equation and calculate T N minus 1?

Student: No.

I can. So, essentially we can start the back substitution right. We have know what we have done is a forward elimination, now I am doing a back substitution right. I would use this equation calculate T N minus 1 and use the previous equation T N minus 2 and calculate T N minus 2 from T N minus 1 and so on. And eventually calculate what?

Student: T 1 T 2.

T 2 from T 3 and T 1 is already known right, it will anyway go. So, that is the back substitution ok. In the in the previous stage we have done the forward substitution, now we have done the back substation ok. You see the essence of the algorithm, of the tri diagonal matrix algorithm or Thomas algorithm right. We are essentially doing the row operations, but in a different way right. This is nothing, but the row operations; like for every cell we have a dependency on the previous cell.

And you keep doing that when you reach the last point, because of the boundary condition you actually get a value for the unknown. Use the value of the unknown and then trace back using the same equations and you will get all the unknowns right, that is all we have done ok; very good. Now, let us actually come up with an algorithm ok. Now, we have talked about the philosophy; questions till now. Any questions?

Student: Sir.

Yeah.

Student: So, that T 1 is not at.

T 1 is not at this cell say T 1 is on the face of the cell right. So, essentially between the distance between the T 1 and T 2 is what?

Student: Delta x E by 2.

Delta x E by 2 that will again get counted in the coefficients right. Now, can you use this algorithm for finite difference? You can use, right you can use the same solution technique or finite element anything right; other questions.

Yes.

Student: (Refer Time: 30:53).

Right.

Student: (Refer Time: 30:54) how did you find out?

Ok. The question is how do you calculate the flux on the boundary faces on the east and west faces for the boundaries? We have used a 1 dimensional formula right.

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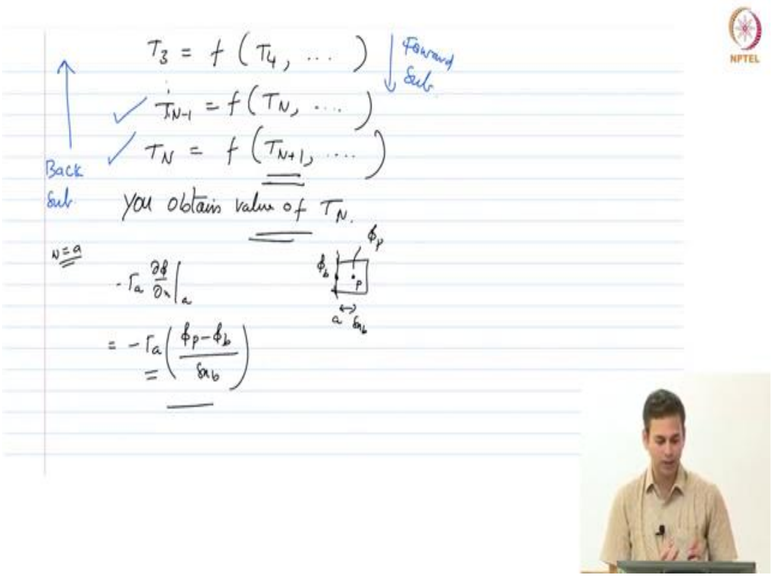

For example on the let us say gamma a, let us call the east sorry the west is a ok. This is the first cell right. How do we calculate gamma a? Let us say gamma a is basically this face right and then this is your first cell p.

Student: Yeah.

Right we would use this value phi b and the phi p and this distance delta x b and calculate it right, you would use a one sided formula right.

Student: Yeah.

That is all kind of a forward formula for the face gamma a and are not actually gamma a this is basically your flux. So, what I am talking about is, minus gamma a partial phi partial x at a right, you would use a minus gamma a times phi p minus phi b by delta x b right; this is how you would calculate. Gamma a now can be calculated. So, you need a kind of a Gauss cell or something right or because now we do not have a value, you could even use gamma a equals gamma p right.

Because, assuming that you do not have anything on the other side; alright essentially you are using a harmonic mean or arithmetic mean and you assume that there is a same material on the other side right. So, what if you use a kind of a virtual cell what will be the value of phi? That is basically same as whatever interpolation you want to use right, if you assume that you have the same material on other side then you can use gamma a as this thing.

Now, if you have that is a good question; now how do I calculate gamma on a on a boundary phase right? Let us say I have a insulated boundary condition, I say Q is 0 right; now what will be gamma a? You do not need to calculate right, because Q is 0, gamma partial T, partial x would always turn out to be.

Student: 0.

0, that is alright; now if it is not the case then you have to assume that there is a material on the other side that will be usually given, because that will form the because the gamma values are the problem only is that the gamma values have to be interpolated in the interior of the cell phases right, but not on the boundaries; because that will be part of a boundary condition. So, that will usually be given ok.

We will come back to that if there are other questions on that; more questions, no clear? The philosophy is clear? Why we are going through the forward substitution and the back substitution right. Sorry, yes.

Student: (Refer Time: 33:44) conservation.

Ok.

Student: Ok, consider one cell.

Ok.

Student: (Refer Time: 33:50).

Yeah.

Student: (Refer Time: 33:53) boundary page and right side, how to find out the (Refer Time: 33:56); how to find out the source term?

The question is how to find out the source term? How to satisfy conservation for the first cell? Now, the source is unknown or known?

Student: (Refer Time: 34:04).

Source is known right.

Student: (Refer Time: 34:07).

So, essentially you want me to solve the assignment here. So.

Student: (Refer Time: 34:10).

The ok. So, as far as the assignment is concerned; how do you calculate? So, where do you write the source for? For which point?

Student: Cell centroid.

For the cell centroid that is all. So, when you write an equation for the first cell, you would use the x p or whatever has what?

Student: (Refer Time: 34:27).

Delta x E by 2, that is alright; other questions, no fine.

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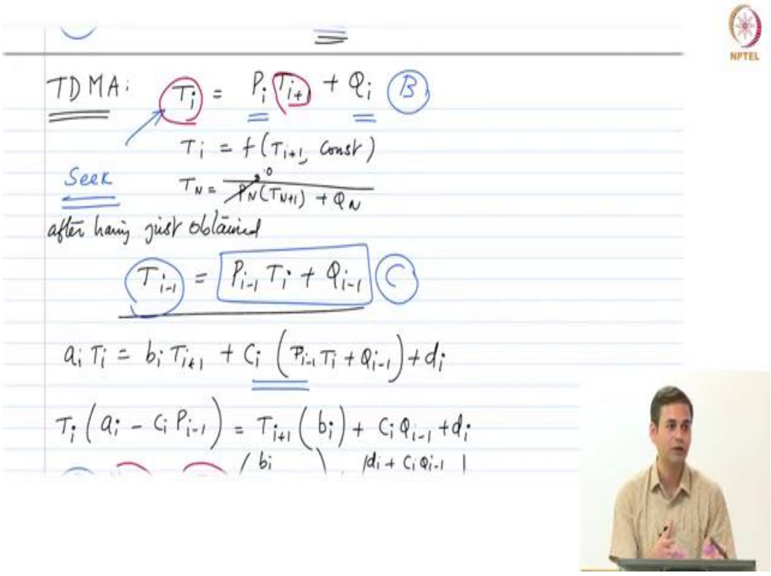


Then let us move on with the algorithm. So, we are looking at a TDMA ok. So, what is the term that we are trying to trying to do? We are trying to do for every T i or phi i, for every T i we are trying we are kind of seeking a relation in terms of T i plus 1 right; that is what we are seeking. So, essentially; that means, we are seeking if I have T i, I am seeking a relation for $T_i = P_i T_{i+1} + Q_i$, where P i and Q i are some constants that depend on the particular cell right. This is basically the functional dependency.

We have just written that T i is a function of T i plus 1 comma some constants right, this is what we have written till now. We said T 2 is a function of T 3, T 3 is a function of T 4 and so on. So; that means, this is what we are seeking right. So, we seek a relation that looks like this T i would be some factor P i times T i plus 1 plus Q i right, that is what we want. And, then after just after having just obtained T i minus 1 ok.

What would be T i minus 1? So, the previous one would be d i minus 1 that would be how much? $T_{i-1} = P_{i-1} T_i + Q_{i-1}$ is that correct? Actually P and Q are now tied to the cell right and right hand side the unknown should be one cell forward to the previous cell right; essentially T i minus 1 is P i times P i minus 1 times T i plus Q i minus 1 ok.

So, let me write down the original equation, original equation was what? We had $a_i T_i = b_i T_{i+1} + c_i (T_{i-1} T_i + Q_{i-1}) + d_i$ a i T i equals b i T i plus 1 plus c i T i minus 1 plus d I right, this is equation we have. Let us call this as equation A ok, let us call this as this as equation B, let us call this as equation C ok; a, b, c, d are they known?

Student: Yes.

They are known from the discretization right, they have these a east, a west, a north whatever right all those things. But, there is one catch here; what would be the difference between b i? Well, in this case there is no catch; is not it? because we always wrote a east phi east equals a you know a phi times east and west. So, there is no problem here ok, I will come back to that question later fine.

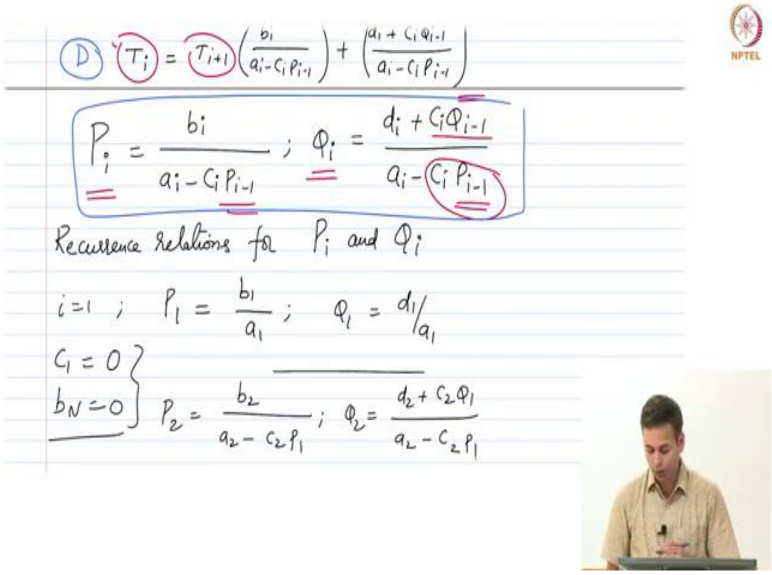
So, now what I want to do is I want to kind of substitute for T i minus 1 ok, for this value into the equation for A right. I want to substitute T i minus 1 into this equation fine, can I do that? Ok. So, I would substitute a i T i equals b i T i plus 1 plus c i times; what is T i minus 1? k minus 1 is this entire expression P i minus 1 T i plus Q i minus 1.

So, I plug in here, this would be c i times P i minus 1 T i plus Q i minus 1 plus d i. Essentially we substituted equation C into equation A for T i minus 1 right, that is all we did. Now, why did we do that? Because, this final equation we got is a function of what dependent quantities T i and?

Student: (Refer Time: 38:11) plus 1.

T i and T i plus 1 only, right ok.

(Refer Slide Time: 38:20)



So; that means, if I go forward, if I simplify this thing; what will be the coefficients for T i? a i and.

Student: c i P i.

c i P i minus 1 minus c i P i minus 1, is that correct?

Student: Yes, sir.

Equals what will be the coefficients for T i plus 1?

Student: (Refer Time: 38:35).

b i that is all.

Student: Yes, sir.

And, then what else the constant.

Student: c i times (Refer Time: 38:42).

c i times Q i minus 1.

Student: Plus d i.

Plus d i ok. So, this is the constant ok, then can I write it as T i equals T i plus 1 times b i upon a i minus c i P i minus 1 plus d i plus c i Q i minus 1 upon a i minus c i P i minus 1, is that fine? I just divided the coefficient of T i for all the terms on the right hand side fine. Now, can we compare this equation which is let us call it as D with B? Can we compare D with B and obtain the coefficients? Right.

Now, on the left hand side you have T i here, here you have T i, this is T i plus 1, this is T i plus 1. So, this coefficient P i should be equal to this coefficient and the Q i should be equal to the constant here ok. So, if I compare; can you help me write what is P i? P i is b i upon a i minus c i P i minus 1 and then what would be Q i? d i plus c i Q i minus 1 upon a i minus c i P i minus 1.

Now, these are the recurrence relations for P i and Q i because, P i Q i now depend on what? P i minus 1 and Q i minus 1 ok. So, because it is kind of a recurring relation, only if you know Q i minus 1 P i minus 1 we can obtain P i Q i and so on right. So, this is nothing, but the forward elimination or forward substitution, we are calculating what is P i Q i from P i minus 1 and Q i minus 1 ok. So, that is the forward substitution that we are doing ok.

Because P i is now depends on P i minus 1 and Q i depends on Q i minus 1 and P i minus 1 ok. So, these are the recurrence relations fine. So, let us now, but we have to start off with somewhere. So, where do you start the forward substitution with what cell? First cell ok, let us go 1. So, if you say i equals 1; what would be P 1? So, what is the conditions we had on the B and C? We had two conditions? What was those?

Student: (Refer Time: 41:26).

c 1 is 0 and b N we have set it to 0, this is to get to kind of disconnect the T 0 and T N plus 1 ok, but this has nothing do with the boundary conditions ok. So now, what will be P i; if you plug in i equals 1 is there will there be some b 1?

Student: b 1 (Refer Time: 41:49).

b 1.

Student: (Refer Time: 41:50).

Upon a 1 because c 1 is 0; so, this will be b 1 by a 1. What will be Q 1? c 1 is 0; so, essentially this term is 0, this term is 0 d 1 by.

Student: a 1.

a 1 d 1 by a 1 right. So, we can start off with P 1 and Q 1, P Q are the new unknowns we have introduced. We said we have to seek a relation that looks like T i relates to T i plus 1 right, that is how we introduced P and Q. But, where do we get these values P and Q? They have to come from the discretization which is nothing but the a east, a west and so on which in this particular notation are nothing, but a b c d right.

So, we have obtained P 1 Q 1. Now, can we use these recurrence relations that we have to obtain P 2 Q 2, once we have P 1 Q 1? Ok. Let us see what will be P 2 from the equation? $P_2 = \frac{b_2}{a_2 - c_2 P_1}$. Now, is b 2 a 2 c 2 known? Yes, they are known as soon as you finish the discretization, P 1 is just now calculated from b 1 a 1.

What will be Q 2? $Q_2 = d_2 + \frac{c_2 Q_1}{a_2 - c_2 P_1}$. Now, is d 2 a 2 c 2 known? Known right for the cell 2. What about Q 1 and P 1?

Student: Known.

Known, we have just calculated them. So, P 2 Q 2 can be calculated right and then we keep calculating P 3 Q 3 and so on to where? P N Q N right.

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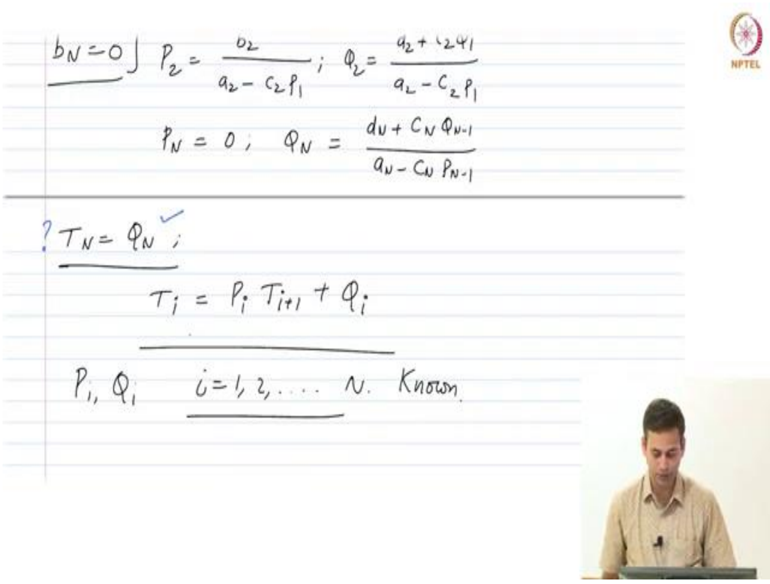

P N Q N, what is P N?

Student: 0.

0, P N is 0. Why is it 0?

Student: (Refer Time: 43:34).

Because b N is 0, P N is 0. Can you calculate Q N? $Q_N = d_N + \frac{C_N Q_{N-1}}{a_N - C_N P_{N-1}}$ It will be

Student: Q N minus.

; it can be calculated right this is essentially d N plus C N Q N minus 1 upon a n minus C N P N minus 1 ok. So, this can be calculated, P N is 0 fine.

Now, if my P N is 0 and Q N is some number which we calculate, some value; then if we go back to the equation, the original equation ok. If we go back to the equation B which is nothing, but; so, if I plug in in this equation i equals N what will this be? T N equals.

Student: (Refer Time: 44:26).

P N times T N plus 1 of course, T N plus 1 does not have any meaning plus Q N. What is P N anyway?

Student: 0.

We got this as 0, we got this as 0 because we have set b N equal to 0 to begin with, we have disconnected the connection like we have kind of disconnected these outer cells, is not it? So, that is P N is 0. What about Q N? Q N can we do we have a value for Q N? We just calculated Q N, then what is T N?

Student: (Refer Time: 44:54).

T N equals Q N. So, you can calculate Q N and you get the value for T N. So, this is your first calculation right of the back substitution process. So, T N is now b N ok. So, what we have is you calculate T N equals Q N, you said this thing. Q N is known, calculate T N from here; then we go back to our original equation which was a $T_i = P_i T_{i+1} + Q_i$ this is our original equation. Now, at this moment do we know P i Q i for 1 to N, all the values?

Student: Yes.

We know right. So, essentially P i Q i for i equals 1 to all the way to N are all known right. Now, what is that that we calculate? We just calculate what is T N right, now can I calculate T N minus 1 from this equation?

Student: (Refer Time: 45:51).

I just plug in i equal to N minus 1, calculate T N minus 1. Next I can calculate T N minus 2 and so on calculate T 2 right, that is your back substitution process.

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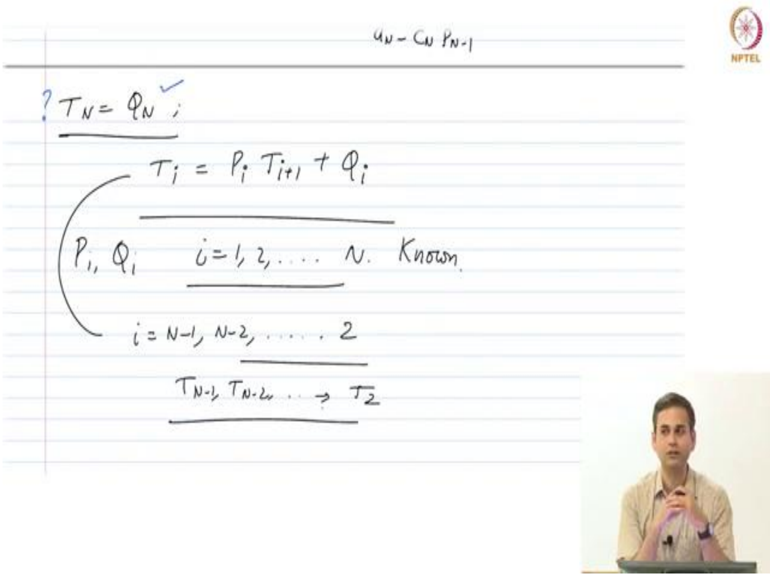

So, use this relation for $i = N - 1, N - 2 \ldots \ldots .2$ all the way to 2 and calculate $T_{N-1}, T_{N-2} \ldots . . T_2$ that is the Thomas algorithm, that is all. We started off with forward substitution that is nothing, but calculation of the new coefficients P and Q and we have used the same recurrence relation.

Once you find the final temperature, you come back and calculate in a back substitution calculate T N minus 1 to T 2. Now, this works both for Dirichlet or Neumann ok. We have not discussed that only as a special case I discuss, but it works for both. Have you heard of a LU decomposition?

Student: Yes.

Also yeah, how is it different from Gaussian elimination?

Student: Yes.

Product of weight is ok, is it different from Gaussian elimination?

Student: It is not different.

It is not different.

Student: It is just storing.

It is just storing, it essentially LU is the same as Gaussian elimination right; instead we are just storing it as two different matrices. Why do we do that, is there an advantage of storing that?

Student: Yes.

Multiples. So, essentially if you if your right hand side changes, you can still use the same system and then solve it right. The coefficients are the same, right hand side is different; that means, the same problem with different boundary conditions you can still use the same coefficients right and then calculate. So, that is why we are storing it; now is this different from LU or Gaussian elimination? It is not.

What we have done basically is a kind of LU decomposition for a tri diagonal matrix right, that is Gaussian elimination applied to tri diagonal matrix; that is all. Do you see that? Ok. Now, this is a direct method which you do not have to worry because, you will all code this in one of your assignments, in the upcoming assignment right. And, then you can see for yourself that this direct method would be much faster than the iterative method that is the Gauss-Seidel ok.

Now, can we extend this to 2 dimensions, this process? We can right, it just little more complicated right; instead of tri diagonal you will get maybe.

Student: Pentadiagonal.

Pentadiagonal. So, then we can do that which we will not do; rather what we do is we kind of look at a different method which is basically combines say the Gaussian elimination with the direct solver that is tri diagonal matrix algorithm; such that you do not have to write a much complicated 2D solver as such ok. We will kind of use this in the coming lectures fine.