Computational Fluid Dynamics Using Finite Volume Method

Prof. Kameswararao Anupindi
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture - 12
Finite Volume Method for Diffusion Equation: Boundary conditions for 2D diffusion equation

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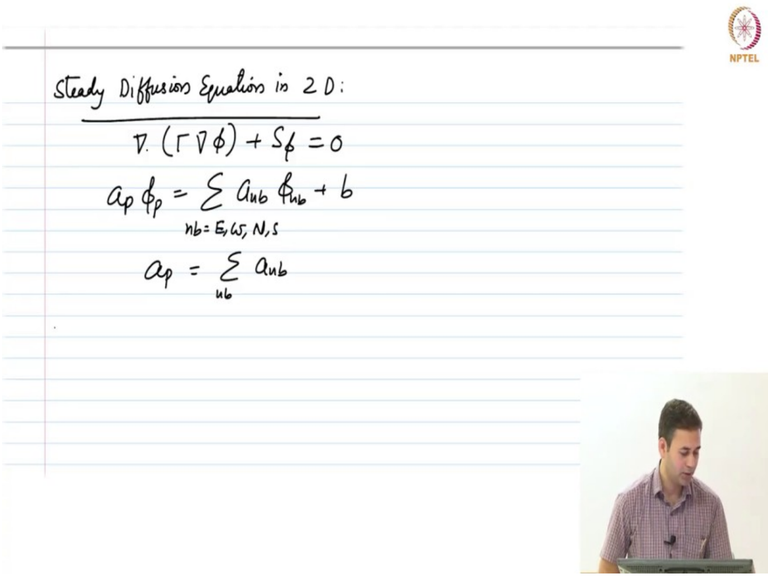

Good morning. Let us get started. So, we looked at the Diffusion equation. We looked at the steady diffusion equation in 2-dimensions yesterday, this was $\nabla.(\Gamma\nabla\phi)+S_\phi=0$ del. We have obtained an equation that is basically $a_p\phi_p=\Sigma a_{nb}\phi_{nb}+b$; where, nb is your east, west, north and south and then, the central coefficient $a_p=\Sigma a_{nb}$ that is what we have written down yesterday.

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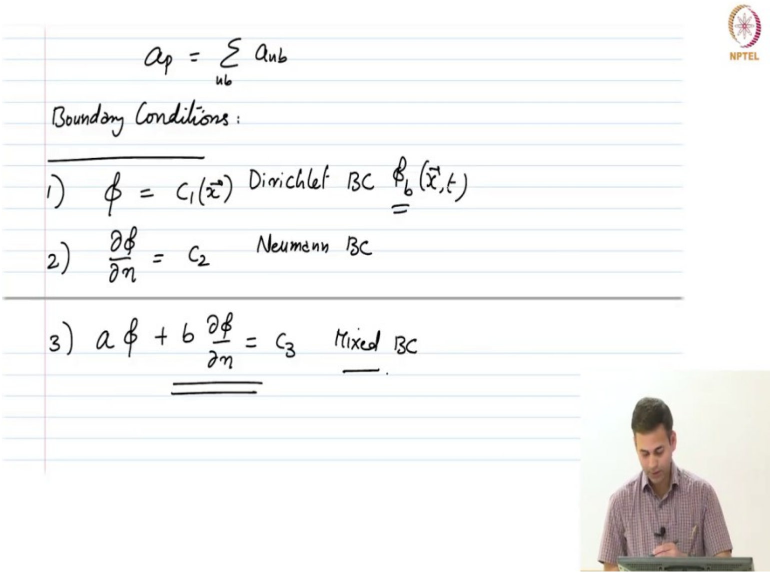

Now, we have to discuss the Boundary Conditions in today's lecture. So, coming to the boundary conditions, we can have three types of boundary conditions. You could specify the value of the dependent variable itself. So, that is you can specify phi equals some constant.

This type of boundary condition is known as what? A Dirichlet Boundary Condition. You can also specify the gradient of the dependent function ok, dependent variable. So, that is $\frac{\partial \phi}{\partial n} = c_2$ So, this would be this is known as a Neumann Boundary Condition ok.

And, can you have any other type of boundary condition? You can have a mixed boundary condition; that means, you can specify a combination of Dirichlet and Neumann that could be for example, $a_\phi + b\frac{\partial \phi}{\partial n} = c_3$. So, this is called a mixed boundary condition. So, let us take a look at each of these boundary conditions and how will that changes our discretization in today's lecture ok.

Student: (Refer Time: 02:20).

Yeah?

Student: (Refer Time: 02:22).

Sure. So, C 1 can be a function of space right. In fact, C 1 can be a function of space or it could be a function of temperature and things like that ok. Essentially, now temperature; C 1 can be a function of let us say of the space x bar because temperature is again here. Now, again for example, here this could be variable as well right, that is true. So, I have just written it as some constant; but that is need not be a just a constant ok.

We will come to that. We will not incorporate a constant there as such ok. But what we call it as phi equals some phi on the boundary will be specified. How phi b varies with? Let us say with space or with time, would be specified by the user right or by you, who is kind of simplifying the problem ok. Other questions? No, ok.

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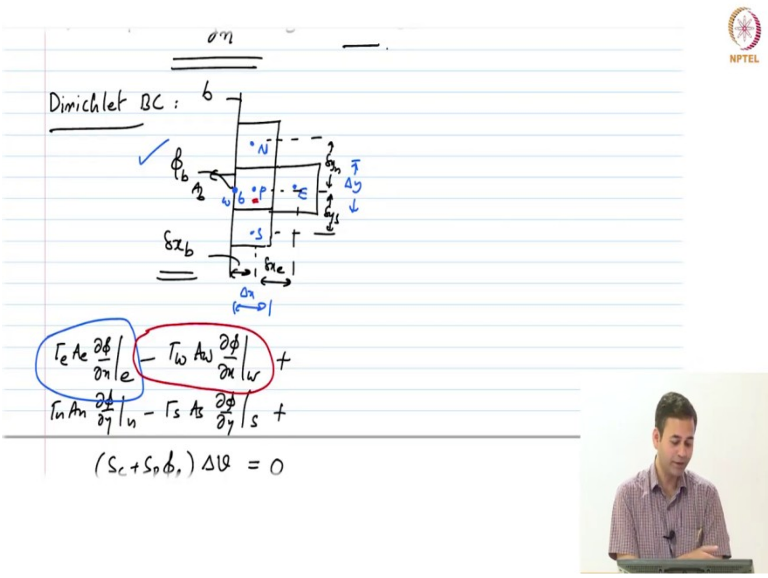

So, let us look at Dirichlet boundary condition. So, essentially, we have we have the domain, I am taking let us say a left side of the domain as my boundary. So, this is my boundary b and we have this is the P cell, this is east cell, this is north cell and this is the south cell and we also have the boundary that is here, this is let us say b and we specify what is the value this is phi b.

So, phi b is what we are specifying, it could be again a function of time and it could be different on different boundaries. I guess is that ok. Now, what about the distances? So, the distances are; so, this this P cell has a distance of let us say delta y in here and delta x in here. And the distance between the cell centroids is delta y north and delta y south and this distance is now del x e between cell centroid of the P cell and the east cell.

And of course, up till now we had this del delta x by 2 as our distance between the P cell and the boundary cell right, in the 1D cases. Here, I am going to call this distance as simply as delta x b ok. So, I am going to just call it as delta x b now we realize that delta x b is actually delta x e upon could be essentially delta x by 2 right. So, remember that two factors would not come in now because we have now removed it, we just call it as delta x b fine ok, that is what we have.

Now, of course, we will not go through all the steps rather we will write down the discrete equation. What is our discrete equation? That was gamma east; anybody remembers? Gamma east, let us say A east that was delta y right; $\Gamma_e A_e \frac{\partial \phi}{\partial x}|_e - \Gamma_w A_w \frac{\partial \phi}{\partial x}|_w + \Gamma_n A_n \frac{\partial \phi}{\partial y}|_n - \Gamma_s A_s \frac{\partial \phi}{\partial y}|_s + (S_c S_p \phi_p) \Delta v = 0$

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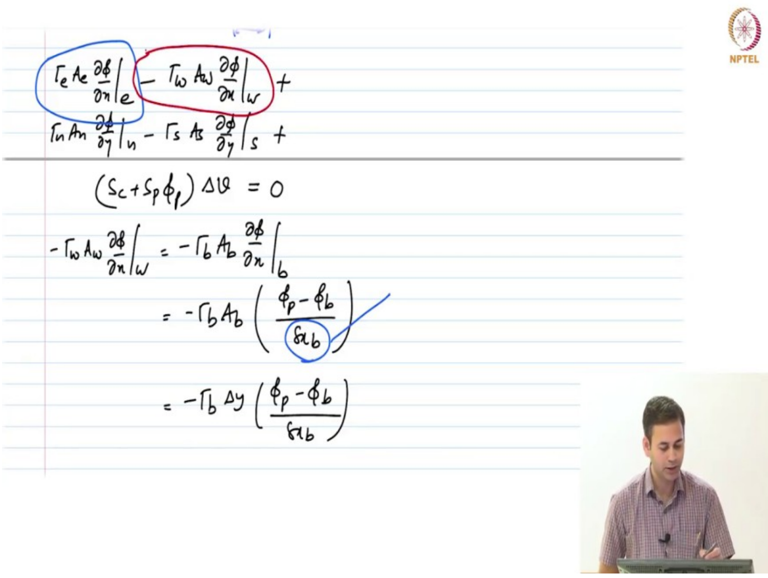

Now, we realize that the west face for the P cell is the same as what? Same as b right, the boundary cell, the boundary face. So, I can write the $-\Gamma_w A_w \frac{\partial \phi}{\partial x}|_w = -\Gamma_b A_b \frac{\partial \phi}{\partial x}|_b$. Now, how do you how do I evaluate the flux on the b face, what I got here? What would this be? $-\Gamma_b A_b \left(\frac{\phi_p - \phi_b}{\delta x b} \right)$ what is partial phi partial x on b face, what are the values I have? I have only phi available at p and phi defined at b right. But phi b, is it known or unknown?

Student: (Refer Time: 06:55) known.

This is known right because it is given to you this is known phi p is of course unknown ok; phi p is an unknown. So, if I were to calculate this, then I would write this as So, this is where you have to remember that the two factor is not there.

Earlier, we had this as delta x e by 2 or something, where you would got a another 2 there ok. Now, that is absorbed into delta x b fine. The distance between the near boundary cell and the boundary face centroid ok, fine ok. We do not have to we would not go into the details here rather what would be now a b? What is the value of a b? Minus gamma b; no ok, we do not have. So, what is magnitude of a b on this face?

Student: (Refer Time: 07:48).

Delta y ok. So, this would be $-\Gamma_b \Delta y = \left(\frac{\phi_p - \phi_b}{\delta xb}\right)$ that is what we have. Now, of course, you know how to write expressions for these terms right. These terms do you know, what is A e? A e is delta y partial phi partial x is nothing but phi east minus phi p by delta x e and so on. So, we know all of those; only for this term, we have now just derived. How to do it?

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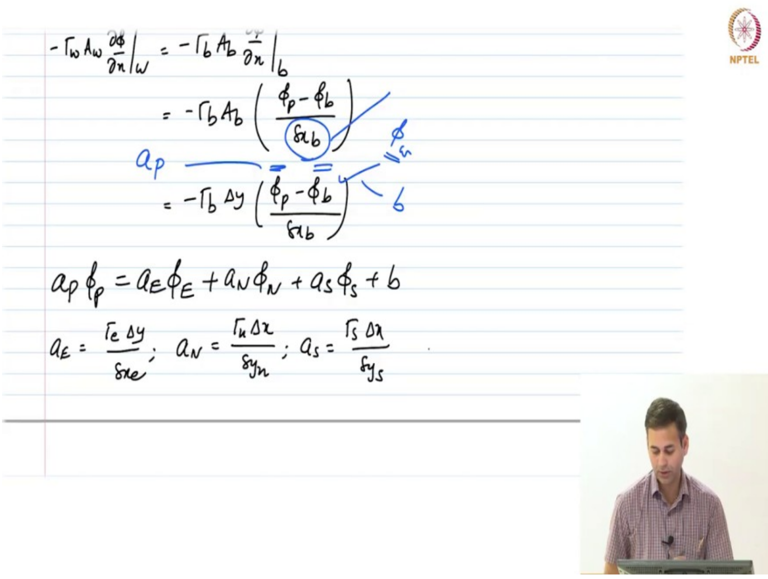

Now, where will the contribution from these terms go? So, the contribution is gamma b delta y by delta x b that will go to one term will go to phi p coefficient, the other term will go to b because phi b is known. So, this is known. So, entire term will go to b and there will be contribution to a p right. Do you see that? Ok. Now, then let me write down the final equation, I will write I will not write down each of the equations. So, essentially, we have $a_p \phi_p = a_E \phi_E + a_N \phi_N + a_s \phi_s + b$ Student: No.

No right. Because we do not have a west cell right. So, this would be there will be a There is no west because we do not have we do not have a coefficient for phi east right. Because phi east does not sorry, we do not have a coefficient for phi west. As a result, we do not have any coefficient for aw right ok. Now, this is what we get alright. Then, what is a east? If you go back in your notes, what you get is a east is gamma east delta y by?

Student: Del x.

Del x e. What would be a north?

Student: (Refer Time: 09:43).

$a_N = \frac{\Gamma_n \Delta x}{\delta y_n}$What would be a south?

Student: (Refer Time: 09:49).

$a_S = \frac{\Gamma_s \Delta x}{\delta y_s}$. Now, what about ah; so, we call this another coefficient as a b right.

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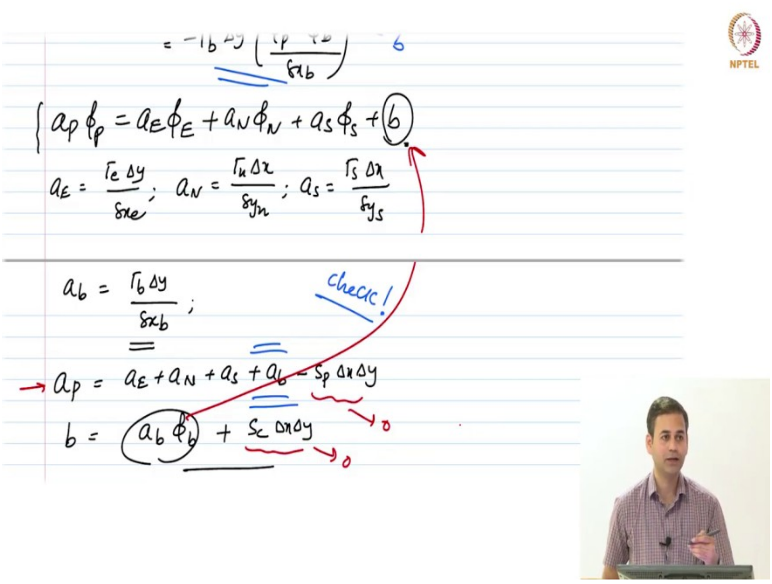

a b is how much? Gamma b delta y by del x b right. So, that two factor is built in here ok. If you follow, follow me; then, what about ap? ap will have contributions from where? From east, north, south and from the from the boundary as well right and from the source also ok. So, what would be ap then? $a_p = a_E + a_N + a_s + a_b - S_p \Delta x \Delta y$; do you see that? Ok. I have not written down.

I hope you kind of follow me on this; essentially, we have written down the linear profile assumption for all the fluxes; partial phi, partial x, partial phi, partial y on all the four faces. In this particular case, it is only the three faces; east, north, south; on the west face, we have already derived. Now, plug all of those back in and then, rearrange the terms to get a p phi p equals sigma nb phi nb plus b and then, we are writing down the equations, the coefficients right? Alright. Then, what about b? What all terms will going to b? Phi b term will go, that will be a b times?

Student: (Refer Time: 11:24).

Minus? Would there be a minus? This should be a plus, is not it? So, phi p has a coefficient with the minus; but phi p is always sent to the right hand side, so that is why it becomes plus. What about the coefficient for phi b? It will be plus right, you have minus times minus. So, it will be plus and it will stay on the left hand side right. So, what will be b? a b phi b plus or minus?

Student: Plus (Refer Time: 11:46).

Plus right; $b = a_p \phi_b + \Delta x \Delta y$

Student: (Refer Time: 11:52).

So, you have to kind of check these expressions ok. I have write down you have to make sure this is correct. Fine is it ok? Ok. Now, let us talk about Scarborough criteria. What is Scarborough criteria saying here? Is my ap equal to sigma a nb or is it greater?

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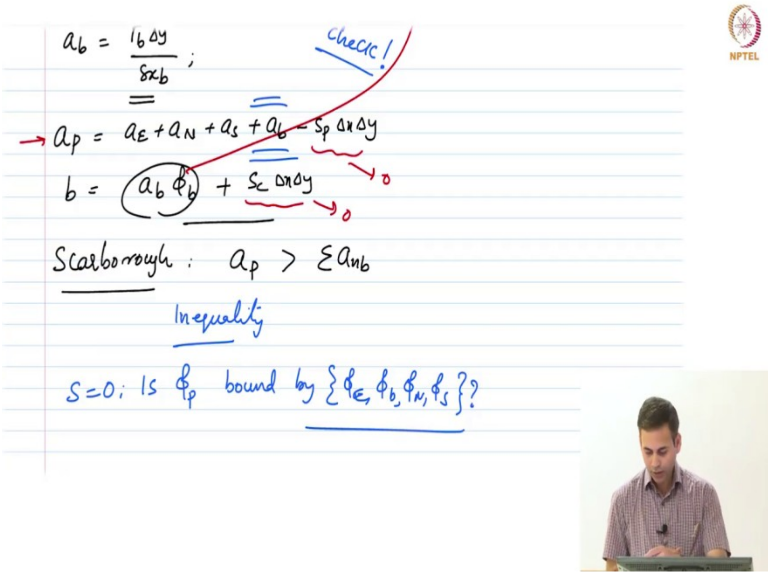

Student: Greater.

So, what are the neighbors here, for ap, for p?

Student: (Refer Time: 12:51).

East, north, south. Is b a neighbor?

Student: No.

No, we do not call it a neighbor. Why? Because phi b ok, it is a face, but still I have this I still have a b phi b built in I still have this let us say a b coming in here. Is not it? Then, why do I why do I not call it as a neighbor? Because phi b is known or unknown?

Student: Known.

Phi b is known. So, if you write down the matrix in the in the particular row, would you have an entry for phi b? You will not because phi b on the right hand side. So, essentially you do not have right. So, the neighbors are nothing but everything that comes up on that particular row of the matrix right and only the unknowns that come up are the east, north and south that is all; only these three.

So, now when I say nb, what am I saying? I am only saying that it is east, north, south. We are not talking about b here. Now, is it now greater or equal or what? Let us say we do not have source terms ok. Let us say source is 0; this is 0, this is 0. But we still have a Dirichlet boundary condition in that case is $a_p > \Sigma a_{nb}$a p equals sigma a nb or is it lesser or greater?

Student: Greater.

Greater right. Why it is greater by what amount?

Student: By ab (Refer Time: 14:00).

By a b ok. So, it is greater by a b value ok. So, it satisfies now Scarborough criteria in inequality ok, satisfies in inequality. We do not worry about satisfaction in equality because all the integral cells anyway satisfy in inequality fine. Let us say we do not have source term, then is the value of phi p bound by the neighbors? So, the question is let us say source is 0; $S=0$e is $\{\phi_E, \phi_b, \phi_N, \phi_s\}$phi p bound by I would call it as neighbors in a different context ok. Is it bound by I would call   is this true or not?

For example, you go back here. We have to check two things; one is ap, a sum of all the coefficients. Next one is this coming out to be a fractions right. So, ap, is it a sum of all the neighbors? Neighbors in a sense, I am considering the face value as well right. So, that is aba east a north a south and then, what will happen to phi p? Phi p would be there is a b phi b coming from b right. So, essentially, this will be a it will be bounded by the neighboring values right. It will be bounded by the three neighboring cells and the face value that we specify.

Do you see that? Yeah, because b contains now a b phi b, if you are going to plug this back in here, then what you are going to get is phi p equals sigma a nb by ap, which are all sum up to 1 because of this fact right ok. So, we can conclude that phi p the near boundary cell value is bounded by a 3 face neighbors and the face value that we have specified fine. Questions still here, I have not I have just assumed it to be 0. It need not be 0. In in case, if it is not 0; then, is it does it satisfy Scarborough?

Student: Yes sir.

It does ok. If source is not 0, will it be bounded by the cell values and the neighbors? It will not be right? It can be greater than or less than, it can be anything depends on the source or the sink ok. Other questions? No, ok. Let us move on to Neumann boundary condition.

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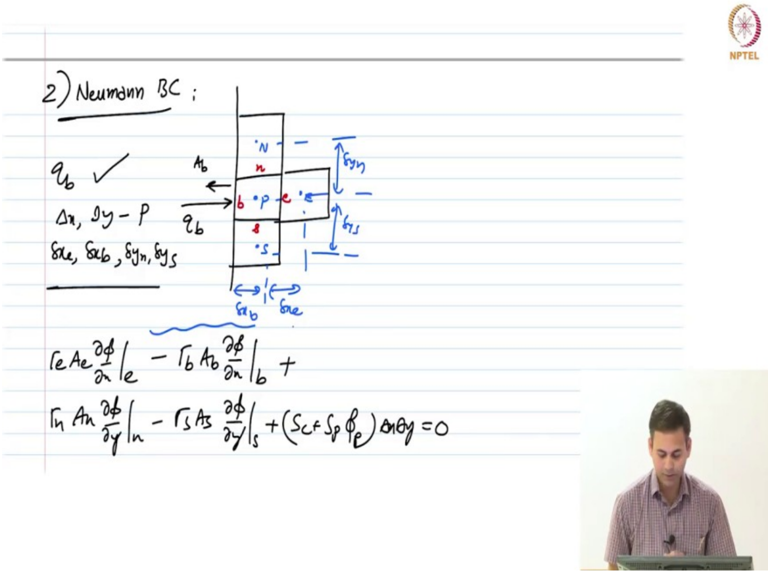

So, in a Neumann boundary condition, what do we specify? We specify the gradient of the dependent variable; so, that means, its specifying. So, my setting remains the same P, east or north, west and then, we specify what is. So, this is a b ok, the faces are east a boundary, north face and south face and we are applying a heat flux boundary condition that is q b ok. So, q b is now what? Is known or unknown?

Student: Known.

Known right. You specify Neumann boundary condition; you know what is q b ok. The remaining delta x, delta y for the P cell and the distance is$\delta_{xe}, \delta_{xb}, \delta_{yn}, \delta_{ys}$ remain the same as before ok. We know where they are. Delta x b, delta x e and these are all the delta y north and delta y south alright.

Now, let us write down the equation again ok. So, we know the discrete equation that is $\Gamma_e A_e \frac{\partial \phi}{\partial x} |_e - \Gamma_b A_b \frac{\partial \phi}{\partial x} |_b + \Gamma_n A_n \frac{\partial \phi}{\partial y} |_n - \Gamma_s A_s \frac{\partial \phi}{\partial y} |_s + (S_c S_p \phi_p) \Delta x \Delta y = 0$Now, let us concentrate on the second term that is the minus gamma b A b partial phi partial x b ok.

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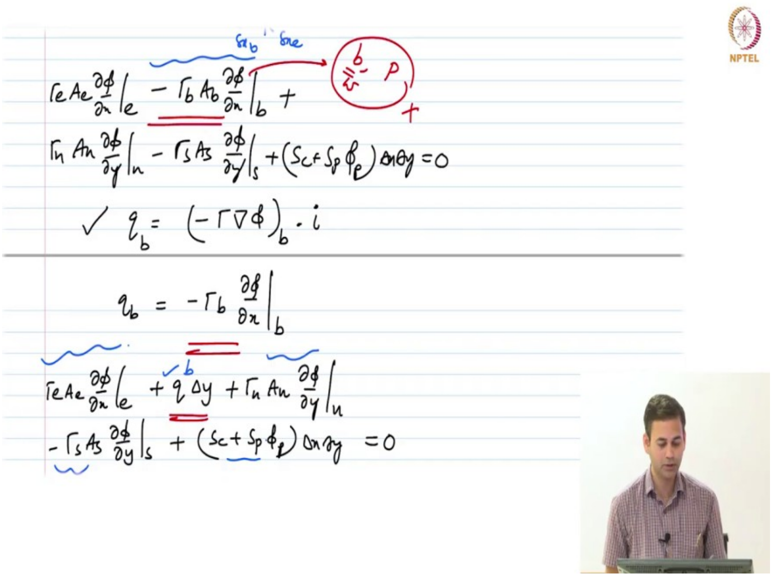

So, what is q? q is nothing but minus gamma grad phi evaluated at b dotted with i alright. Thatis my q that I have specified, let us call it as $q_b = (-\Gamma \nabla \phi)_b . i$that is my heat flux vector, that we have specified. What will this amount to? This will be how much? This will be

$q_b = -\Gamma_b \frac{\partial \phi}{\partial x} |_n$minus gamma b. What is grad phi b dot i?

Student: (Refer Time: 19:10).

. Then, can I substitute for this term for this term from this known value? I can right. So, what would this equation look like if I substitute this? It will be the same as before

$$\Gamma_e A_e \frac{\partial \phi}{\partial x} |_e + q \Delta y - \Gamma_n A_n \frac{\partial \phi}{\partial y} |_n - \Gamma_s A_s \frac{\partial \phi}{\partial y} |_s + (S_c S_p \phi_p) \Delta x \Delta y = 0$$

So, what remains is a b and what is the value of a b?

Student: Delta y.

. So, far so good now because we have specified a heat flux boundary condition, the flux term that is partial phi partial x b is now completely wiped out right. It just got replaced with q b. Now, as a result, what happened?

Earlier, we used to have two contributions coming from here right; either one going to the b, one going to p or if this was not a boundary, it would have gone into w and p right. Now, all these things are gone right because we just replaced it with q, which is known right. As a result, now can you write a discrete equation for this? Now, you can write in discrete equation. Now, where will q times del dot y will go, delta y will go?

Student: b.

b right. Because this is known. This will go into b term; this will go into b term. Now, what will contribute to the coefficient a p? What coefficients will contribute? A east, will it contribute? It will contribute, of course yeah, because it will come from here, A north.

Student: Yes.

Yes, A south.

Student: (Refer Time: 21:13).

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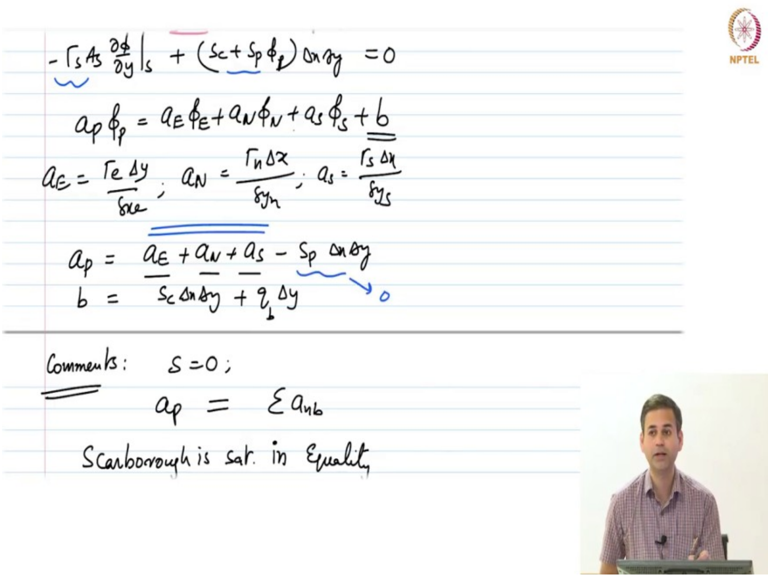

And then, the source term that is all ok. Now, let us write down that equation. Let us say we plug in to all the algebra, we will write $a_p \phi_p = a_E \phi_E + a_N \phi_N + a_s \phi_s + b$. Of course, what is a east? $a_E = \frac{\Gamma_e \Delta y}{\delta_{xe}}$; $a_{N = ¿ \frac{\Gamma_n \Delta x}{\delta_{yn}} ¿}$;¿ $a_S = \frac{\Gamma_s \Delta x}{\delta ys}$. Now, the important point is what is a ap? ap is what? $a_p = a_E + a_n + a_s - S_p \Delta x \Delta y$

Student: Plus a north.

Plus a north.

Student: (Refer Time: 21:54).

A south.

Student: (Refer Time: 21:57).

Minus Sp.

Student: (Refer Time: 21:59).

Delta x delta y. Anything else? Nothing ok. What about b? $b = S_c \Delta x \Delta y + q \Delta y$

.

Student: (Refer Time: 22:07).

Minus or plus?

Student: Plus (Refer Time: 22:12) plus.

Plus because it stays on the left hand side itself right, only the p is gone to the right hand side. Of course, we are writing it in the reverse way right. So, this will be plus or minus?

Student: Plus.

Plus q times what?

Student: Delta y.

Delta y that is your b ok. So, let us call q as q b because we say this is a boundary heat flux that is specified ok. So, this is q b; a p is known that is good. Questions till now, right if you do not follow, all you have to do is go back and put in the flux terms and then, just derive it ok. If you do it once for Dirichlet, you will get it for everything else; it is very straightforward ok. Now, let us say few comments, let us say we have a source is 0; source is 0, then what happens to ap? Is ap equals or greater than sigma a and b?

students: Equals (Refer Time: 23:10).

Equals right. If source is 0; it is only a sum of a east a west and sorry a east a north and a south; only these three terms will come fine. So, it is only satisfies Scarborough in what?

Student: Equality.

Equality. So, Scarborough is satisfied in equality ok; that means, if you have all Neumann boundary conditions, will you ever satisfy Scarborough in inequality? You will not ok. We have discussed this thing.

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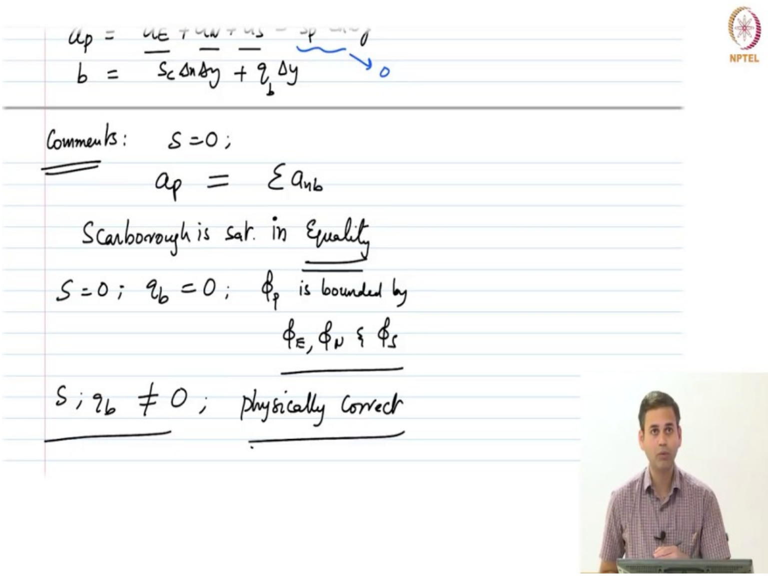

So, then, let us say if you have $S=0$and $q_b=0$; that means, you have an insulated boundary condition and you have no source, then is a phi p bounded by the neighbors?

Student: (Refer Time: 24:05).

It is right because b is 0. So, as a result, a p is sum of a east north south and this is 0. If we say source is 0, then phi p is bounded by the neighbors; east north and south right ok. ϕ_p is bounded byϕ_E, ϕ_N, ϕ_s . In case if source are $q_b \neq 0$, then will it be bounded?

It will not be bounded and it is it is possible right. If you have a source term, your phi p can be higher than the neighbors and similarly, if you have a heat flux boundary condition, you are adding heat on one face the temperature for the near boundary cell can be higher than the other cells ok, that is physically correct.

Now, what we are interested in is we have now obtained an expression, we will put in this equation into the Gauss-Seidel or any other solution technique that we would use; but we have not actually calculated what is the value of phi on the boundary right. Because somebody has told you that this is the heat flux I am applying; but in most cases, you might be interested in calculating what is the temperature or what is the phi on the boundary itself right.

So, we should be able to calculate that from the heat flux boundary conditions right. If it was Dirichlet, we know that value; if it was Neumann, we should be able to calculate from the numerical technique.

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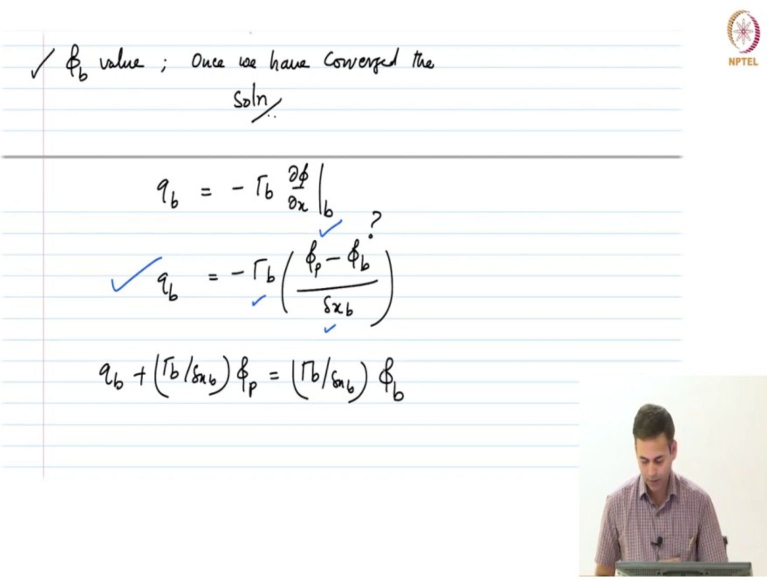

So now, we will just calculate phi b value. Once, we have converged the solution ok. Once, the solution is converged ok, then you have a converged value for phi p right. From there, you can calculate what is phi b value. How do you calculate? You just apply the same boundary condition that you have started off with that is what was the definition for

$q_b = -\Gamma_b \frac{\partial \phi}{\partial x} |_b$, that is what we have.

This is $q_b = -\Gamma_b \left(\frac{\phi_p - \phi_b}{\delta xb} \right)$q b is known, heat flux is known. Now, phi p now, once you have converged, your solved where your phi p known?

Student: Yes.

Right, it is known ok. Delta x b and gamma b are also known. What do we have to do? Calculate phi b that is all ok. So, what will be phi b? Phi b would be $q_b + \left(\frac{\Gamma_b}{\delta xb}\right)\phi_p = \left(\frac{\Gamma_b}{\delta xb}\right)\phi_b$

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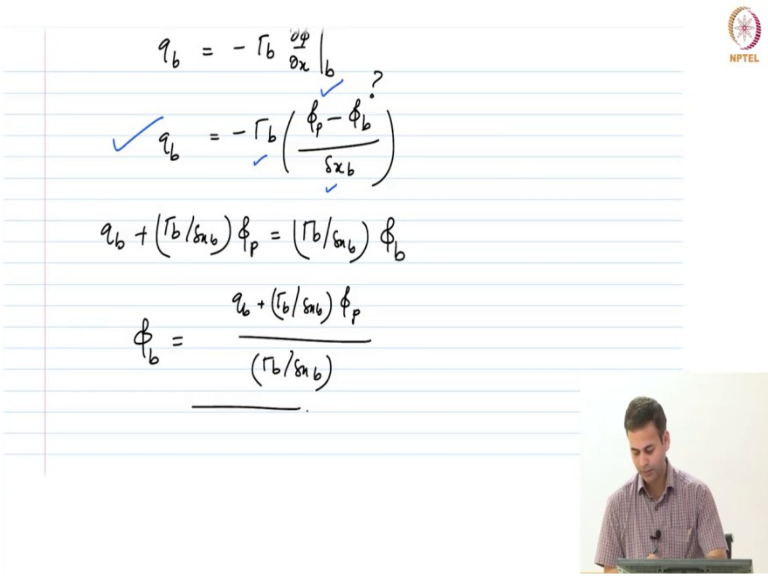

So, your phi b is the value of the dependent variable on the boundary is nothing but q b

$q_b + \frac{\left(\frac{\Gamma_b}{\delta xb}\right)\phi_p}{\left(\frac{\Gamma_b}{\delta xb}\right)}$. You can calculate what is the value of dependent variable on the boundary or a Neumann boundary condition, that is given ok. Questions still now? So, we have covered the Dirichlet boundary condition and Neumann boundary condition ok. Questions any? Clarifications? No, very clear ok. So, you can easily implement alright.

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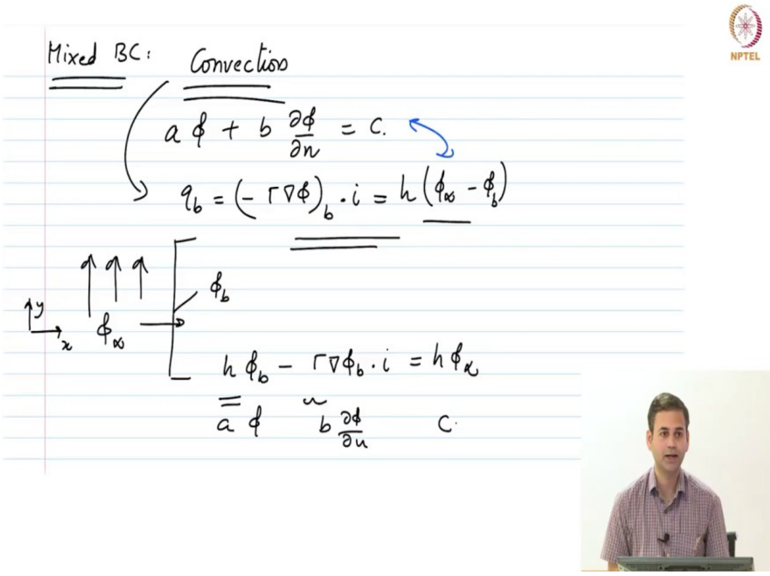

So, the next one, we have is the mixed boundary condition, what would be a typical physical application, where you get mixed boundary conditions? Any? For example, Dirichlet you can tell me that ok, I have a heater which maintains at constant temperature right.

Neumann, you can say ok, I have insulated it, there is a heat flux 0 or I have some kind of a flux that I am applying through some kind of a heater. Then, that is Neumann boundary condition, where exactly do we get this kind of mixed boundary condition. Is it just out of curiosity, we are doing it or do you get it somewhere? Phase change? No, phase change is what? Constant temperature right, constant temperature ok.

Student: (Refer Time: 28:36).

Sorry?

Student: (Refer Time: 28:38).

Convection right, convection. So, if you have convection, the kind of boundary condition you would apply is a mixed boundary condition right. Why do we say that? Essentially, we what we are looking at? We are looking at a boundary condition that is a linear combination of Dirichlet and Neumann that is we are looking at $a\phi + b\frac{\partial \phi}{\partial n} = c$ ok.

So, if I have convection, what do we have? We have $q_b = \left(-\Gamma \nabla \phi\right)_b . i = h\left(\phi_\infty - \phi_b\right)$ Equals h times the heat transfer coefficient h times phi infinity that is for the fluid that is outside right minus phi right. Let us say if I have a boundary, this is maintained at phi b right.

We are looking at conduction. So, this is some kind of a solid and we have some flow that is happening here. So, this is at t infinity right. So, there is a heat flux that is flowing in this direction. So, this is what we get. Now, fine, I have said convection is the application; but is it now similar to this equation or not or they are not related?

Student: (Refer Time: 29:57).

It is similar. It is the same thing right. If I rewrite this thing, if I call this as phi b, this is phi b; then, I can write this equation as bring h phi b to the left hand side, we have $h\phi_b - \Gamma \nabla \phi_b . i = h\phi_\infty$. This is similar to $a\phi + b\frac{\partial \phi}{\partial n}$ equals some c right. This is the same thing. So, if we have convection on a solid wall, then what we are talking about is a mixed boundary condition alright.

So, how do we deal with the finite volume method, if we have a mixed boundary condition that we will see in the next class. Fine, alright.