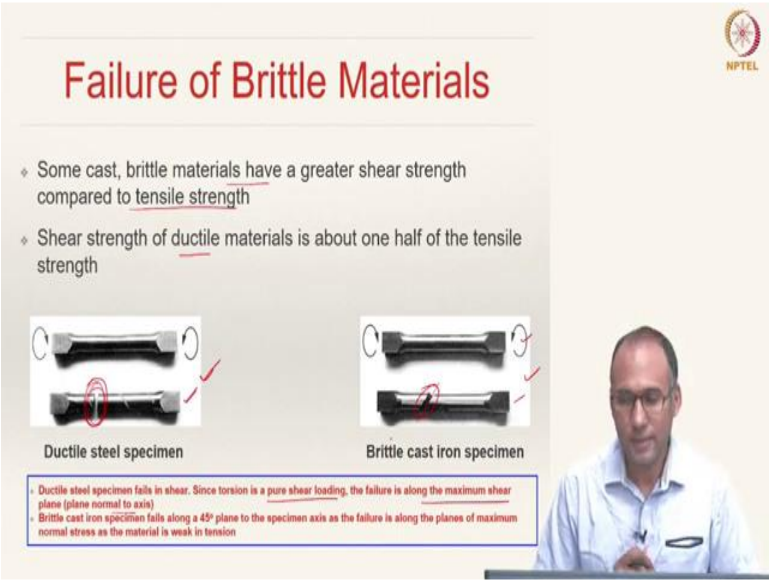
Basics of Materials Engineering
Prof. Ratna Kumar Annabattula
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture - 36
Static Failure Theories (Coulomb-Mohr and Modified Coulomb-Mohr)

(Refer Slide Time: 00:13)



Welcome back. In today's class, we will look at the failure theories for brittle materials. As we have discussed in the last class, the brittle materials fail through a different mechanism as compared to ductile materials. For instance, some cast brittle materials have greater shear strength compared to tensile strength.

Normally for ductile materials the shear strength is one-half of the tensile strength, but some materials, particularly cast brittle materials may have a higher shear strength than the tensile strength.

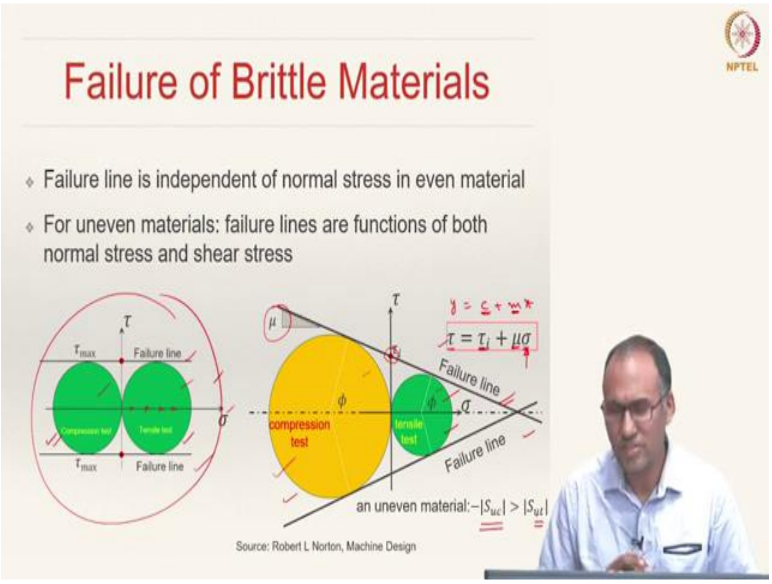
For instance, in the last class we have looked at how a brittle material fails when you are applying uniaxial tension. It fails across the normal surface to the applied loading direction. When you apply a torsional load on a ductile material, it fails normal to the axis of application of the torque. However, if it is not mentioned that torsion is applied to break the ductile specimen, it is possible that it may be also be thought of as brittle material which has failed by applying tension, because the failure surface looks very similar to that.

If you apply a torsion on a brittle material it fails at an angle 45°. Why does that happen? We know that ductile materials fail in shear, right? When you are applying torsion, the maximum shear stress is on this plane since torsion is a pure shear loading and the failure is along maximum shear plane which is normal to the axis of application of load.

We know that brittle materials however fail due to normal strength. Although you are applying pure shear, as we have discussed in the context of Mohr circle, there are planes in the material which experience normal stress.

The plane along which the normal stress is maximum happens to be this 45° plane and because the brittle materials are known to fail due to their normal strength, you will see the failure surface to be at an angle 45°.

(Refer Slide Time: 03:00)



Let us now see how the failure surface looks like. We have discussed about the 2 kinds of materials in the last class, i.e., even and odd materials. What do we mean by an even material? An even material has both the tensile and compressive strengths to be equal in magnitude.

If the tensile strength and the compressive strength of the material are different, then such a material is called uneven material. And brittle materials are in general, uneven materials.

Let us now look at the failure lines. This particular figure represents the Mohr circle for an even material which has same strength. So, let us say we are doing uniaxial tension test and uniaxial compression test; it fails at the same magnitude and hence this is the Mohr circle for uniaxial tension and this is the Mohr circle for uniaxial compression. The common tangent to these two failure surfaces is going to be the failure line.

That is the failure line that you would take as the behavior of the material or the response of the system. However, if the same exercise is done for an uneven material, wherein the material has higher compressive strength compared to tensile strength, you see that the green circle is uniaxial tension Mohr circle and the orange circle here is the uniaxial compression Mohr circle. And if we have higher compressive strength, the radius of the Mohr circle is going to be higher.

These two stress-states are extremes and any other stress-state is somewhere in between the pure uniaxial tension and pure uniaxial compression. The tangent joining the orange circle and the green circle is the failure line. For an even material, the failure line is independent of the normal stress; for instance, here $\tau_{\max}$ which is the failure stress is independent of the normal stress. No matter what normal stress is applied, all of them have the same failure line.

For an even material, the failure lines are independent of the normal stress. For an uneven material, the failure line is a straight line on the $\sigma - \tau$ plane which can be written similar to $y = mx + c$ as

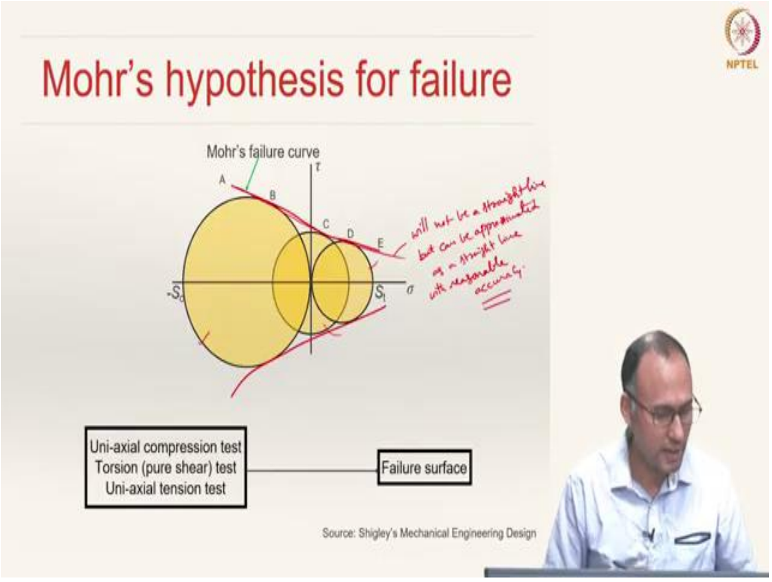
$$\tau = \tau_i + \mu\sigma$$

The slope of the failure line is known as friction angle which is represented by μ and σ is the normal stress. τ_i is the position where he failure line intercepts the τ axis. τ_i and μ are material properties. It is to be noted that the failure line depends on the normal stress as shown in the above equation, unlike in the case of even material.

Note that when we are saying uneven material, the magnitude of ultimate compressive strength is greater than the magnitude of ultimate tensile strength; this is typical of brittle materials.

A similar response is also observed in non-continuum materials like granular materials. The granular materials also show a behavior very similar to what we are discussing here.

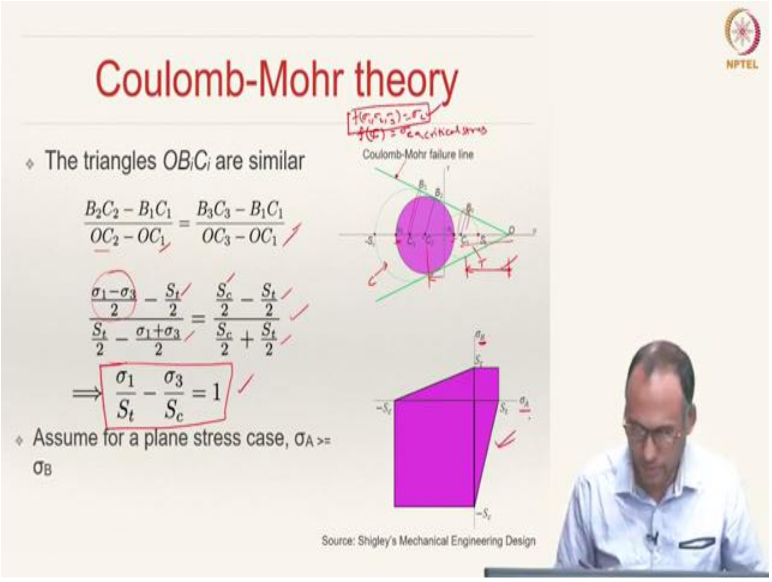
(Refer Slide Time: 07:17)



The left most circle is uniaxial compression, the right most yellow circle is uniaxial tension and the middle circle is basically a pure shear case, right? Then you can draw a common tangent to all these three circles. This common tangent represents the failure surface. This was proposed by Mohr and hence this is what is called Mohr's failure theory for brittle materials.

In principle, the common tangent will not be a straight line usually, but can be approximated as a straight line with reasonable accuracy. That is how you represent the failure surface for an uneven material which was proposed by Mohr.

(Refer Slide Time: 08:46)



Let us now look at this failure theory; how do we go about prescribing this failure theory. We need to write a functional form to describe the failure theory. In the very beginning, we have said any failure theory can be expressed as

$$f(\boldsymbol{\sigma}) = \sigma_c$$

$\boldsymbol{\sigma}$ in the above equation is a tensor and σ_c is the critical stress. Basically, that is a description of the failure surface.

In this figure, the leftmost dashed circle is pure compression and this is pure tension. S_c is the compressive strength, S_t is the tensile strength and in between, you may have some generic stress-state with principal stresses σ_1 and σ_3.

Here, we are dealing with plane stress case and hence $\sigma_2 = 0$. Because this is the normal and then you can join the centers of the circle. The leftmost circle center C_3 is joined with the point at which the Mohr failure line is a tangent, which is B_3.

C_2 is the pink circle that is actually the loading circle and which has a tangent at point B_2. The circle corresponding to pure tension is denoted by C_1. OB_iC_i are similar triangle; that means, OB_1C_1, OB_2C_2 and OB_3C_3 are similar triangles. We can write,

$$\frac{B_2C_2 - B_1C_1}{OC_2 - OC_1} = \frac{B_3C_3 - B_1C_1}{OC_3 - OC_1}$$

B_2C_2 is basically the radius of this circle which can be written as $\frac{\sigma_1 - \sigma_3}{2}$.

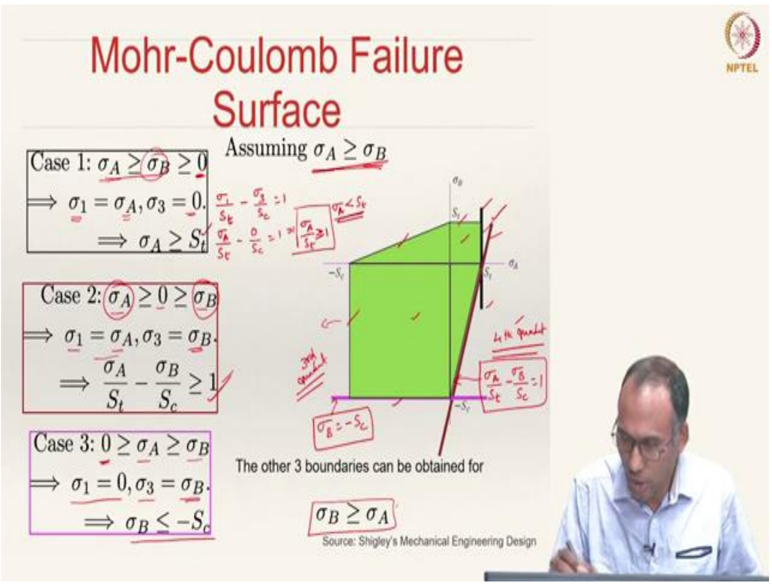
B_1C_1 is that radius equal to $\frac{S_t}{2}$. Similarly, $B_3C_3 = \frac{S_c}{2}$. Please keep in mind that here we are only writing magnitude. Substituting and simplifying, we get,

$$\frac{\frac{\sigma_1 - \sigma_3}{2} - \frac{S_t}{2}}{\frac{S_t}{2} - \frac{\sigma_1 + \sigma_3}{2}} = \frac{\frac{S_c}{2} - \frac{S_t}{2}}{\frac{S_c}{2} + \frac{S_t}{2}}$$

$$\Longrightarrow \frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

This is the equation of the failure surface. Let us now assume that the principal stresses are given by σ_A and σ_B. For a plane stress case if $\sigma_A > \sigma_c$, then you will have something like this. We will see how this failure surface will be obtained by using this equation.

(Refer Slide Time: 12:40)



In case 1, let us assume that $\sigma_A > \sigma_B > 0$. That means, the stress-state lies in the first quadrant.

According to our definition in the previous expression, σ_1 and σ_3 are the maximum and minimum principal stresses. In this case, the maximum stress will be σ_A and minimum will be 0. Hence, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$.

Substituting these values in the equation $\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} \geq 1$, we get

$$\frac{\sigma_A}{S_t} \geq 1$$

This is the failure surface and the failure of the material occurs when the value $\frac{\sigma_A}{S_t}$ is greater than or equal to 1.

This is actually representing failure domain; if we want to say that it is safe, then $\sigma_A < S_t$; that is safe, equal to 1 is actually on boundary and greater than S_t is actually outside the boundary; that means, that is unsafe region.

So, $\sigma_A = S_t$ is that black line. We are always assuming that $\sigma_A > \sigma_B$. In case 2 $\sigma_A > 0 > \sigma_B$. Then what happens? Out of these two, this will be my maximum principal stress and that will be my minimum principal stress.

So, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$ and then our equation would be

$$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1$$

Which quadrant are we talking about? Fourth quadrant, right? Here, $\sigma_A > 0$, $\sigma_B < 0$ and then you would draw that and that would be the line.

And then there can be another case where in both $\sigma_A, \sigma_B < 0$. In case 3, we have $0 < \sigma_A < \sigma_B$. Both the principal stresses are negative; that means, we are in the third quadrant. What is the maximum principal stress? $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$.

So, that gives rise to

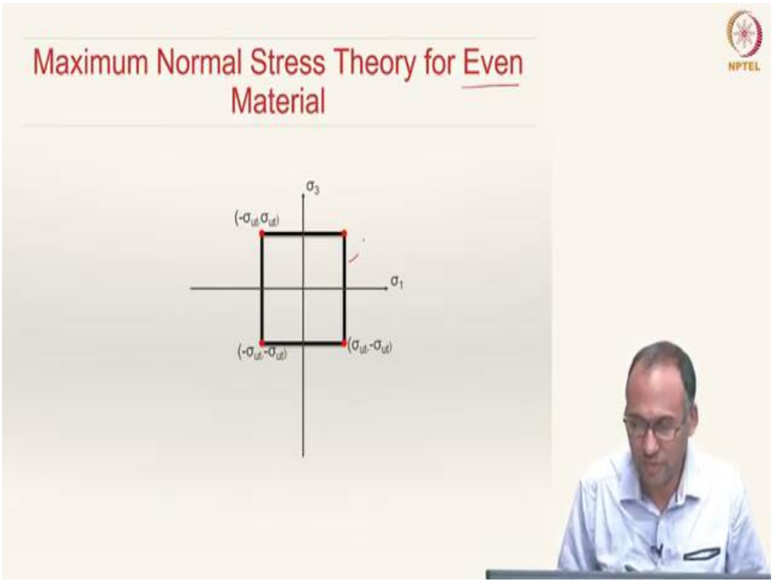
$$\sigma_B \leq S_c$$

That means, on this side if you further reduce then that will become the failure surface. So, equal to $-S_c$ is the boundary. So, that will be that line that pink line.

We got three boundaries by assuming $\sigma_A > \sigma_B$. If you assume $\sigma_B > \sigma_A$, then the other three boundaries can be obtained and that is how your failure surface looks like. This is very similar to what you have seen for the maximum shear stress theory in principle, but because of the fact that it is uneven material, on the tensile side and the compressive side you will have asymmetry.

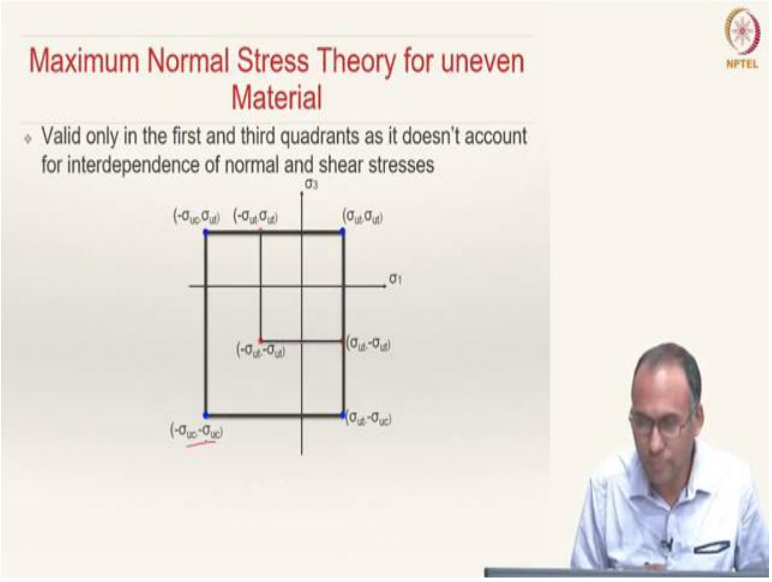
It is also a hexagon, but it is not a regular hexagon in some sense, right? By drawing these boundaries, the green region represents the safe region. So, if the stress-state is within this green region, then the material is safe.

(Refer Slide Time: 17:32)



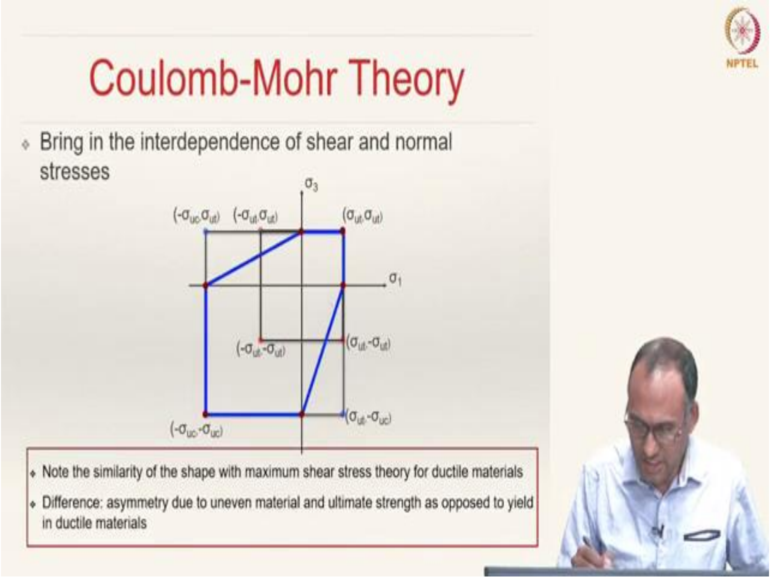
Let us now compare all the theories that we have discussed for brittle materials. The failure of brittle materials can also be represented by maximum normal stress theory which is also sometimes used. For an even material $\sigma_{ut} = \sigma_{uc}$. Hence, it is a square and is symmetric about the $\sigma_1 - \sigma_3$ plane; that is the failure surface for maximum normal stress theory of an even material.

(Refer Slide Time: 18:12)



For an uneven material $\sigma_{ut} \neq \sigma_{uc}$. The inner square is for even material and the outer square is for uneven material because only on the compressive side your strength is changing tensile strength is the same. So, please note that this theory is valid only in the first and third quadrants as it does not account for interdependence of normal and shear stresses.

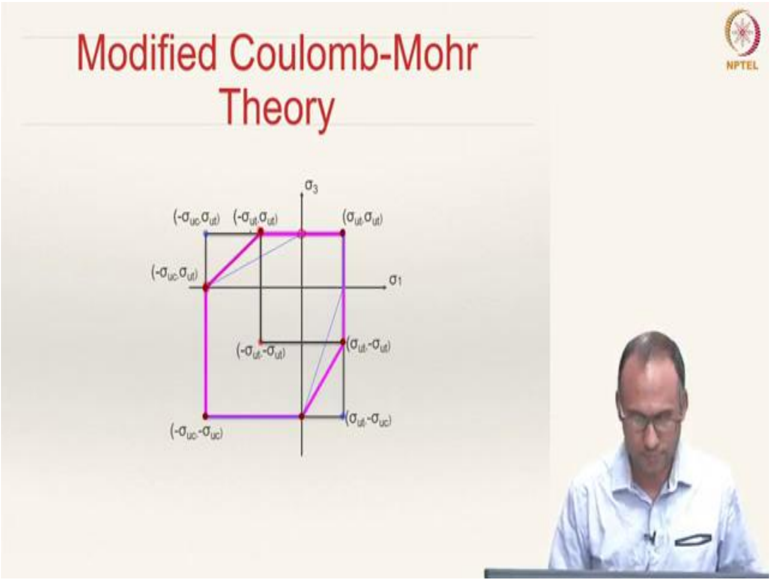
(Refer Slide Time: 18:45)



The Coulomb-Mohr theory brings in the interdependence of shear and normal stresses. And please note the similarity of the shape with maximum shear stress theory for ductile materials. The only difference is that asymmetry due to uneven material and ultimate strength as opposite to yield in ductile materials.

In ductile materials, the shape is the same, but here you are not using ultimate strength, but you are using yield strength. For brittle materials we are using ultimate strength as there is no yielding whatsoever that we can define for brittle materials; there is no plastic deformation. So, this is what we call Coulomb-Mohr theory and that is mostly used. This theory is much better than your normal stress theory for these materials.

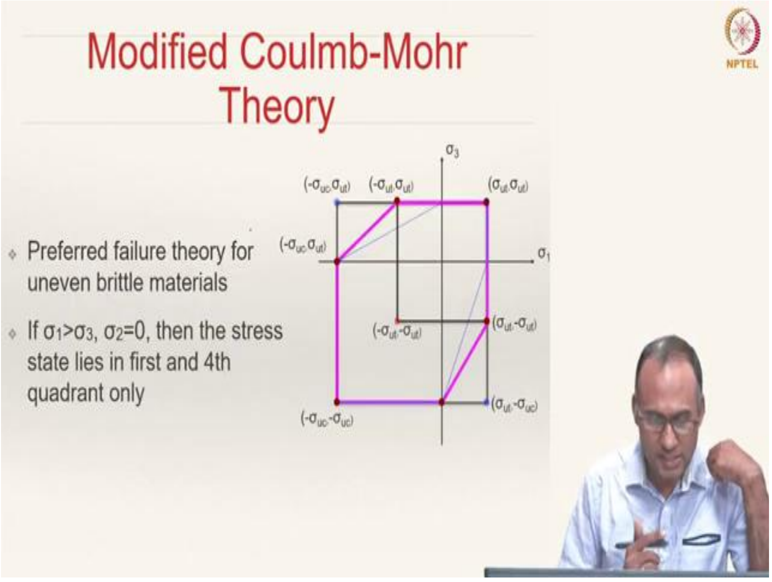
(Refer Slide Time: 19:34)



People later actually looked at the experimental data and they have modified the failure theory a little bit, called this as modified Mohr-Coulomb theory wherein, instead of connecting this line and this point, you actually connect the σ_{ut} and this is what we call modified Mohr-Coulomb theory. This has actually been modified from the Mohr-Coulomb theory by looking at the experimental data.

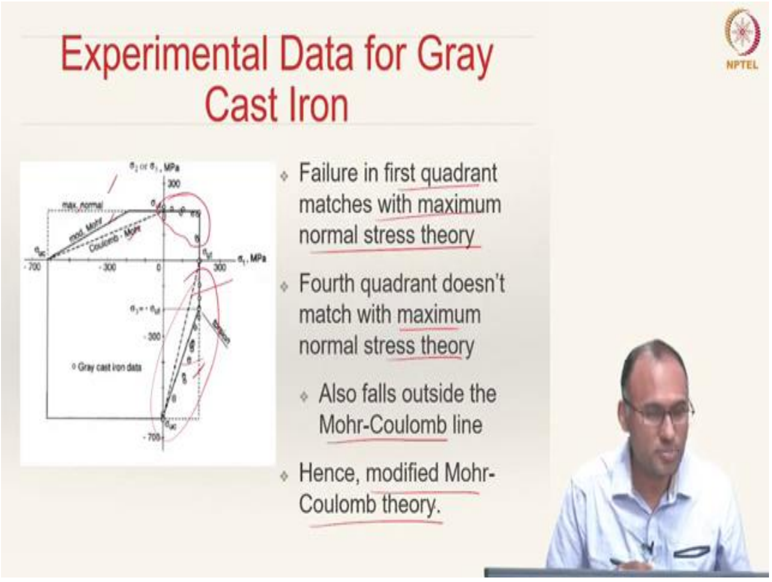
After looking at the experimental data, it turned out that the modified Mohr-Coulomb is a better way to compute the failure or estimate the failure for a given material. So, it is a better theory compared to Coulomb-Mohr theory.

(Refer Slide Time: 20:19)



The Modified Coulmb-Mohr Theory is the most preferred failure theory for uneven brittle materials and if $\sigma_1 > \sigma_3$ and $\sigma_2 = 0$, then the stress-state lies in first and fourth quadrant only.

(Refer Slide Time: 20:40)



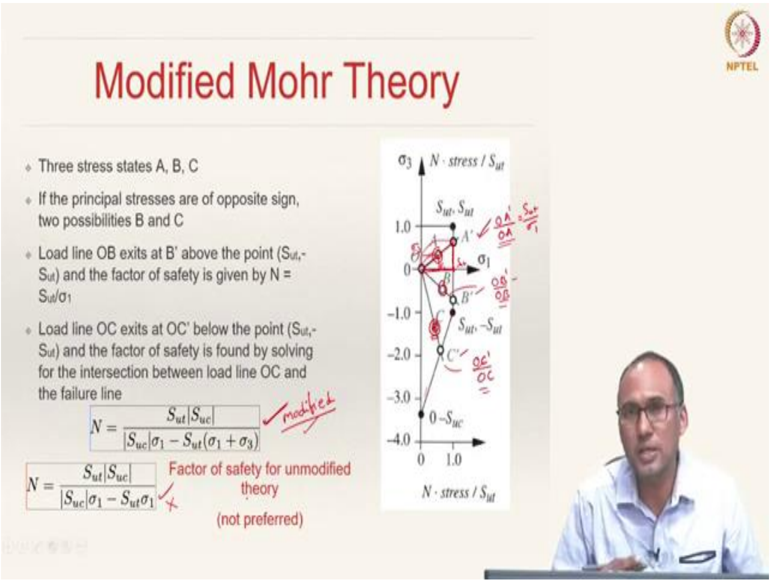
Here, we are seeing experimental data for gray cast iron plotted on all the four theories; actually 4 theories that we have shown in the previous slide, right? Here, you have maximum normal stress theory (uneven materials), Coulomb-Mohr theory and modified Mohr-Coulomb theory. The solid lines are modified Mohr-Coulomb theory. Note that, in the first quadrant all these theories are equivalent; the first quadrant and third quadrant, the maximum normal stress theory, Mohr-Coulomb theory, modified Mohr-Coulomb theory are equivalent; they differ only in second and fourth quadrant.

Failure in the first quadrant matches with maximum normal stress theory, as we can see the failure data points are same as the maximum normal stress theory because the failure surfaces themselves are the same. However, if you come to the fourth quadrant, the experimental data matches better with modified Mohr-Coulomb theory compared to pure Coulomb-Mohr theory, right? So, fourth quadrant actually does not match with maximum normal stress theory first question and also it falls outside the Mohr-Coulomb line.

That is the reason why people have actually looked at these experiments and they have improved the theory saying that modified Mohr-Coulomb is a better approximation. And please note that all these theories are only our abstractions and our attempt to describe the failure in a mathematical form.

There are several assumptions behind deriving these mathematical forms and hence, it is often necessary to actually correct for these theories after looking at the experimental data and that is how this modified Mohr-Coulomb theory has been developed.

(Refer Slide Time: 22:42)



We have looked at modified Mohr-Coulomb theory and so let us say we are looking at first and fourth quadrant and we have 3 different kinds of stress-states. Let us say you have stress state A, stress state B, stress state C and how do you go about calculating the factors of safety for these things.

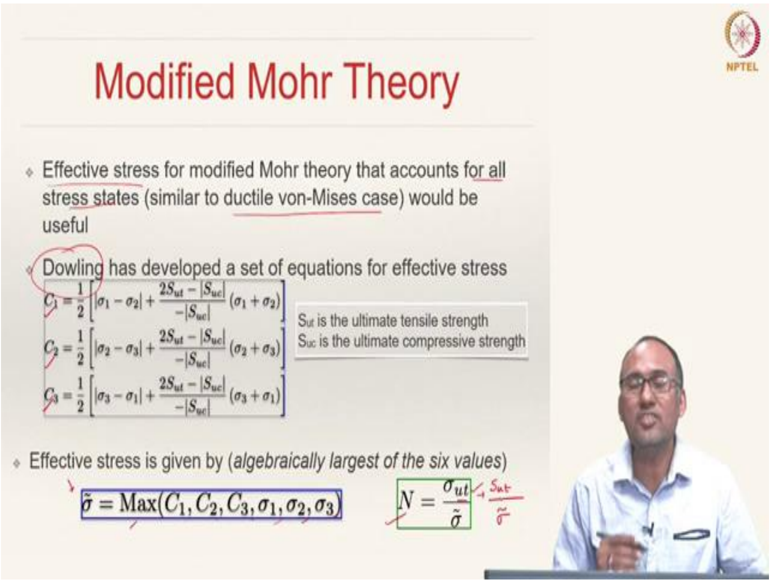
Normally, for instance for this, the factor of safety would be $\frac{OA'}{OA}$. You just compute the distance and then you can calculate. So, similarly for this it is $\frac{OB'}{OB}$ and for this it should be $\frac{OC'}{OC}$.

Note that the equations of the lines are little different in each of these regions; the boundaries that it is hitting are little different and hence you need to be little bit careful in estimating their factors of safety. There is another way to do that is because this is a similar triangle, you could actually say that $\frac{OA'}{OA} = \frac{S_{ut}}{\sigma_1}$.

The same could actually have been done for OB'. However, when you come to OC', it will not work in a such a nice fashion.

The factor of safety for point C can be derived by using by knowing the equation of this line. This is for modified theory. For unmodified theory, this is the equation which is not actually very much preferred. You should always try to restrict yourselves to modified Mohr-Coulomb theory because that matches much better with the experimental data at least in fourth quadrant.

(Refer Slide Time: 25:15)



In the case of ductile failure, we have a very nice way of calculating factor of safety; what was that? Irrespective of the complexity of the stress-state may be, we have been calculating something called equivalent stress or von-Mises stress.

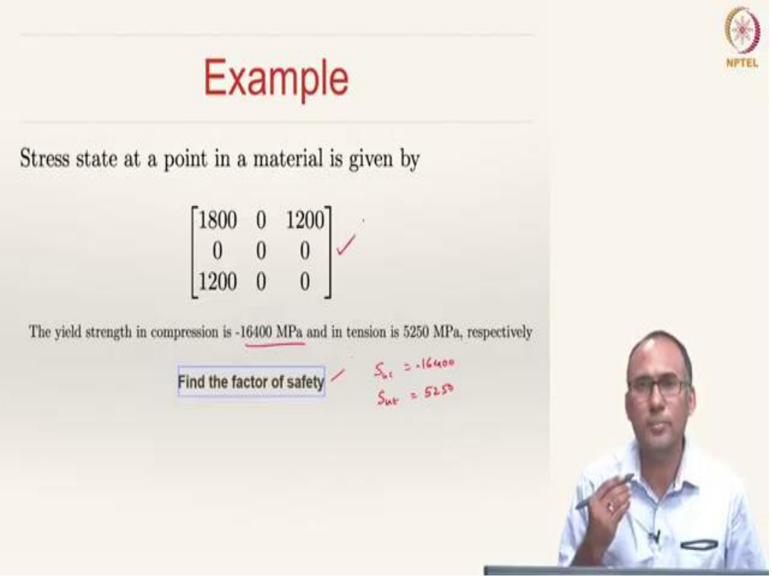
Then, the factor of safety was defined as the yield strength of the material divided by the von-Mises stress or equivalent stress; that means for a 3-dimensional state of stress, equivalent stress or von-Mises stress is actually a 1-dimensional representation; an equivalent 1-dimensional representation. Similarly, can we also do a similar kind of a 1-dimensional representation?

Can we write an equivalent stress or an effective stress from a complex 3-dimensional state of stress even for brittle material or modified Mohr theory? Norman E Dowling proposed an approach to do this. He came up with the definition of an effective stress for modified Mohr theory that accounts for all stress-states similar to ductile von-Mises case.

How did he calculate? His prescription is, you calculate C_1, C_2, C_3 by knowing $\sigma_1, \sigma_2, \sigma_3, S_{ut}$ and S_{uc}. From there, you calculate effective stress as

$$\tilde{\sigma} = \text{Max}(C_1, C_2, C_3, \sigma_1, \sigma_2, \sigma_3)$$

(Refer Slide Time: 26:48)



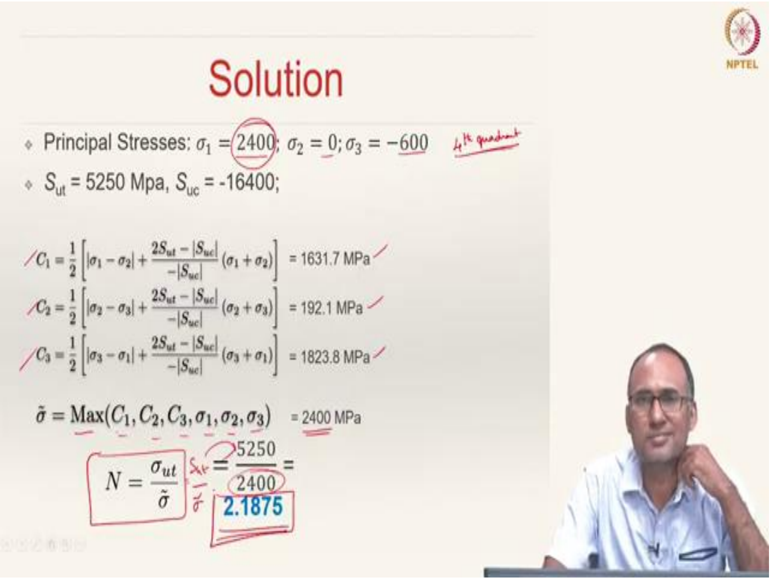
The factor of safety is defined as

$$N = \frac{\sigma_{ut}}{\tilde{\sigma}}$$

Please keep that in mind to use σ_{ut} and not σ_{uc}. Let us look at one example problem and see how do we go about calculating the factor of safety using modified Mohr-Coulomb.

This is the stress-state at a given point and $S_{uc} = -16400$ MPa, $S_{ut} = 5250$ MPa and we need to calculate factor of safety. So, how do we go about doing that? So, from the given stress state we can always calculate our principal stresses.

(Refer Slide Time: 27:37)



The principal stresses turn out to be 2400 MPa, 0 MPa, -600 MPa; that means, this is in the fourth quadrant. If you are in the first quadrant, you actually do not need to do such a complicated calculation because in the first quadrant your maximum normal stress theory itself will work.

Here, you are in the fourth quadrant. So, you need to account for modified Mohr-Coulomb theory. So, we have calculated C_1, C_2, C_3 using the formula and that comes out to be 1631.7 MPa, 192.1 MPa and 1823.8 MPa.

Then, you find out the maximum value of $C_1, C_2, C_3, \sigma_1, \sigma_2, \sigma_3$. That turns out to be 2,400 MPa. Now,

$$N = \frac{\sigma_{ut}}{\tilde{\sigma}} = \frac{5250}{2400} = 2.1875$$

That is how one would go about calculating the factor of safety for a given component.