Basics of Materials Engineering
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Lecture – 32
Static Failure Theories (Distortion Energy Theory)

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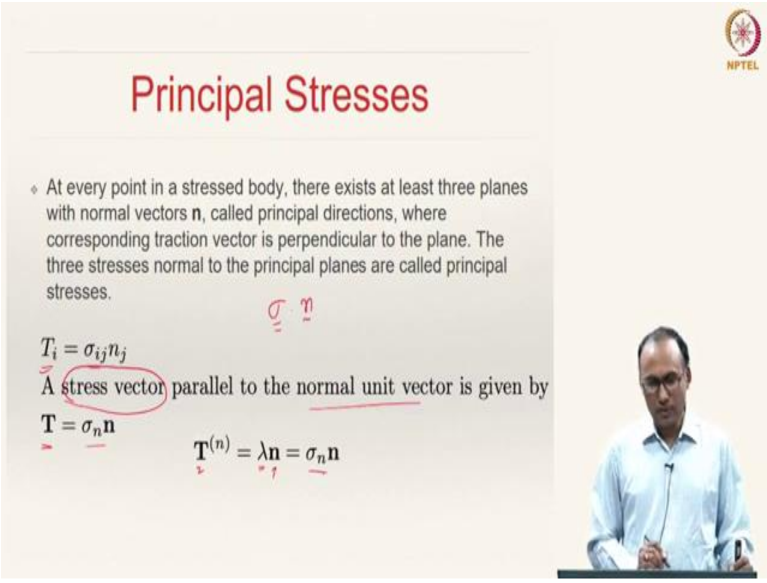

Welcome back. In the last class, we were looking at the concept of principal stress and then we have written that the traction vector can be written as

$$T_i = \sigma_{ij} n_j$$

$\boldsymbol{n}$ is the normal vector to the surface on which you are trying to find the traction.

Certain textbooks in the mechanics literature also refer to this traction vector as stress vector.

If the stress vector or traction vector is parallel to the normal unit vector, then the traction can be written as

$$\boldsymbol{t} = \boldsymbol{\sigma n} = \lambda \boldsymbol{n}$$

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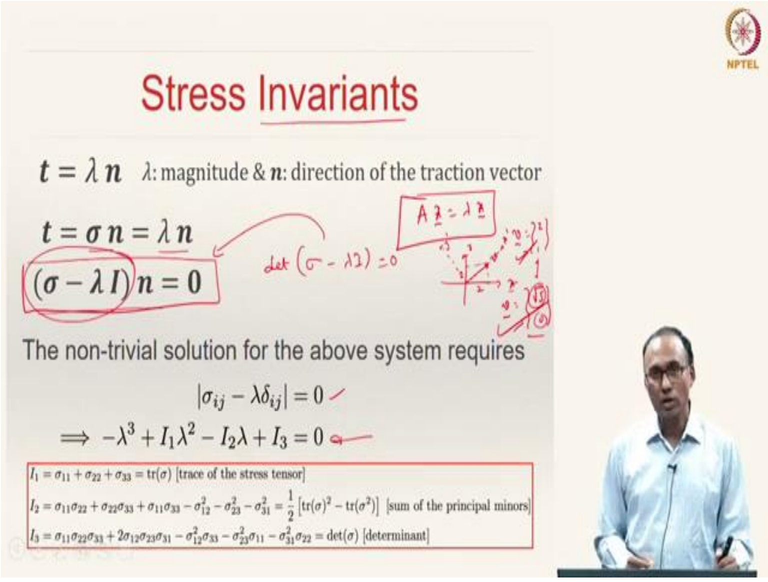

This can be casted as an eigenvalue problem as,

$$(\boldsymbol{\sigma} - \lambda \boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$$

similar to the $\boldsymbol{Ax} = \lambda \boldsymbol{x}$ problem; such a problem is called an eigenvalue problem.

The solution of this equation requires that this matrix is singular which means that its determinant must be equal to 0 i.e.,

$$|\boldsymbol{\sigma} - \lambda \boldsymbol{I}| = 0$$

$$\Longrightarrow -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

The roots of this characteristic equation are the eigenvalues of the system and they represent the principal stresses of the stress tensor.

We now introduce a concept called stress invariants. For instance, if you take a vector $\boldsymbol{v}$ in 2D, which has two-units projection onto x axis and one-unit projection on to y axis. So, the vector $\boldsymbol{v} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$, with respect to the x and y axes.

Let us say that I am defining a new axis system x' and y'. Now, $\boldsymbol{v} = \begin{Bmatrix} \sqrt{5} \\ 0 \end{Bmatrix}$ with respect to x' and y', because this is the magnitude of the vector. So, you can see that the same vector with respect to the original coordinate system i.e., the $x - y$ system is represented as $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$ and in a new coordinate system i.e., the $x' - y'$, is represented as $\begin{Bmatrix} \sqrt{5} \\ 0 \end{Bmatrix}$.

So, in these two coordinate systems, we are representing the same vector; but the elements of the vector are different. And hence, you might think that they are two different quantities; but they are not, only what we have changed is the frame of reference from which we are looking at this vector.

However, in both the frames of reference certain quantities do not change. What are those quantities? For instance, the length of the vector in both the frames of remains the same; so that means, the length is an invariant of your vector.

Similarly, if you take a second order tensor, you can define invariants to second order tensor; that means irrespective of the frame of reference with respect to which you are writing your components of your stress tensor, certain quantities remain invariant, that means they do not change. So, for a second order tensor, you will have three invariants, represented as I_1, I_2, I_3.

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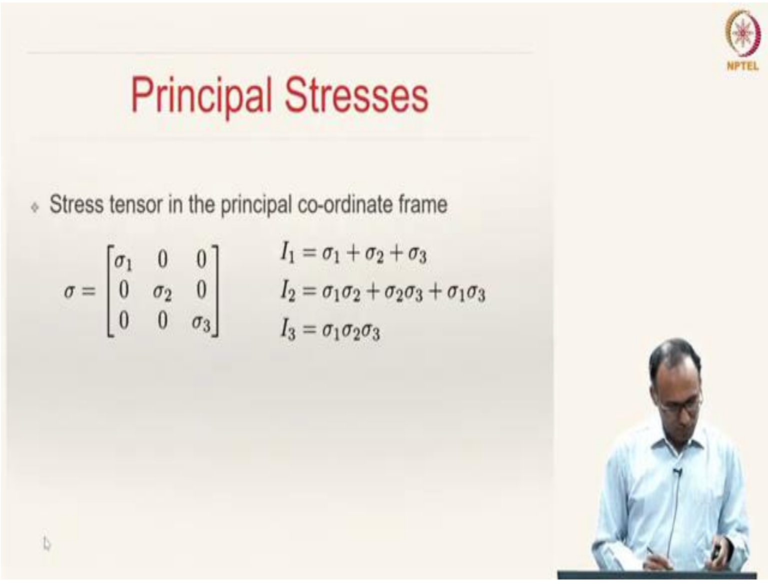

I_1 is the trace of the stress tensor which is the sum of the diagonal elements. That is what is called as the first invariant; that means, the trace of a tensor does not change by coordinate transformation. The second invariant I_2 is the sum of the principal minors and the third invariant I_3 is the determinant of the tensor.

These three parameters do not change under coordinate transformation. This is what we mean by stress invariants. We will use them in the failure theories and hence we are introducing the concept of stress invariants. Basically, what it means is that, whether you work with full stress tensor or a principal stress tensor, you can write your failure theories in terms of these invariants I_1, I_2, I_3 or in terms of principal stresses.

For instance, if we take the stress tensor in principal coordinate frame, you will only have normal stresses and the shear stresses will be 0, and hence you will only have diagonal componentsσ_1, σ_2 and σ_3. From this, we can see that the first invariant is sum of the elements on the diagonal, that is the trace of the tensor and the second invariant is the sum of the principal minors and the third invariant is the determinant of the stress tensor.

Even if you express stress in a different frame of reference and then you may have full stress tensor; that means all the elements may be nonzero, even then for that particular stress tensor, these three quantities should remain the same; that is what is the meaning of invariant.

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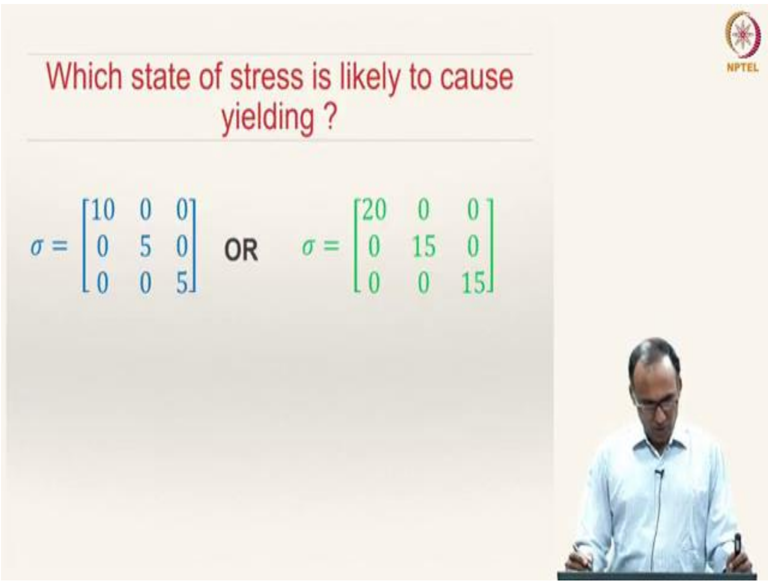

Now, the question is the following, and we try to answer this question towards the end of today's lecture. So, you are given two different states of stress both in the principal frame of reference,

$$\sigma = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{OR} \quad \sigma = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Which of these stress-states is likely to cause yielding first? Think about it.

The second stress-state looks like it has higher values of the stress, so it is possible that this is probably going to cause yielding first. Let us see whether or not that is the right answer towards the end of this discussion.

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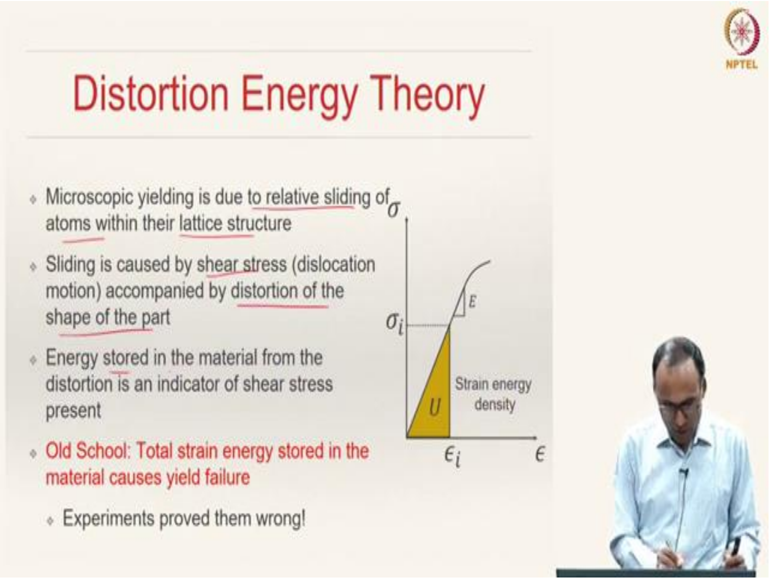

We will now start discussing about the distortion energy theory. The microscopic yielding that we have discussed in the mechanical behavior module is due to relative sliding of atoms within the lattice structure, right? The sliding is caused by shear stress or in other words the plastic deformation is caused by dislocation motion, which is accompanied by the distortion of the shape of the part.

That means, whenever you are having shear deformation or the sliding of atom planes one past the other, you have distortion of the crystal lattice or there is a shape change to the crystal lattice or the part.

Due to the work done by the applied external forces, there is some energy stored in the material. The energy stored in the material from the distortion part just by shearing is an indicator of the shear stress present. That is what is causing the deformation or the plastic deformation.

People were earlier thinking that the total strain energy stored in the material causes yield failure; that is what causing plastic deformation or the onset of plastic deformation.

What do we mean by total strain energy? It is the area under the stress-strain curve up to yield point. Let us say σ_i is the yield point yield and corresponding strain is ϵ_i; the area of this triangle is what we call total strain energy. However, experiments proved them wrong. They saw that, it is actually not the total strain energy that is responsible for distortion or the plastic deformation.

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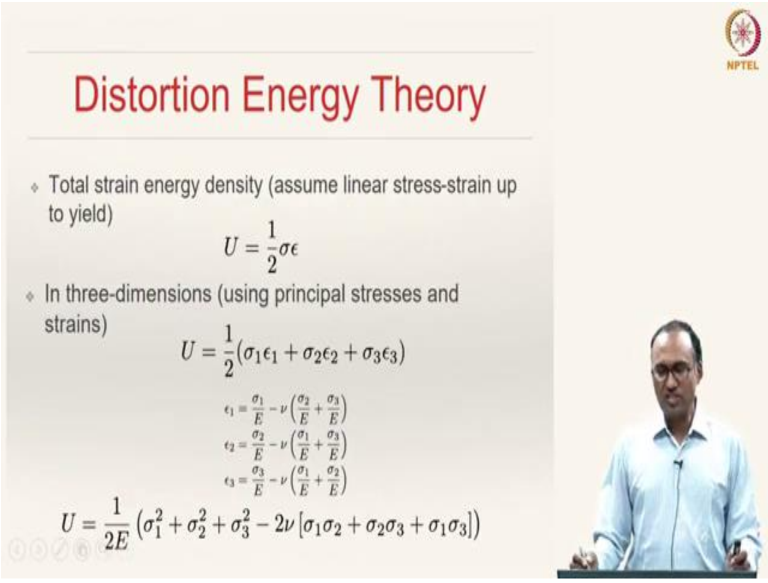

The total strain energy density for a one-dimensional case assuming linear stress-strain relation up to yielding is

$$U = \frac{1}{2}\sigma\epsilon$$

If it is in full 3D,

$$U = \frac{1}{2}(\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3)$$

It is good to work in principal frame of reference, as it is much easier to write this expression. In 3D, the stress-strain relations can be written as,

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\left(\frac{\sigma_2}{E} + \frac{\sigma_3}{E}\right)$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \nu\left(\frac{\sigma_1}{E} + \frac{\sigma_3}{E}\right)$$

$$\epsilon_1 = \frac{\sigma_3}{E} - \nu\left(\frac{\sigma_1}{E} + \frac{\sigma_2}{E}\right)$$

We are writing in the principal frame of reference. Now, you can plug in the value formula for ϵ_1 into this equation for total strain energy. Then, you will be able to write the total strain energy U in terms of the principal stresses σ_1, σ_2 and σ_3 , Poisson's ratio ν, and Young's modulus E as,

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1])$$

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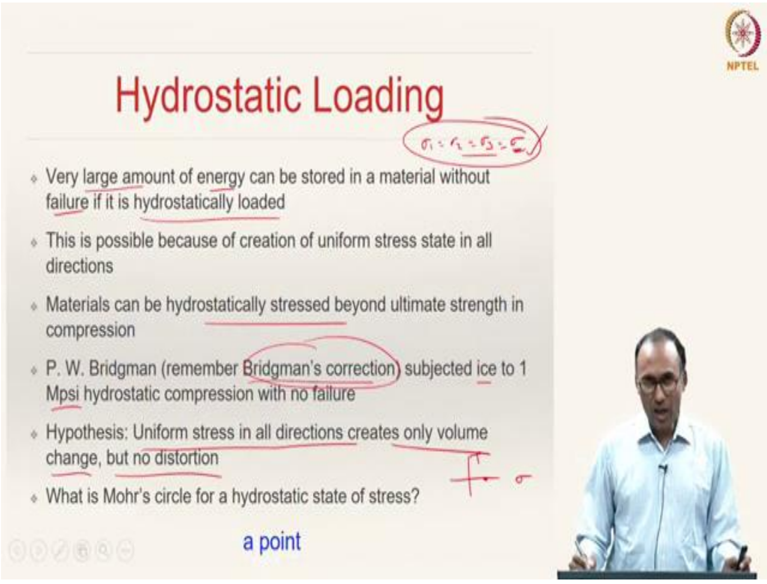

Let us now talk about the concept of hydrostatic loading, because that is going to be useful for us to understand how materials actually plastically deform under the action of total stress.

It is found that a very large amount of energy can be stored in a material without failure, if it is hydrostatically loaded. What do you mean by hydrostatic loading? Applying hydrostatic state of stress means, having the normal stress in all the three directions being equal, i.e., if $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, then such a state of stress is called a hydrostatic state of loading.

If you have this kind of a loading, then the material can be loaded without actually having to plastically deform; that means you can store large amount of energy without failure.

This is possible because, this uniform stress-state is exists in all the three directions i.e., x, y and z. Because of that, you will not have any shape change, as it is only size change; the size will be reducing, but the shape remains the same, because of the uniform stress in all the three directions.

Materials can be hydrostatically stressed beyond ultimate strength in compression. If you are only applying hydrostatic stress, they can be deformed beyond ultimate strength in compression. We have discussed the Bridgman correction when we were discussing the stress-strain curves, right?

Bridgman conducted an experiment on ice, where he has subjected solid ice to 1 Mpsi hydrostatic compression; the material did not fail even at such very high compressive loads.

Normally, if you are not subjecting it to hydrostatic compressive loads, it is much easier to break the ice, right? So, the hypothesis is that the uniform stress in all directions creates only volume change; but no distortion; distortion means shape change. Why are we talking about shape change? Because the plastic deformation causes relative shearing of atom planes which is in turn cause shape change.

If you are applying uniform stress in all the three directions, you cannot impart this shape change, but you can only impart volume change; that means size may reduce, but the shape becomes invariant.

How does your Mohr circle look like for a hydrostatic state of stress? The radius of the Mohr circle will be zero because $\sigma_1 - \sigma_2 = \sigma_3 - \sigma_2 = \sigma_1 - \sigma_3 = 0$.

So, the radii are zero and hence your Mohr circle will be a point on the σ axis, because hydrostatic stress is a normal state of stress.

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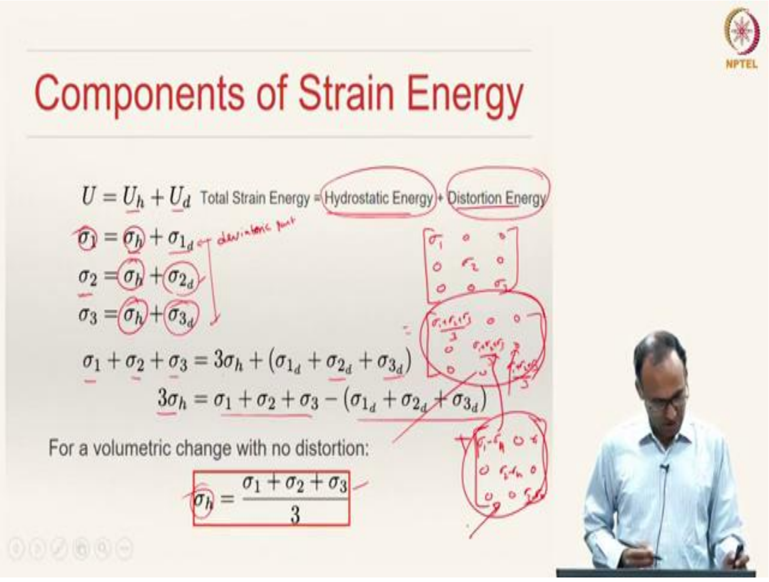

The total strain energy that is stored in a material can now be decomposed into two parts; the hydrostatic energy and the distortion energy. The hydrostatic energy U_h is responsible for size change or volume change, and the distortion energy U_d is responsible for shape change.

$$U = U_h + U_d$$

σ_1, σ_2 and σ_3, which are the principal stresses can be written as the sum of hydrostatic part and a deviatoric part. σ_{1d}, σ_{2d} and σ_{3d} are called as the deviatoric stresses which are responsible for shape change. The hydrostatic part should be same in all the three directions, that is why I am writing σ_h for all the cases.

$$\sigma_1 = \sigma_h + \sigma_{1d}$$

$$\sigma_2 = \sigma_h + \sigma_{2d}$$

$$\sigma_3 = \sigma_h + \sigma_{3d}$$

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h + (\sigma_{1d} + \sigma_{2d} + \sigma_{3d})$$

$$3\sigma_h = \sigma_1 + \sigma_2 + \sigma_3 - (\sigma_{1d} + \sigma_{2d} + \sigma_{3d})$$

If you do not have any distortion and there is only volumetric change, then the hydrostatic stress can be written as,

$$\sigma_h = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3} & 0 & 0 \\ 0 & \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3} & 0 \\ 0 & 0 & \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3} \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_h & 0 & 0 \\ 0 & \sigma_2 - \sigma_h & 0 \\ 0 & 0 & \sigma_3 - \sigma_h \end{bmatrix}$$

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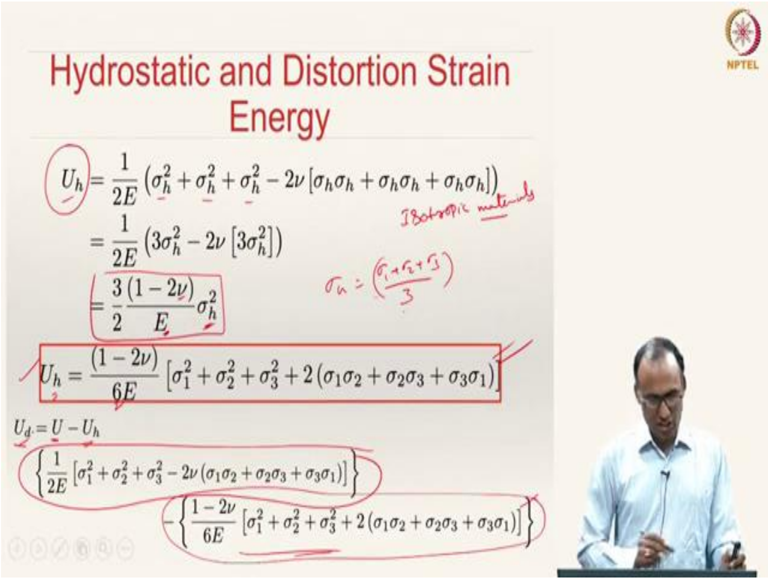

We have said the total strain energy is the sum of hydrostatic energy and distortion energy, right? So, hydrostatic energy can be calculated using the same formula that we have derived here. When we are talking about hydrostatic energy, then $\sigma_1 = \sigma_2 = \sigma_3$.

The hydrostatic part of energy turns out to be this simple expression,

$$U_h = \frac{3}{2}\frac{(1 - 2\nu)}{E}\,\sigma_h^2$$

Here, ν is the Poisson's ratio. Please note that we are always discussing about isotropic materials and hence, there are only two elastic constants ν and E; we are not discussing about anisotropic materials.

Substituting $\sigma_h = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}$ in the previous equation gives hydrostatic energy in terms of the principal stresses

$$U_h = \frac{(1-2\nu)}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

This is the expression for the hydrostatic part of the energy, when you have a stress tensor in principal stress space.

The distortion energy will be the difference between the total strain energy and the hydrostatic part of the energy. We have already calculated the total strain energy as well as the hydrostatic part. The distortion energy can be expressed as,

$$U_d = \left\{\frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1]\right)\right\} - \left\{\frac{(1-2\nu)}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]\right\}$$

Only the distortion energy is responsible for plastic deformation; the hydrostatic part is not responsible. Whenever we are going to assess plastic deformation, we should subtract this hydrostatic part of the energy and only consider the distortion part of the energy.

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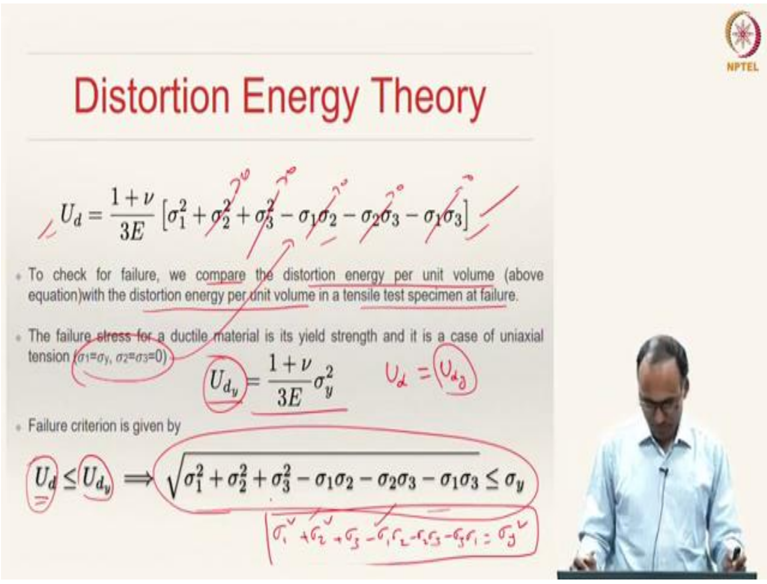

If you simplify this equation, the distortion energy is given by this simple expression,

$$U_d = \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]$$

This is the expression for the distortion energy for a three-dimensional state of stress. To check for failure, we compare the distortion energy per unit volume given by this equation, with the distortion energy per unit volume in a tensile test specimen at failure. When you are doing uniaxial tension test, that is when you are actually measuring yield strength.

So, you measure the distortion energy that is stored until the yield point in your uniaxial tension specimen with the distortion energy of this three-dimensional state of stress. For a uniaxial tension test, you have $\sigma_1 \neq 0$ and $\sigma_2 = \sigma_3 = 0$.

So, in this expression if you substitute $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_y$ because that is the distortion energy stored at the yielding of uniaxial tension specimen, you will get U_{d_y} as,

$$U_{d_y} = \frac{1+\nu}{3E}\sigma_y^2$$

This is the distortion energy for the uniaxial tension specimen. By looking at the property of the uniaxial tension specimen's energy; so now, we are comparing energies. That is a good situation, because these two are scalars.

When the distortion energy stored in a material due to a complex state of loading becomes equal to the distortion energy stored in the material subjected to uniaxial tension test at the yielding, that is when the material subjected to compressed state of loading also undergoes yielding; that is the statement of distortion energy theory.

The material is safe as long as the distortion energy stored in the material under complex state of stress is less than U_{d_y}. The failure happens when $U_d = U_{d_y}$, where U_{d_y} is the distortion energy at yield point for the uniaxial tension specimen.

$$U_d = U_{d_y} \Rightarrow \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \leq \sigma_y$$

This is the mathematical expression for the distortion energy theory. If it is less than, then the material will not yield and if it is equal to, then that is when the we say that the material will start yielding.

The boundary of the failure surface is represented by the expression,

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = \sigma_y^2$$

Within that boundary the material is safe, outside the boundary or on the boundary the material is unsafe.

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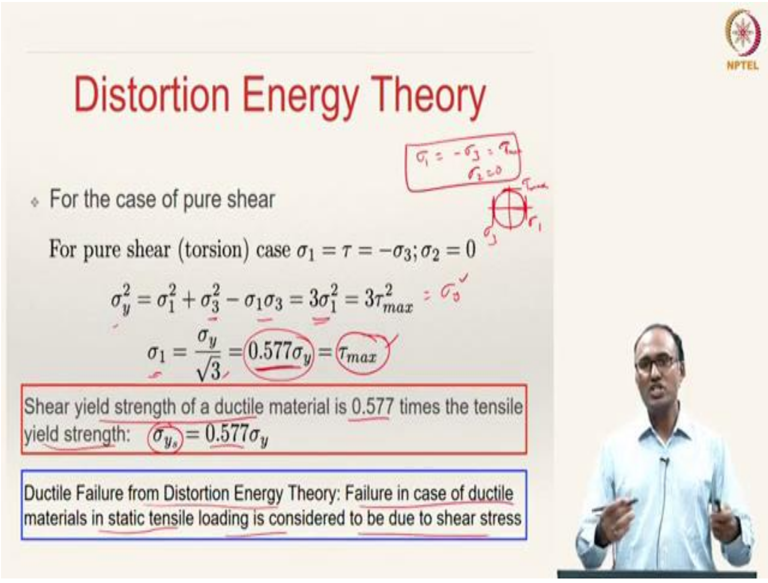

Let us now take a simple case of loading called pure shear. The condition for pure shear is,

$$\sigma_1 = -\sigma_3 = \tau;\ \sigma_2 = 0$$

For pure shear; that means when you are applying torsion, you plug that in your expression here. We get,

$$\sigma_y^2 = \sigma_1^2 + \sigma_3^2 - \sigma_3\sigma_1 = 3\sigma_1^2 = 3\tau_{\max}^2$$

This is how the Mohr's circle looks like for pure shear loading scenario.

$$\sigma_1 = \frac{\sigma_y}{\sqrt{3}} = 0.577\sigma_y = \tau_{\max}$$

During a pure shear loading, the material is going to yield when the principal stress is only 0.577 times the yield strength of the material.

Yield strength is measured from a uniaxial tension test. When the maximum shear stress in the material at any point reaches 0.577 times the yield strength of the material, that is when yielding happens or failure happens. Here when we say failure, it means the onset of plastic deformation.

In other words, this means that the shear yield strength of a ductile material is 0.577 times the tensile yield strength of the material, i.e.,

$$\sigma_{y_s} = 0.577\sigma_y$$

As we have already discussed, the distortion energy theory is a failure theory that is applicable for ductile materials that show yielding; it is not really a good theory for brittle materials.

Ductile failure from distortion energy theory: Failure in case of ductile materials in static tensile loading is considered to be due to shear stress.

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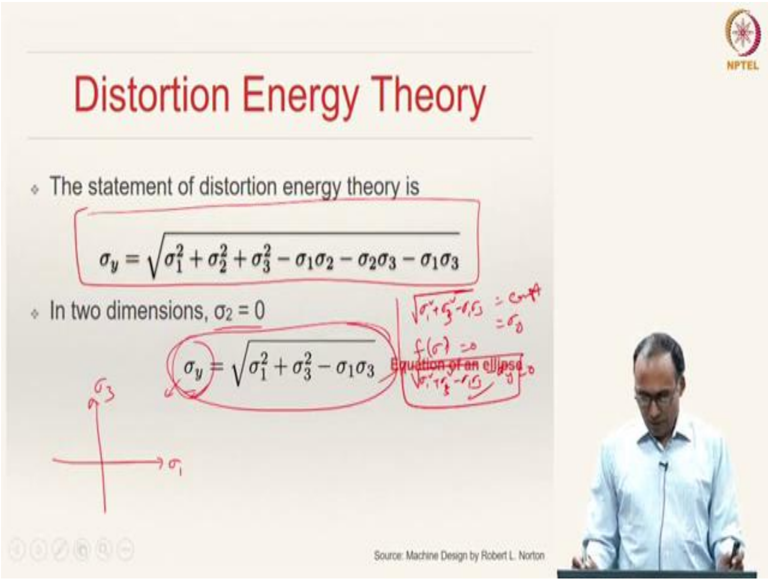

The mathematical statement of distortion energy theory is this, which basically represents the boundary of the failure surface. In 2D, for a plane stress case, $\sigma_2 = 0$,

$$\sigma_y = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3}$$

The general form of any failure theory is given by

$$f(\sigma) = 0$$

For the plane stress case,

$$f = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} - \sigma_y = 0$$

For distortion energy theory, $f(\sigma)$ can be written that way. It can also be expressed as,

$$f = \sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 - \sigma_y^2 = 0$$

If you would plot this particular function in $\sigma_1 - \sigma_3$ space, where $\sigma_1$is the x axis and σ_3 is the y axis, because you know the value of σ_y, how does it look like? This represents the equation of an ellipse, which looks something like this.

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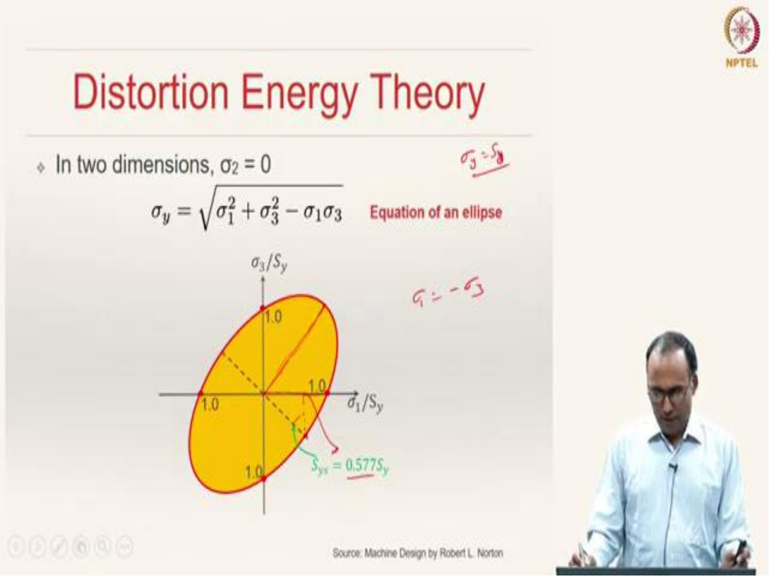

Here, I am normalizing both σ_1 and σ_3 with S_y which is the yield strength of the material.

Failure happens in uniaxial tension when $\frac{\sigma_1}{S_y} = 1$. Please note that here, this distance is larger compared to this distance; that means, if we are loading in this fashion, you have much more room for failure to happen.

This dashed line represents $\sigma_1 = -\sigma_3$ which corresponds to pure shear loading. Then, we will see that this point corresponds to $0.577S_y$.

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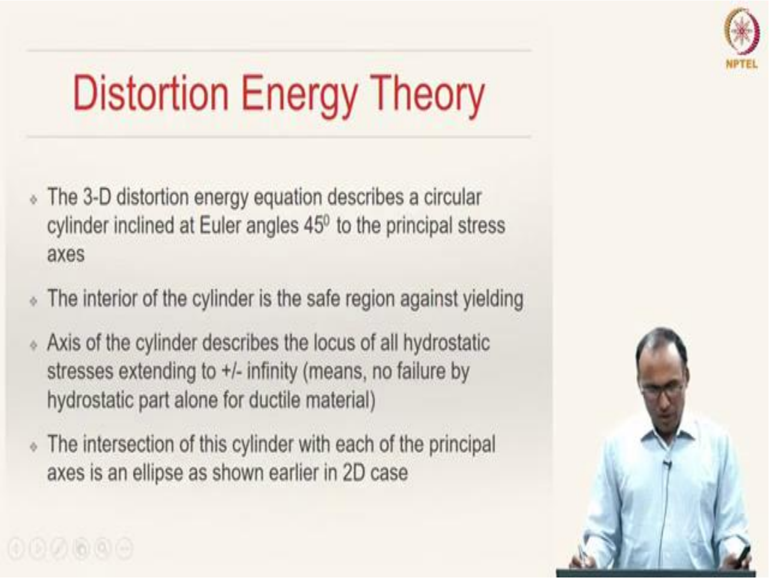

How does the distortion energy theory equation look in a 3-dimensional space where your axes are σ_1, σ_2 and σ_3? The 3D distortion energy equation describes a circular cylinder inclined at Euler angles 45° to the principal stress axis, and interior of that cylinder is safe region against yielding.

Even here, the yellow region is safe, and the red is the boundary at which it is not safe. The moment the stress-state falls outside, that means it is already beyond yield stress which is not a good thing.

You should ensure that whenever you are designing a component, the stress state when you are representing on $\sigma_1 - \sigma_3$ plane should be within the boundary of the red curve. It should be always within the yellow region.

In 2D, it looks like an ellipse and in 3D it looks like a circular cylinder inclined at Euler angles 45° to the principal stress axes. So, the axis of the cylinder in that case, describes the locus of all hydrostatic stresses extending to $\pm\infty$.

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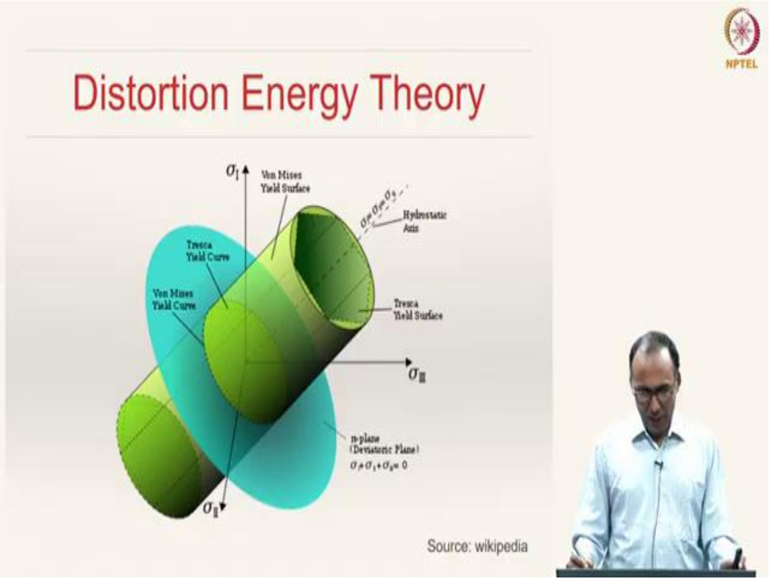

Ignore the hexagonal shape that is shown there, and focus on the circular cylinder; you see that this is the axis. Here you have σ_1, σ_2 and σ_3 and the cylinder is inclined at a 45° angle with all the principal axes.

This line that you are seeing here, represents a state of stress on which $\sigma_1 = \sigma_2 = \sigma_3$. As long as your stress-state is on that line, it will never intersect the boundary; that means that is the hydrostatic state of stress.

The failure or the distortion of the material is not caused by the hydrostatic stress. This yield surface is called von Mises yield surface, after von Mises who is responsible for the distortion energy theory. The intersection of the cylinder with each of the principal axes is an ellipse as shown in the 2D case.

If you cut the cylinder with respect to one of the axes, then you see that it is going to be an elliptical shape. That is the shape that you are seeing in the 2D case.

In the distortion energy theory, the failure surface or the yield surface is an ellipse on $\sigma_1 - \sigma_3$ plane inclined at an angle 45°. In 3D, it is a circular cylinder inclined at 45° Euler angles to the σ_1, σ_2 and σ_3 axes.

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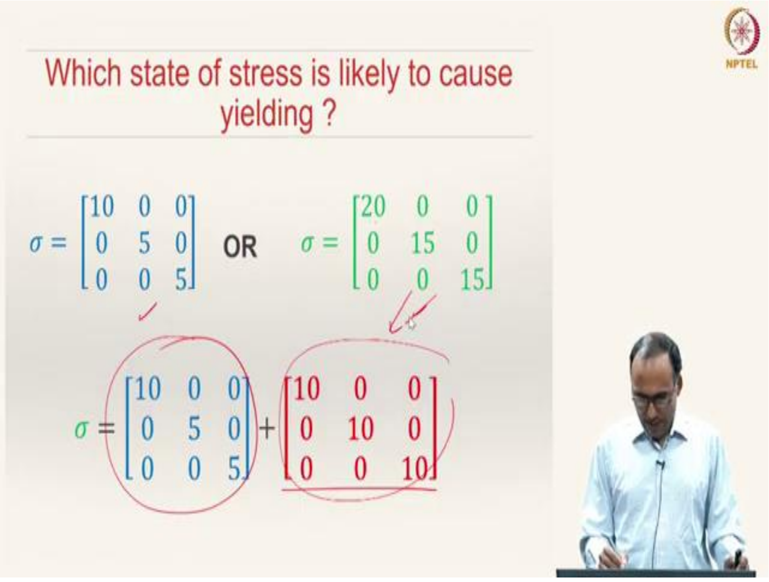

So, that is about distortion energy theory. Now, this is a question that we have started off with, right? Which state of stress is likely to cause yielding? This is one state of stress, this is another state of stress.

If you calculate the hydrostatic part of the stress tensor and deviatoric part of the stress tensor, you see that the second stress tensor can be written as the sum of the first stress tensor and some hydrostatic part, right?

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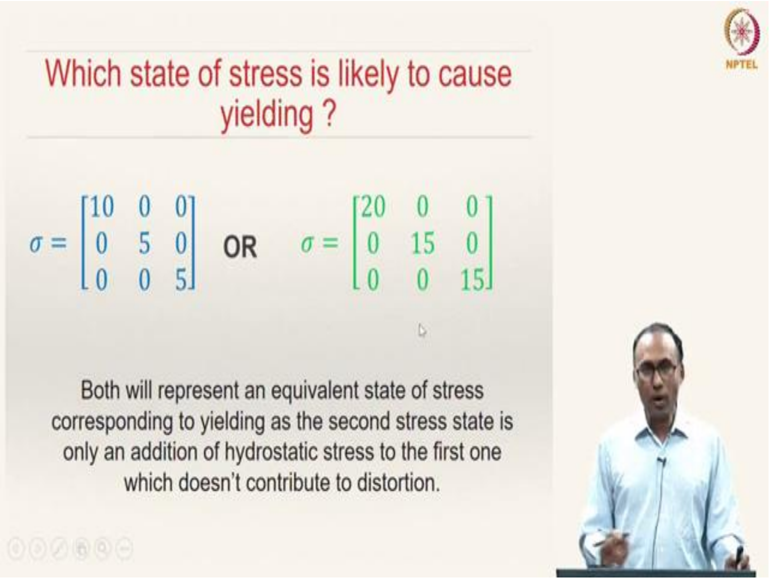

You can write this tensor like this. As a result, both will represent an equivalent state of stress corresponding to yielding, as the second state of stress is only an addition of hydrostatic stress to the first one. You are only adding an additional hydrostatic stress, which does not cause yielding and hence both the stress tensors represent the same state of yielding.

If you would draw the stress-state on $\sigma_1 - \sigma_3$ plane, both of them will show the same point, because it is a 3D state of stress.

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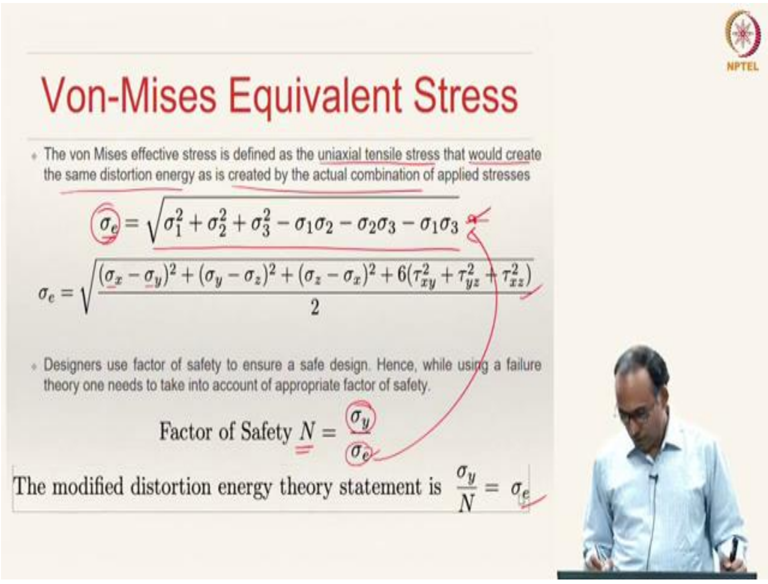

The distortion energy theory states that, this stress which is a complex equation taking into account the elements of the principal stress tensor -- this value is compared with a scalar which is σ_y.

This value can be given a name called equivalent stress. This is what we call von Mises equivalent stress or von Mises effective stress and is defined as the uniaxial tensile stress that would create the same distortion energy as is created with the actual combination of applied stress.

$$\sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

If you are doing a uniaxial tension test, what would be the stress that you would apply in order to have same amount of distortion energy as compared to this combined state of stress, so that will be this value. So, that is what we call equivalent stress.

Basically, the equivalent stress is a scalar quantity; please keep that in mind, it is not a tensorial quantity, because it is a one-dimensional equivalent of your combined state of stress. You can write the equivalent stress both in terms of principal stresses or using your full stress tensor, which is $\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$. So, both the expressions are possible and you must have learned this in your strength of materials class or mechanics of materials class.

When you are designing components, designers typically use something called factor of safety; that means they do not want to design their component such that under the action of external loads it is going to experience yield strength. They would want to design their component, such that the component experiences a stress lower than the yield strength of the material, right? Hence, they use factor of safety to ensure safety of design.

The factor of safety for the distortion energy theory can be written as

$$N = \frac{\sigma_y}{\sigma_e}$$

The modified distortion energy theory statement compared to the previous long mathematical expression can be written as

$$\frac{\sigma_y}{N} = \sigma_e$$

If the equivalent stress reaches the value $\frac{\sigma_y}{N}$, when you are taking factor of safety into consideration that is when failure happens; otherwise the failure does not happen.

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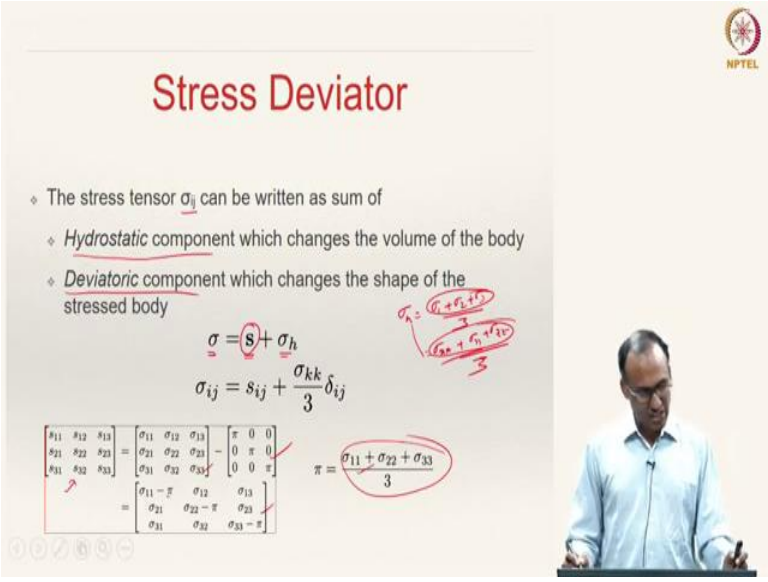

Let us now look at the deviatoric part of the stress tensor, that is also called stress deviator. So, the stress tensor σ_{ij} can be written as a sum of hydrostatic component and deviatoric component.

$$\sigma_{ij} = s_{ij} + \frac{\sigma_{kk}}{3}\delta_{ij}$$

The deviatoric stress tensor is usually represented with the symbol s. So, the full stress tensor sigma can be written as the sum of deviatoric stress tensor and hydrostatic stress tensor.

Please note that

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

Both are the same as the trace of stress tensor which is the sum of the diagonal elements, is invariant.

The deviatoric stress can be written as $\sigma_{11} - \pi$; I am calling that as π.

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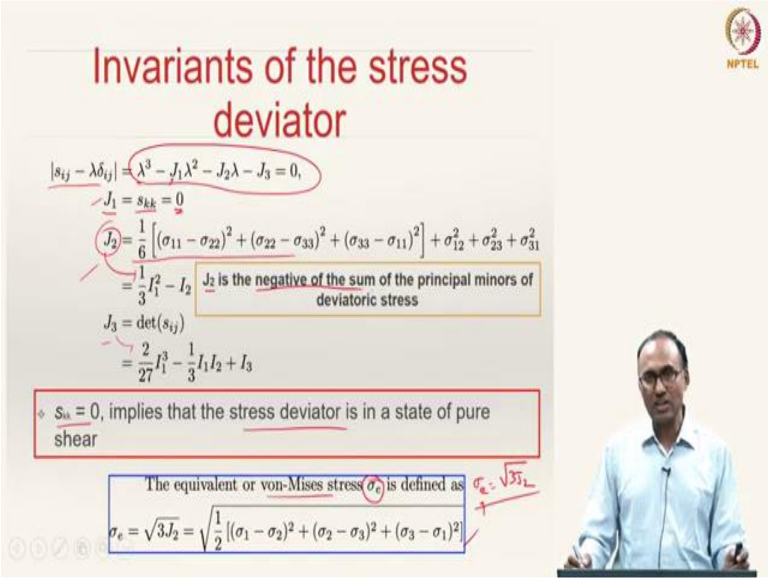

Now, let me try to write the invariants of the stress deviator.

$$|s_{ij} - \lambda\delta_{ij}| = \lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0$$

This is the characteristic equation written in terms of the invariants of the stress deviator, which are represented by J_1, J_2 and J_3. I_1, I_2 and I_3 are the invariants of the full stress tensor.

J_1 is the trace of the stress deviator. What is the trace of the stress deviator? We can clearly see, because it is a stress deviator; it will only cause distortion, no volume change and the diagonal elements actually represents the volume change. So, first invariant J_1 is always 0.

J_2 which is the second invariant of the stress deviator is given by this expression.

$$J_2 = \frac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 = \frac{1}{3}I_1^2 - I_2$$

While I_2 is the sum of the principal minors of the full stress tensor, J_2 is the negative of the sum of the principal minors of deviatoric stress tensor.

$$J_3 = \det(s_{ij}) = \frac{2}{27}I_1^3 - \frac{1}{3}I_1I_2 + I_3$$

So, $s_{kk} = 0$ means, the volumetric part is 0; which actually means that the stress deviator is in state of pure shear, because it is only creating distortion. So, now, the equivalent stress or von Mises stress; this is also called von Mises stress also be written using this expression as

$$\sigma_e = \sqrt{3J_2} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}$$

So, if you know the second invariant of that stress deviator; then you can calculate the equivalent stress right away.

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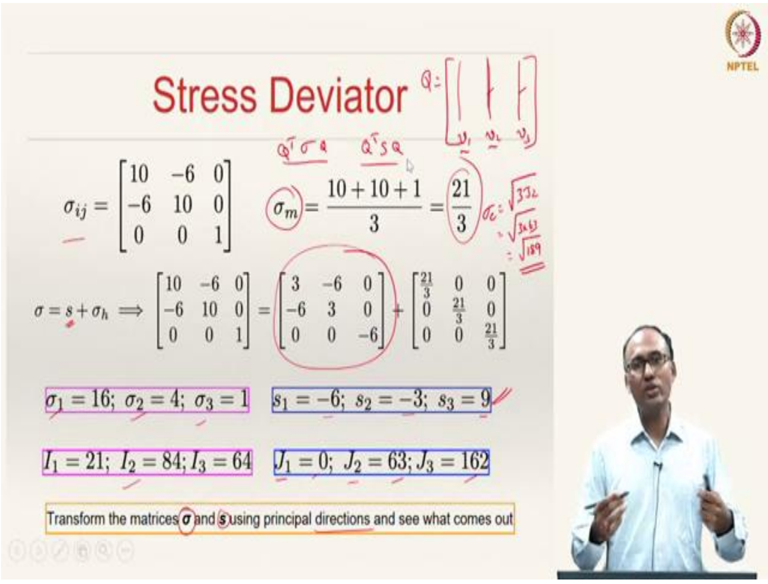

If you now look at this σ_{ij}; from that you can calculate the mean stress or hydrostatic stress to be $\frac{21}{3}$, from that I can calculate my stress deviator, right? The stress deviator is basically this. The principal stresses for the full stress tensor are,

$$\sigma_1 = 16, \sigma_2 = 4, \sigma_3 = 1$$

The principal stresses for the stress deviator are,

$$s_1 = -6; s_2 = -3; s_3 = 9$$

Note that the trace of the deviatoric matrix in the principal frame of reference is $s_1 + s_2 + s_3 = 0$.

And you can see that

$$J_1 = 0; J_2 = 63; J_3 = 162$$

$$\sigma_e = \sqrt{3J_2} = \sqrt{3 * 63} = \sqrt{189}$$

Now, if you know the principal stresses, you can find out principal directions, right?

The eigenvectors will be the principal directions; if you know the eigenvalues, you can always find the eigenvectors. So, the eigen vectors will be the corresponding principal directions for the eigenvalues or the principal stresses.

And if you would use the principal directions to transform σ and s -- let us say you are constructing a matrix Q and let us say the principal directions are $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ corresponding to σ_1, σ_2 and σ_3 respectively. Then, do the transformations $Q^T \sigma Q$ and $Q^T s Q$, and see what you will get, ok?

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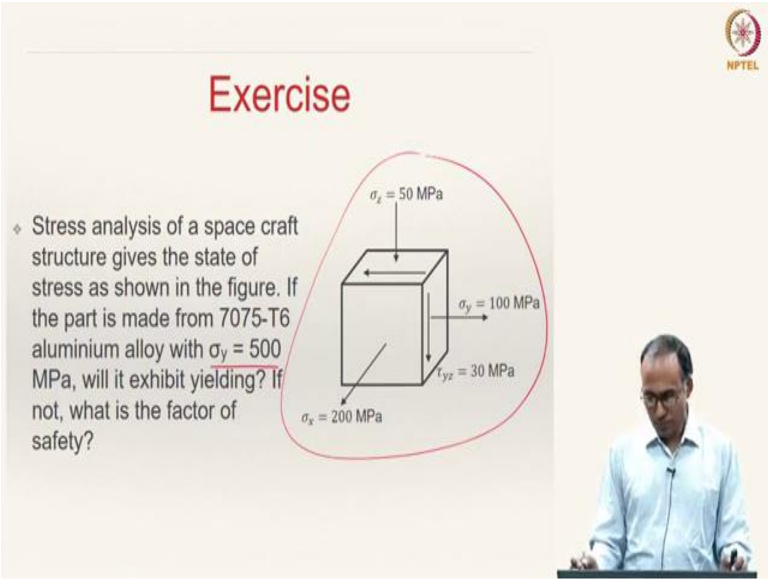

Let us do this problem by applying distortion energy theory and try to understand. So, the stress analysis of a spacecraft structure gives the state of stress as shown in this figure. If the part is made from an aluminum alloy, whose yield strength is 500 MPa. The question is, will this stress state cause the material to yield? If not, what is the factor of safety? It is extremely important to write σ_x, σ_y and τ_{xy} with proper signs.

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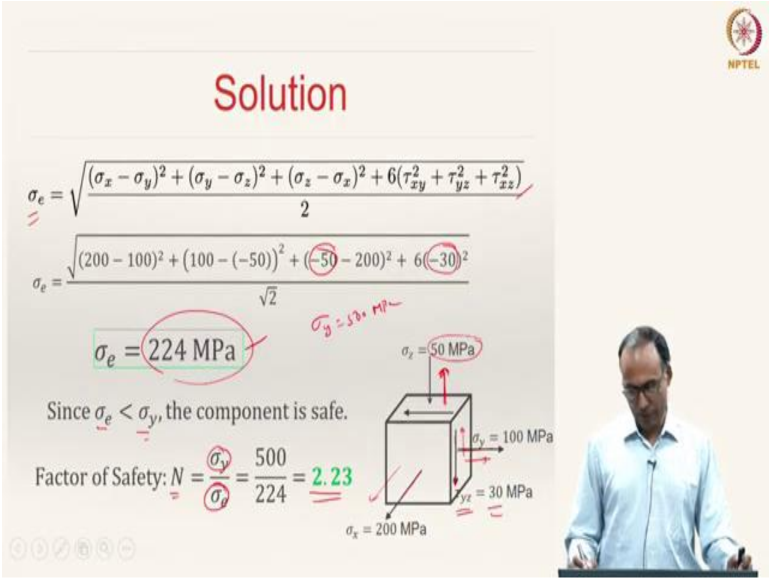

This is our equivalent stress. It is good to write in the x, y, z frame of reference rather than writing in the principal frame of reference; here we are given x, y, z data and not the principal data, because you have a shear stress component here. So, that is not a principal state of stress.

I am writing this in this way. *x* axis is in this direction, *y* axis is in this direction. So, z axis should be in this direction, positive z in that direction.

When you are writing σ_z, it should be -50 MPa because it is compressive, normally this is the direction of your axis, it is going in the negative direction, it should be compressive. So, that is why we are taking it to be negative.

If the shear stress is positive, then that should have been like that; but here your shear stress is negative, because it is in the negative *z* direction. And hence $\tau_{yz} = -30$ MPa. This is important, if you do not take the signs correctly, you will get a different equivalent stress; that is extremely important to realize.

So, you are getting the equivalent stress by this calculation to be 224 MPa and $\sigma_y = 500$ MPa. Since this value is less than σ_y, the component is safe; and the factor of safety is yield strength divided by the equivalent stress, i.e.,

$$N = \frac{\sigma_y}{\sigma_e} = \frac{500}{224} \approx 2.23$$