Surrogates and Approximations in Engineering Design
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Lecture - 12
Radial Basis Function – 1

So, even before we get into the math part of Radial Basis Function, I just wanted to draw an analogy with some stuff that you might already be aware of. How many of you know finite element here or has some exposure have taken a course on finite element?

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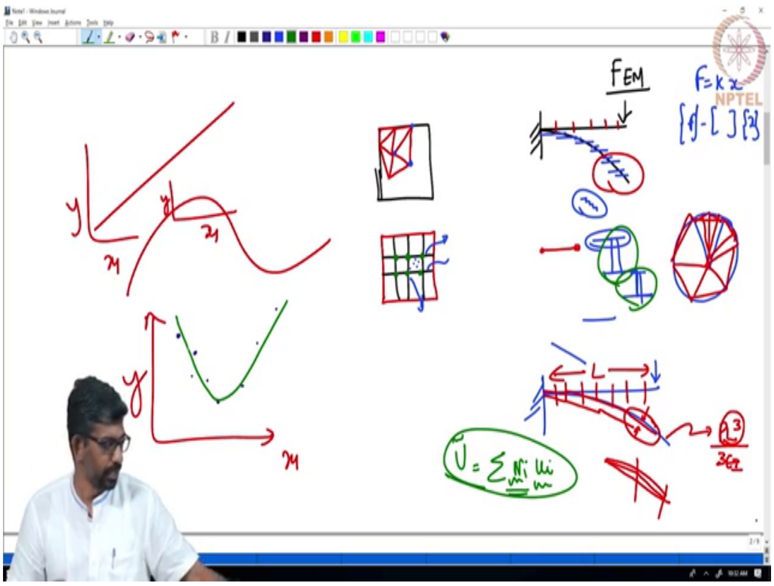

Surprising because I thought that all of you have a structural background, anyway the idea behind finite element is very simple. I mean I will try to explain it with the context that what we want ok. Let us not get into finite element as such. Finite element is also a numerical approximation, where you do not have a close form solution. So, a simple candle over beam of course, finite element can solve more complex problems ok.

The idea is this is your deflection function, but this function also could be complex that is one thing. The second is I do not have an explicit, pretend, imagine that I do not have an explicit equation to compute the deflection. In this case, I know it is p l cube over 3 a. The moment this structure becomes complex, I do not know what the deflection is; there could be multiple loadings. So, in such situations what is advised is you divide this structure into finite number of elements that is what is finite element methods are analysis right.

After this, what you do? Using the non-mechanics, not for the entire structure, but for each of these guys, you can solve for the displacement and the what you call the stresses or whatever the quantity that you are interested in. Usually, we solve for the nodal displacements because at the nodes is what you have the information. The moment I say nodes, it mean there are different types of elements that you can use; the finite element that I am talking about. You can use different types of elements; since this was just beam, you need to use something called a beam element in this particular case.

Let us say that you are interested in something else that is going to vary across the cross section ok. And if you had a surface let us say, then the surface can be approximated by triangles for instance yeah. So, I can keep approximating the structure into triangles and there are also I can also use rectangular approximation, if there is a need. The point is each of this is an element ok; the point that I want you to appreciate is each element has a node. What I will do is I look for the solutions that each of these nodes. I will find out the displacement at each of these nodes, but that is not that that might limit my solution right. I know on the solutions only at the few locations. We know a simple way by which I can interpolate these four to find out the displacement or stresses or strains wherever I want across the structure. This is over all idea finite element.

So, it is a simple stuff that you want to know is F equals k x is what you are going to use; k comes from the stiffness matrix, it is a stiffness matrix. Basically based on the geometry and the material property, you can get it. Based on how you are discretized as it that is all; x is something that you are going to solve for, F is something that you usually know you know what the displacement is sorry what the forces and you know where you are applying it.

Accordingly, there were certain elements will go off in this k and basically you need to do a matrix inversion to find this x, but it is not the straight forward because I could you have million nodes here. Doing a million by million inversion could be a problem and there are some mathematical tools that let us you do this ok. This is the most simplistic way of looking at it.

This is a nice geometric way of looking at the same problem. What you are actually trying to do is; what is the difference between a triangular element and a rectangular triangular element? The triangular element requires only 3 nodes to describe it; whereas a rectangular element requires only 4 nodes. What does that mean? Though I give 4 nodal information for this guy, meaning I will have to calculate 4 nodal information, which means I need a higher order equation. There are 4 unknowns, whereas, in this case there are only 3 unknowns. Whereas, just now we saw with 3 unknowns or with with 2 unknowns for a linear line, for a linear, you can only approximate the linear fitting. If it is a quadratic fitting, then there are 3 unknowns.

So, similarly when you are trying to solve for more unknowns, it means that your approximation is higher order. It is not necessarily true that your approximation is better, it is of higher order that is all; just because you are getting a masters does not mean that you are knowledgeable than undergraduate student ok. But getting a master's degree enables you for a higher order degree that is all; that is what is the idea. But you will have to subject yourself for another couple of years of education. So, more the information or more the unknowns that you need to compute, the larger should be the number of points. So, that is the idea. So, there is more competition for a rectangular element compared to a triangular element, but this is likely to let you model higher order compared to the triangular element.

So, now what happens is the whole idea is I am going to approximate this curved line which is which represents my deflection using multiple approximations ok. I am not going to just construct the one line I am going to approximate it for each of these elements right. So, if I use a linear approximation or let us say a constant approximation that is what a bar elements; the bar element does not vary along your the length ok. It does not take bending right. So, let us say that I use a bar approximation. What it means is I will have my displacements like this ok.

So, which is not correct; so, my displacement should be continuous. It cannot be like this ok, but that is ok, this is ok. This is the overall idea. If you have a circle, let us say you do not know how to find the area of the circle or the perimeter of the circle, but you know how to find the perimeter of triangle. So, what you can do is you can use multiple triangles and you can find what the perimeter is. It is likely to be lesser than the actual perimeter ok. There are two things; this is the lower element because your this is inscribed, whereas if the circle is inscribed, you can do it like this also. You can go out and inscribe the circle inside the triangle. So, that will be your upper element.

What might be what should I do here to get a better approximation of my circles perimeter?

Student: Increase (Refer Time: 08:12).

I should increase the number of triangles. yYou get the idea? This approximation will be better than what I got earlier. Now if I further divide my triangle, I will get even better approximation ok. So, but then I will have to solve for 2 n, then 4 n type of elements not type number of elements ok. So, the number of computations increases, but I will get a better approximation. How good of an approximation you want is what the designer takes a call it ok. And that at some point in time it will converge. Should I use 100? Should I use 1000 or should I use million elements or million triangles in this particular case? You can look for the perimeter convergence or area convergence whatever the quantity that you are trying to do, you can look for a convergence ok. After beyond a percentage it will given to the total number of computation that you are doing; the gain in the error or the percentage by which you are reducing error are will go down insatiably then you stop at that point that is what is called.

So, in a similar sense I am trying to do this, what could be a better approximation is yes, I could use more number of elements that could be one stuff ok. Still what you are trying to do is your trying to do this. I do not know whether you are understanding, if I am going to zoom this; what it means is it is actually not even that it should be like this, just step function. What I could do still because, what do I need to know to represent this particular line? No points is ok, but unknown noise not even 2 points in this case. You need only this is a constant line.

So, primarily, you just need to know x equal to c or y equal to c that is all, but the c varies it is c 1, c 2, c 3, c 4 like that ok, but it is only one that you need to find. Naturally what is my higher order information? I need to be able to find 2 unknowns. So, 2 unknowns mean’s what? I can do this correct. So, now, I am going to redraw that guy here so that so, what it means is, I can do this.

Actually speaking I should draw on top of it, so that you understand. Still I might have this issue. Why? Because this guy is P L cube over 3 E I, he is a cubic. What am I doing? I am trying to represent that guy using a linear line ok. It is varying this is the L. So, L 1, L 2, L 3, L 4; L 4 is still varying in a linear sense for me. So, I made still get this error. If I keep zooming that error will be large actually, ok. So, how can I reduce? There are 2 ways in which I can reduce; one is increase a number of elements ok. So, it is still linear, but then you will that is one way of doing it. The other way; obviously, is to.

Student: (Refer Time: 11:56).

Go for a higher order element ok. So, instead of using a linear approximation, I will use a quadratic approximation. But then quadratic approximation requires?

Student: 3.

3 unknowns in 1D; it requires 3 unknowns. So, which means that you need, 3 nodal points within an element that is a concept of finite element.

The whole idea here what I want you to appreciate is of course, it is great if you appreciate the nodal points then finite element and all that. But what I want you to appreciate is we are eventually interested in the shape of my deflection, shape of my stresses. Yes, there is a map that is inbuilt into it, but what is it that you are trying to look at from a end user perspective is there is some shape that this guy is going to deflect. I want to approximate that shape. So, that I can visualise I can I can make decisions. So, I do not need to approximate the entire deflection, but I will go and approximate it in the linear sense.

So, if the approximation itself is bad, then I will use more number of elements that is one way of doing it or I will enable a better approximation. So, that I can use a lesser there is always an argument ok. Because you still the number of coefficient versus the number of competitions that you need to do which is better; there is always a debate on all the stuff. But the idea that I wanted to you to appreciate here is you are trying to approximate a shape. That is exactly what we are trying to do in the metamodels also ok.

So, you are there is a shape and you are trying to model the shape. This shape could be like this or the shape could be like this. You want to model these x 1 and y, x 1 and y, x 1 and y. The only point is you do not have these, here also you do not have; you are only creating by interpolation. You do not have this curve instead what you have? Sorry and then it is likely that this is going to be the fit for this guy ok. I just give you a set of points, can you give me the curve that is what we are going to look at.

And in a similar, way this can meaning, but in this case we are likely we are looking at a continuous function. We are not looking at a, but this continuous function see eventually, here also what in finite element also what you do with this though your estimating it in an every element, there is also some continuity equation that we enforce ok.

For instance, this step function cannot is not allowed ok, there is some continuity equations, c 1 continuity, c 2 continuity something that you need to bring in because you will not stop with displacement. You want to identify strains; displacement is just the entry point. You want identify strains for which you it should at least be c 1 continuous ok. So, there are some continuity conditions are enforced.

In a similar sense here you want to find a smooth function ok, but the only point is may be each of these functions can have different basis functions, you are only going to linearly sometime. Here also the idea is only linear summation, what we do is usually we write it like this N i U i, U is the displacement, N i U i means it is the shape function; shape function is a interpolation function for each of these guys that is all ok. How we are going to interpolate? It is in a linear sense or in a non-linear sense or in a quadratic, whatever non non-linear quadratic sense. U i is the displacement at each of the nodes that is all.

So, once you find the nodal displacement is only a linear superposition; in the end these wholes could this is it, any complex. So, you go for any kind today people are the using radial any you know nubs non uniform radial basis systems to model this displacement. There also it is still N i U i because I am eventually going to find my nodal displacements and I will do linear superposition of it to find what my total displacement.

In a similar sense here, I want to approximate this function at not at around each of those points, but I will add that that is the overall and that addition will be a linear superposition is was the overall ideas, but you can also do a different types of superposition ok.

So, this is something that I want you to appreciate and we will go into radial basis function. What I will do is I am going to skip a couple of slides to begin with and then yeah. So, we will start with this because right now I gave you the analogy of finite element, we will start with this and then I will go to the math part. So, you will I hope you will be able to appreciate this better ok.

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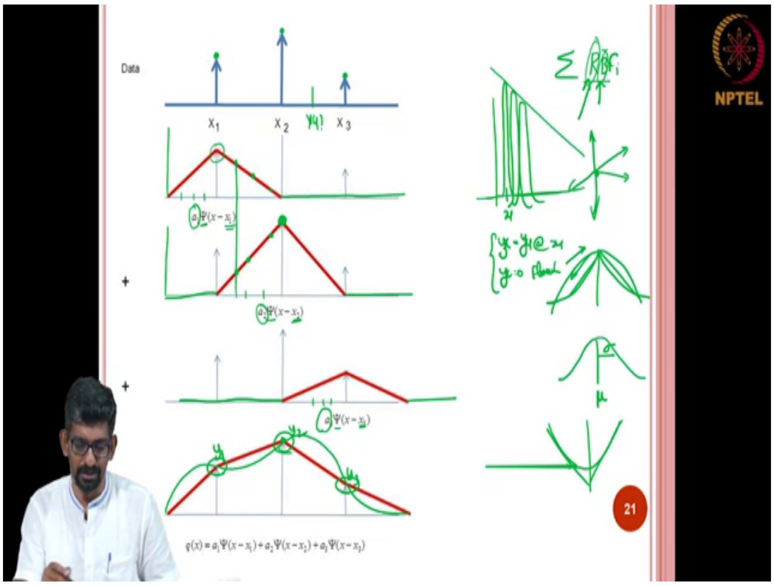

So, I have 3 data conditions here. So, this is a value of the point at x 1 , this is the value of the point at x 2, this is the value of the point at x 3. I have only 3 data points.

Now, what I am saying is this is what I told you right. I want to have some function here and let us call this is the radial basis function i. The form of the function is the same because as you remember, when I did the approximation for the finite element beam, the form of the functions were the same. There are multiple such functions right, it was just linear line I mean it was a either constant line or it was a linear line the value. They took were different, but they were the forms were the same they did not change. In a similar sense, the form of this we will see what this form is ok.

The form of this radial basis function is the same, but I am going to use many of these radial basis functions, how? For instance, this psi describes the function the form of my function. Yes, it is measured in terms of the x wherever you want to estimate minus the x 1 along which you centre your radial basis function. See, if you just thought about it for a couple of seconds, it is very simple it says radial basis. It is a basis function it is in a radial sense radial means what is ok. This is the point and it is in a radial sense. In a radial sense it increases or decreases ok, so, this is in a top view perspective.

If you look in the front view kind of stuff, this could be one stuff ok. It is around the centre and it either decreases or increases towards the centre. I mean it decreases away from the centre or increases towards the centre. And I could also have a slightly complex description, but this requires a little bit more parameter. It is says here is your centre and here is your standard deviation, this is very similar to your Gaussian function and that is why they are also the Gaussian functions are also used in radial basis. And you can say no I do not want this I want some kind of an approximation like this. Yes you can do that you can also do this kind of an approximation ok. So, it is the radial basis that is what I want you to appreciate ok.

So, the function is in a radial sense meaning it increases or decreases monotonically towards your centre that is something that I want you to appreciate. And of course, you can have it like this also if you want ok, the same stuff just a mirror you can imagine the there is a mirror of the thing.

So, now I am just taking this guy ok. If you look at it the only thing, that is going to vary in this case will be a function of the height number 1 and where your x is otherwise it is varying in the same fashion all these 3 guys. There are 3 points, there are 3 data points that I know and I am going to use 3 radial 3 radial basis functions. The form of the functions are the same, in this case if you just call it, it is a linear variation in a triangular sense linear variation in a triangular sense that is all.

So, the psi, the form of the psi is the same and it is dependent on where you are going to estimate with respect to x 3, where you are going to estimate with respect to x 2, where you are going to estimate with respect to x 1 that is all. So, the function is centred this is one way, but you do not need to centre it at that point. One easy way of solving, this is you centre the function and the data points that you have, but there is no need you can centre this here also ok.

Then what you need to do is you need to find the distance with respect to that centre that is all [noise. Here if you centre it, then the mean at the central point is x 1 minus x 1 . So, it will be 0 in this case, x 2 minus x 2 in this case, it is x 3 minus x 3. So, but for at any other location, what you need to do is you need to plug in the x. I know what my x 1 is, I know the form of these equation. So, I will be able to tell you what psi is. So, the only thing that I need to find is my coefficients. These are equal into our ws. Now what do you think the original function is? It will be, because the overall point is I am trying the approximate a generic function.

So, that if you ask me sir, can you tell me what x 4 value is, I should be able to tell what my x 4 value is right. So, now, I want to offer a generic solution to you. So, that you can put the x anywhere between x 1 and x 3 and you should be able to get your y output.

So, what should I what it slightly my next step to get the generic function?

Student: (Refer Time: 22:17).

Sorry.

Student: Linear (Refer Time: 22:19).

That is all. It is just a linear super position just go at this guy this guy and this guy and how is it likely to look like? It is going to likely to look like this. So, if you understand what is going to happen is if there are values in this, see plus ok; from here to here only this function exists. So, only that function existed; from here, whatever this value is I will have to add these values also to it. Then for this particular value, I need to add this value also.

You understand what I am saying right whatever this value was to that I need to add that value also, then whatever this value. I will add this value whatever this value I will add this value, then when it comes here, this value will go to 0, but; however, this guy will start. So, if you look at it whatever value was here is what will be here which is nothing but your x 2's corresponding y 2. This will be your y 1, this will be your y 3 you understand right.

So, I am just adding up see to even exemplify our understanding, you can imagine that this function goes like this. So, you are fitting functions across your domain centred at each point between the point that your fitting and the adjacent point it will vary beyond that it will be equal to 0. You know what a direct delta function is you will have heard what a direct delta function is? The idea is simple what direct delta function says is I will take a particular value at x i, exist across the design space that we are talking about, but I will take a value at x 1 at any other point l b equal to 0. So, the way you write and direct delta function is x takes a value of x 1 x equal to 0 elsewhere or y equals sorry yeah y equals y 1 at x 1 , y equal to 0 elsewhere ok.

So, for instance if I it is very simple if I want to do this, it is the overkill. What you can do is you can have multiple direct delta functions and keep adding all these guys, you will still get this approximation you understand right.

So, this is what we are going to do. This is how the function approximation is achieved. So, this is a function that I wanted to, I do not know see the point here is a point. I do not know whether this is the actual function. This is my output function my actual function could be this, my actual function could have been this, but the 3 points that I estimated it was like this. And I assume a linear in this case, linearly varying radial basis function is what I have assume.

So, this is something that you have to appreciate because you have decided the form meaning I gave you the form. As a designer, we have decided what the form of the equation is. If at all you believe that it could be this kind of a function, then you should use maybe more number of nodes that is one thing or you should use a slightly complex radial basis function that we will see next ok. But we will take a simple case here, we have taken a simple case and we do this. Kiran do you have a question, you are about?

Student: Does it pass through does approximation, does it pass through out like.

In the case is in which we have built it is like that, there is a concept of noise ok.

So, let us say that these data are experimental and you believe that there is noise into it then you might want to average it.

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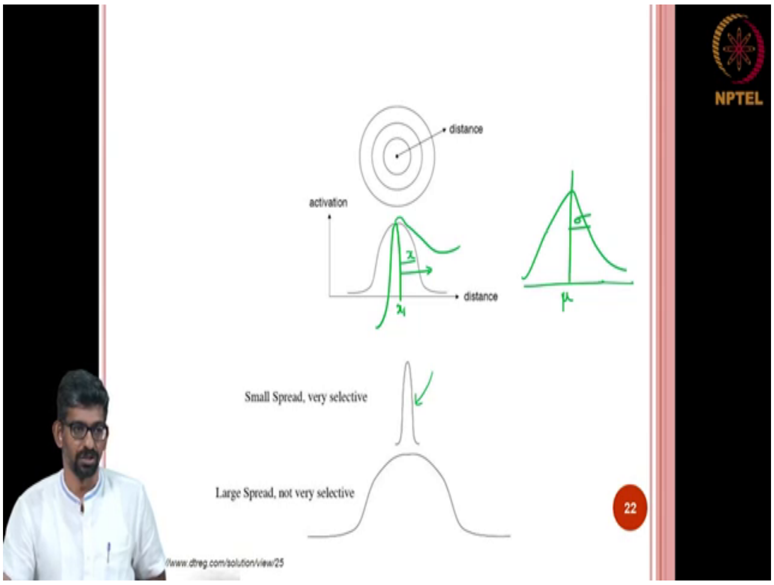

For instance, this is what I meant. When you see from the top, this is a dot at which you want to centre the approximate function that you I mean that the radial basis functions and this could be a radial basis function. So, this is the thing, x i and whatever you want x; this could be an x.

So, you are going to estimate the function with respect to its centre, you understand right? You say I want to estimate the function at this point x; x with respect to the centre. There could be another point x 2, but to the right of it you say from x 2 ok. In this case, it is x 1 let us say that is why x i is ok, the top view of this is this the front view of the function is this ok. This kind of 2 D imagine that there is a function here. So, the top view of this is like this imagine your thumb imagine your thumb and then like Gaussian distribution ok. There are 2 distribution that intersects and then that forms here thumb ok. If you look at this, it is a marginal distribution. If you look at this, it is a marginal distribution that is what it is. If you see from the top it is that is what ok. It is radial varying and of course, you can have a smaller spread, you can have a larger spread, but the form is governed ok.

And usually in radial basis, what do you do is; for instance in this one the mean and the standard deviation right. So, I need to tell the mean is something that I need to tell where you are going to position this guy. He can go anywhere in this. So, he can go here he can go here. So, this is one thing and the other one is the standard deviation. So, in radial basis function, what you usually do is you kind of fix these standard deviation, but in kriging, you can change your standard deviation ok, but it allows for lot of other advantages.

You have to be you will have to do little bit more math, but the idea is you have control on your standard deviation. You understand right, for instance in this is standard deviation is the same. It is really not a Gaussian function, but imagine that the way the slope by which it is reducing is the same. But what kriging might say is I could use different standard deviation. This one could be thin, this could in the same problem; the next function could be fat. The other one could be very thin ok.