Mechanics of Human Movement

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Lecture – 30
Kinetics: Angular Motion Part II

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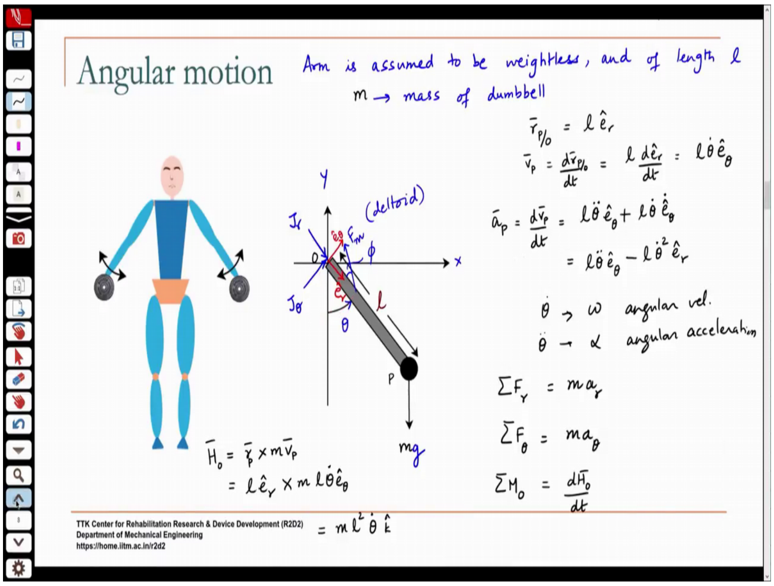

So, yesterday we looked at the angular momentum principle, and so that combined with the Newton's laws will give you the equations of motion for a rotating body. We will use that today to analyze this case of a person using their shoulder muscles to perform an exercise using dumbbells ok. So, let us say we have the arm is assumed to be weightless, and of length l, M is the mass of the dumbbell.

And we will assume that you know at the shoulder, you have reaction forces. So, if this is X, Y, this is J x, and J y, and there is also a resistive movement. So, this is not a free motion. So, it is not like a pin joint about which this arm is going to be freely moving. There is some resistance that is a resistance moment that is being provided by let us say an F m ok, an abductor muscle abductor muscles of the shoulder. In this case, most likely the deltoid muscles, and let us say it acts as an angle phi to the long axis of the arm ok.

So, we will first in some cases of rotational motion, it is actually simpler to use a coordinate system, instead of the x, y Cartesian coordinate system. We will use a coordinate system that is the r theta system, so that is let us say I have a coordinate system that is one aligned along the rod. So, let me call that unit vector e r, and I have e theta. So, one for rotational motion, it is convenient to have e r is your radial direction, and e theta is the unit vector. So, unit vector the caps denote the unit vector along radius direction, and e theta is the unit vector along the tangential direction.

And these can be related to so basically my coordinate system I have if this is i, this is j ok, then this is this is e r, this is e theta, and this angle is theta. So, I can relate the unit vectors of the two coordinate systems. I can say e r what would be e r, so if I take, I can project it on the i, j coordinate system, so I can say e r equals.

Student: (Refer Time: 05:05).

Yeah. So, this is sin theta, so I have sin theta along the i direction, it does not matter how, so here I am taking it.

Student: (Refer Time: 05:21).

In this, so depending on where how I take theta, I can write sin theta i minus cos theta along j ok, this is the y component. So, this is cos theta it is a unit vector, so this is sin theta. So, e r is sin theta along i minus cos theta along j; e theta is these vectors will change with theta e r and e theta will change. So, they are attached to the rotating body, and as the body rotates the coordinate system changes. So, e theta I can write it as if this angle is theta then.

Student: (Refer Time: 06:15).

This angle is also theta. The angle between e theta and i, this angle will also be theta. So, I have cos theta along i plus sin theta along j. Now, these are changing vectors in direction right, unit vectors the magnitude remains the same, but the direction changes. So, I can find e r dot e r dot is cos theta into theta dot, because I do this by d theta times theta dot along i. Differential differentiating this term, I get plus sin theta into theta dot j.

Now, if you look at this, cos theta i plus sin theta j is nothing but e theta, so d e r the rate of change of this vector is nothing but theta dot along e theta ok. Similarly, d e theta by d t equal to e theta dot if I differentiate that, I get minus sin theta into theta dot along i plus cos theta into theta dot along j this is nothing but.

Student: Minus.

Minus theta dot into e r ok so, we will use these, when we try to find the velocities. So, I can write the position of this mass ok. So, r of let us say this is P relative to O is nothing but l along e r ok. So, the velocity of this point P just d I will omit the O, because we are doing it with respect to it is understood that it is with respect to this inertial frame.

We are talking about the absolute velocity, so this is if I differentiate this l dot does not change, there is no change in l the mass is located in that, so l remains l and I get d e r by d t, which I know is theta dot e theta ok. So, l theta dot e theta. This is your omega r that you are familiar with in rotational motion ok. Now, let us look at formulating.

Student: When will this is.

Yes.

Student: (Refer Time: 09:35).

That is an assumption we make. We assume that phi does not.

Student: Change.

Change, we assume the insertion angle of the muscle does not change in the course of the motion. So, now if I find the acceleration of P, I have to differentiate this again. So, I have l is constant first I differentiate theta dot, I get theta double dot along e theta, then e theta is also varying. So, I get l theta dot e theta dot ok. So, this is l theta double dot along e theta plus I know that e theta dot is minus theta dot e r, so this becomes minus l theta dot square e r. Again familiar from circular motion right so, this is theta double dot is generally designated as.

Student: Alpha

Alpha your angular acceleration, and theta dot is your omega ok. So, theta dot is omega angular velocity and theta double dot alpha angular acceleration ok. So, you can see that I can always express this now back in terms of sin theta and cos theta by going back to my i and j coordinates, because I know r, but I will just keep it like this for now ok.

So, now my equations of motion are so sigma F. So, let us say instead of J x and J y ok, it is more meaningful for me, because I am expressing in terms of e r and e theta. I will just call these reactions J r and J theta ok. I am taking two other orthogonal components for the joint reaction J r and J theta. So, now I can write sigma F r my equations become sigma F r equal to mass times the acceleration in the radial direction. Sigma F theta equals mass times the acceleration in the tangential direction. And then sigma M naught, because o is a point that is fixed I can write that as d H naught by d t, where H naught is the angular momentum about that point.

So, let us compute the angular momentum. I have only one particle here, I am assuming the rod is weightless ok. So, my H naught is only r cross m v of p r p cross m v p ok. I am using small m everywhere, so I will just change this. So, this is r p is l e r cross m, v p is l theta dot e theta ok. So, what does this give me? M l square theta dot along e r cross e theta is again just k m l square theta dot k ok. You can also look at it as the moment of inertia of this mass about the point o mass times that l square here ok. So, now if I do d H naught by d t ok, I get d H naught by d t equals m and l square do not change with time, so I get theta double dot k. This is your familiar I alpha I naught alpha, this is alpha.

So, now I can write these three equations to solve for so I have to know theta, theta dot, theta double dot or theta as a function of time ok. Knowing theta as a function of time, I can use these three equations to solve for the three unknowns, which are J r, J theta, and F m ok. So, let me just write down those three equations. Sigma F r this gives me if I do that sigma F r, first I can just solve for if I take moments about o, I am eliminating two of the unknowns. So, it is easier for me to write this equation first this equation.

So, I will do that I will do the sigma M naught that equation and I get F m sin phi times, I should know where the point of insertion is ok. So, let us say that is this distance let us say is a. Again known from anthropometric data ok so, I have F m sin phi into a minus m g ok. The component perpendicular this is theta, so this is m g cos theta, and this component here is m g sin theta. So, and it is creating a clockwise moment. So, taking counterclockwise as positive, this becomes minus m g sin theta into l that is equal to ml square theta double dot. I can write this as a scalar equation because everything is in the k direction, the planar case.

Then, so I can solve for F m from this relation, then using the summation of forces in the r direction in the radial direction. I have J r plus m g cos theta minus F m cos phi is equal to m a r, which is minus m l theta dot square. And summing the forces in the tangential direction, I have J theta minus m g sin theta plus F m sin phi. And what is the acceleration in the tangential direction?

Student: (Refer Time: 19:42).

M into l theta double dot so, theta as a function of time theta, theta dot, theta double dot can be measured in a lab setting. So, using that because these are things that cannot be directly measured, your internal forces cannot be directly measured. So, you can use a model like this to estimate the internal forces based on external measurements of the kinematics, and the known external forces known forces ok.

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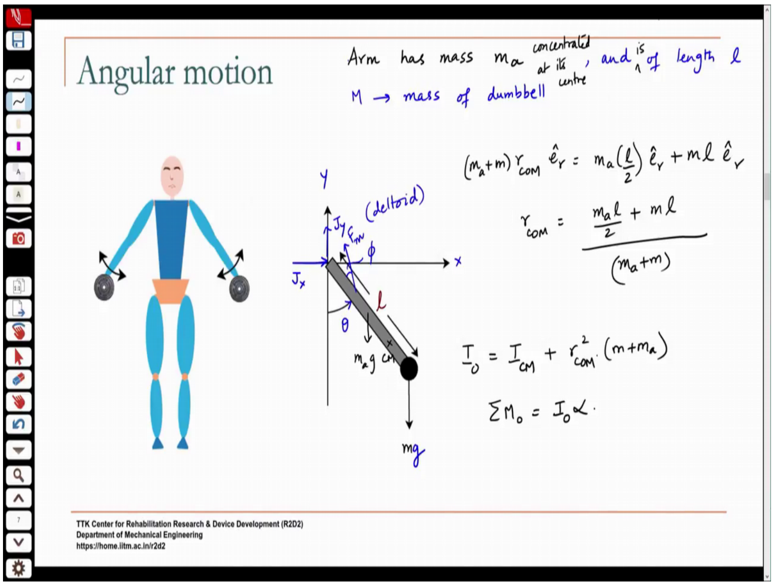

Now, if you consider the arm to have a mass m a, then what would you do? Here we assumed arm is weightless, arm has mass m a, and is of length l. Then you would have let us say it acts at the middle, you have another m a g ok. Now, what would you do? Let us say we assume the mass is concentrated here ok. Then what you would do is you would basically just find the center of mass of these two masses. What would be my if I take r COM ok, along e r right that this would be m a plus m this would be equal to.

Student: (Refer Time: 21:41).

M a into.

Student: (Refer Time: 21:43).

L by 2 e r plus m l e r. So, my r COM would be m a l by 2 plus m l by m a plus m. So, I would just have to find the resultant center of mass, assuming these are point masses ok. We will come to a problem, where instead of assuming it as point masses we consider it a uniformly distributed mass ok. So, here it is you will just

Student: (Refer Time: 22:28).

Lump these to find the center of mass lump these two masses there, and perform the same analysis. So, instead of L you would have r COM, and the new mass of m a plus m in the previous equations, because the velocity everything you would compute for the center of mass. So, when you apply Newton's laws here, when you say sigma F x or sigma F equal to m a the acceleration is for the center of mass of the system.

Student: (Refer Time: 23:04).

Inertia you will have to calculate, inertia you would have to calculate ok. But, again if you are assuming that the sum of these two point masses, so if so if you say arm has a mass m a concentrated at its center ok, then your moment of inertia will be different from if you assume its uniformly distributed, and we will do that the next we will look at a case when where the mass is uniformly disputed, because that you will take into account in your in computing the angular momentum.

Student: Ma'am, even this case when you will be substituting a point center of mass.

Student: Then also you have a movement which is about of these two masses about the center of mass which will a, which will have to add in that rate.

Yes.

Student: So, that also we have to calculate in this case.

Yes, in this case you will have to calculate the center of mass. So, the center of mass will be the center of mass about the c g of this these two about the center of mass of these two bodies.

Student: (Refer Time: 24:17).

Plus the you will use the.

Student: Parallel

Parallel axis theorem to then.

Student: (Refer Time: 24:24).

Compute it about o. You compute the I about the so if this is say the center of mass, you will compute I about the center of mass, which will basically be you know m a times this distance square then.

Student: (Refer Time: 24:44).

M times this distance square that will be your moment of inertia about the center of mass of the system. And then to compute the I about o, you will do this plus r COM square into m plus m a ok. When you compute the center of mass about the if you do the angular momentum formulation, it will kind of come in to the it will automatically come into the equation. This you need to do separately only, when we are when you are directly using sigma m naught equal to I naught alpha ok; if you use the angular momentum formulation, and then differentiate that to get it will automatically come into the equation.

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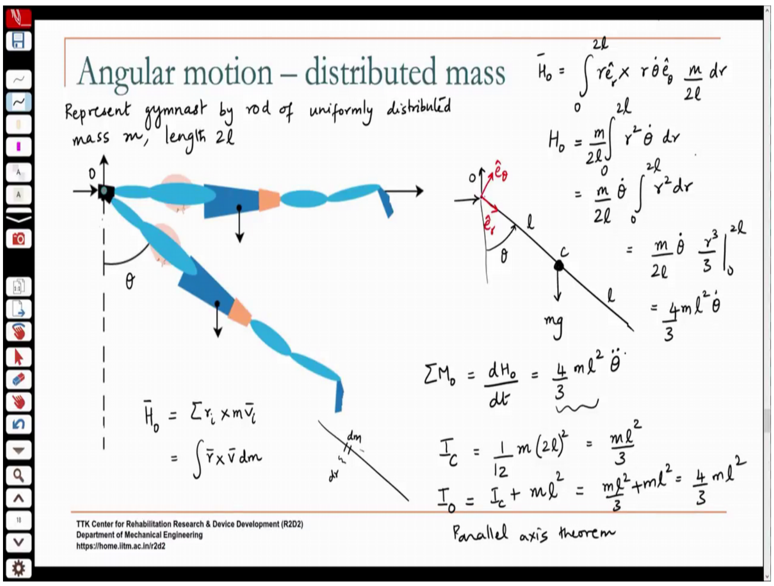

Now, let us look at the case of angular motion. So, you have a gymnast ok. Here you want to really find out, how the gymnast is going to move under the influence of gravity. So, the gymnast is holding on to the bar with the wrists ok. So, it is essentially there is no moment being supported ok, you are not resisting any moment. The pin joint cannot support cannot resist a moment. So, they are essentially going to move down under the action of gravity. And we can try to find the equations of motion of this movement ok.

In this case, let us say that you can represent the gymnast by a rod by a rod of uniformly distributed mass m, and let us say the length is 2 l ok. Let us say, so if I look at the cg, so if this is o, I have the cg will be at l. So, let me draw the this would be m g this is point o, I have the reactions at the pin joint, which is at the wrist, and then this distance is l, this distance is l. So, the wrist acts as a pin joint about which the gymnast rotates.

And let us say the gymnast is let go from a horizontal position. So, somebody you know you have seen that in the Olympics, they will hold them in that position and then let them go. Let go from the horizontal position, they sometimes do this upside down, they have their feet into the rings. And then they will have their arms outstretched, and then move down like that ok. So, they can be modeled as a rod like this.

So, if I look at this point c representing the center of mass, I can write. So, again I can choose a coordinate system e r e theta, and I can write r of the center of mass is l e r. The velocity of the center of mass is l theta dot e theta ok. And the acceleration of the center of mass is l theta double dot e theta the tangential component minus l theta dot square e r; the centripetal component and the tangential component of the radial component, and the tangential component.

So, let us formulate the equations. So, first let me try to compute the angular momentum in this case, where it is a uniformly distributed mass as opposed to a concentrated point mass that we had in the previous case. So, if I look at the angular momentum about o, you know for discrete particles we said it is sigma r i cross m v i ok. Now this is a

Student: (Refer Time: 31:47).

Continuous rod of mass m so, we will write this as an integral, we will say this is r cross v d m ok. So, I take a slice of mass d m, because the velocity in rotational motion the velocity of every point is going to be different. So, for each d m the velocity is going to be different, it will depends on its

Student: Radial.

Radial distance from the center of rotation ok so, r cross v d m is my angular momentum about o ok. Now, how can I express this d m, The total mass of the rod of length 2 l rod of length 2 l has mass m. So, if I take a small mass d m, I can write this as m by 2 l.

Student: D l.

D l ok so, this d m has a length d l or d r sorry I should write it as d r. This element has mass d m, and is of length d r ok. Now, I can integrate this H naught equals integral 0 to 2 l d r varies from 0 to 2 l of m by 2 l, let me put it r cross v is what is v r theta dot e theta right, and d m is m by 2 l d r. So, r e r times e theta is just k.

So, I can remove the just call this r e r in vector form it is that or in scalar form because everything is along the k direction. I can write this as 0 to 2 l m by 2 l is a constant, it comes outside. And I have r square theta dot d r. Theta dot is also a constant for all the points on the rod the angular velocity is the same ok.

So, I can bring theta dot out as well 0 to 2 l r square d r. This would be r cube by 3 0 to 2 l. So, I get 4 by 3 m l square theta dot that will be my angular momentum. The sum of the moments equals d H naught by d t. If I differentiate this, I get 4 by 3 m l square theta double dot ok. Now, let us see if whether this is your familiar I alpha ok, you have this rod of length 2 l ok. It is moment of inertia of a uniformly distributed rod about its center of mass is 1 by 12.

Student: L square.

M into l square in this case it is

Student: 2 l.

2 l. length square ok so this gives me m l square by.

Student: 3.

3. Now, the moment of inertia about the point o by the parallel axis theorem equal to I c plus m times the distance here is

Student: L square

L square ok so, I have m l square by 3 plus m l square, which is 4 by 3 m l square. And theta double dot is my alpha ok. So, this shows that this is equal to I naught alpha ok.

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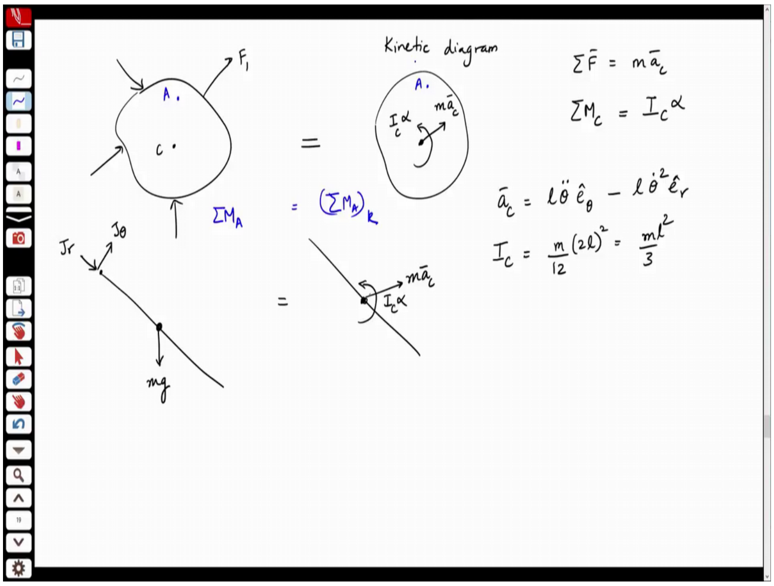

So, now I can find the reactions ok, I have are you guys familiar with the kinetic diagram, have you heard that term before in a dynamics problem. If you have a rigid body ok, acted on by multiple forces ok and let us say its center of mass is here ok. And this is equivalent to this the rigid body, you have mass times the acceleration of the center of mass. And I alpha I c alpha, this is called the kinetic diagram.

The resultant of all these forces equals this; it is essentially using the neutral Euler equation. So, you have sigma F equal to m a center of mass, and sigma M about the center of mass equals I c alpha, where alpha is the angular acceleration. So, this is how you determine the linear and angular acceleration said that. So, using the so instead of formulating angular momentum each time, you can always use this kinetic diagram. If you know the moment of inertia about the center of mass, then you can use that to find you can use this diagram to set up your equations of motion ok. So, in this case here, we have J, J r, J theta I have m g those are the only forces, acting on this ok. This is equal to this is the center of mass, I have some acceleration and I c alpha.

Student: (Refer Time: 40:53).

So, this I should not draw this neatly perpendicular to it, I should draw it some random m a center of mass m a center of a center of mass equals both the radial and the.

Student: Tangential.

Tangential components so, if you look at the previous slide, this is c right. We found the acceleration, can you tell me what, we found the acceleration to be we found acceleration of c as.

Student: L.

L.

Student: Theta double.

Theta double dot.

Student: E theta.

Along e theta minus.

Student: L theta dot.

L theta dot square along e r that is a c. And what did we find I c to be.

Student: (Refer Time: 41:47).

M by 12

Student: 2 l.

2 l square so.

Student: (Refer Time: 41:53).

M by m l square by 3 ok; remember we have taken l as 2 l here that is why, usually you will see m l square by 3 for the moment of inertia about the fixed pivot.

Student: It will be change.

Yeah, that will change.

Student: Where writing of c then we will find that.

Yeah. So, now when you have the kinetic diagram, the reason it is helpful to draw the kinetic diagram is like this equations are applicable to the center of mass. Now, if I want to take moments about some other point A ok, then I have to make sure that this the moments about A equals the moments about A in the kinetic diagram. So, that way I will not miss out this m a c taking the moment of that m a c force about this other point.

Because, otherwise you will just I will show you in the next class, why that is the same as doing the I naught alpha, when he is a fixed point ok. We will stop here, because we are out of time. But, using the kinetic diagram, we will prevent you from making some of these common mistakes, when solving dynamics problems ok.

So, we will get to that in the next class.