Transport Phenomena in Materials.
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Lecture - 09
Derivation of Navier Stokes Equation

Welcome to the session on Navier Stokes equation as part of the NPTEL MOOC on Transport Phenomena in Materials. In this session we are going to derive the Navier Stokes equation starting from the Newton's second law of conservation of linear momentum.

So, this session may look a little tedious; however, I advise that you pause at appropriate locations, do some of the algebra yourself and then get back because it could be quite tedious if you are not familiar with the subscript notation, the tensor operations and the material derivative concept etcetera. So, all the things that we have done till now will all be converging into this particular derivation, so do pay attention what is coming at what stage go back and refresh those concepts if necessary.

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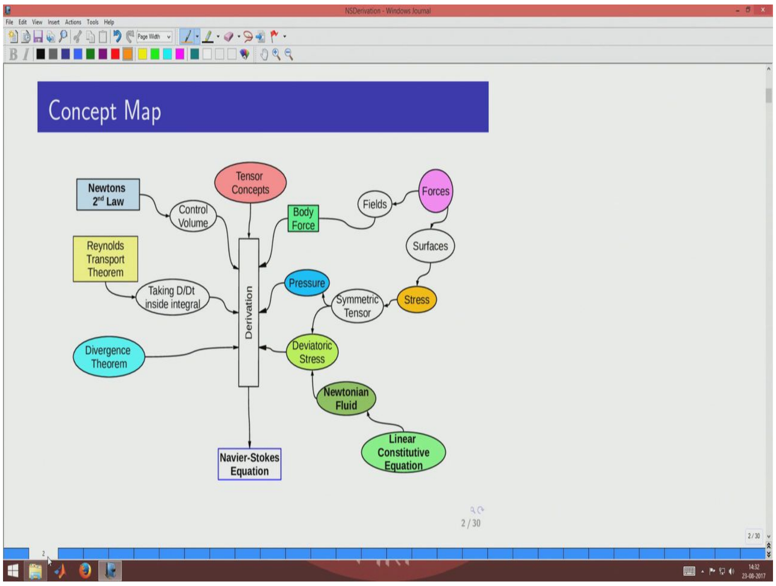

So, the concept map for the derivation of Navier Stokes equation is given here. The starting point for the Navier Stokes equation is nothing, but the Newtons second law the conservation of linear momentum and we see that it basically is talking about the velocities and accelerations and how they are related to the forces.

Now, the difference between how we did that in high school and how we are going to do it in Navier Stokes equation is that we are going to apply this to a control volume and the moment we say that then we write expressions as integrations because we do not want to be constrained by the shape of the entire body. So, we are going to write integrals over the control volumes. And we also will be borrowing the concept of Reynolds transport theorem because at some point the rate change will be then expressed in Eulerian specification. So, we have the need for the $\frac{D}{Dt}$ to go inside the integral. So, we use the Reynolds transport theorem there. And we will have quantities which are specified with dot products with the surface vectors and then we can convert them to volume integrals using the divergence theorem and the flow is basically governed by forces and we see the forces as two different types. So, we see forces as fields which will be then coming as a body force and we can also see the forces applied on surfaces which are basically stresses and we see why the stress tensor can be called as symmetrical tensor we will see very briefly one discussion to convince you on that. And then we are going to decompose the symmetric stress tensor into two components the pressure component the way we know the pressure and the deviatoric stress component.

Then we are going to relate the deviatoric stress to the velocity gradients through a linear constitutive equation introducing the concept of a Newtonian fluid. And then we put this all into the derivation along with the tensor concepts like a symmetric tensor when it multiplied with asymmetric tensor and then summate them you get a 0 and how the isotropic tensor of order 4 can be written and so on. So, all these things put together will give you the Navier Stokes equation.

So, you can see that in this concept map each of these concepts can be studied separately and we convinced about before we start this derivation. Make sure that you are familiar with all this background concepts and the basic point that we must never forget is that it is no different from Newton’s second law because that is the starting point for us if there is no more physics than what is there in the Newton second law that is going into the Navier

Stokes equation. So, we start of by looking at these terms here we saw that the forces are necessary and so we are going to describe them.

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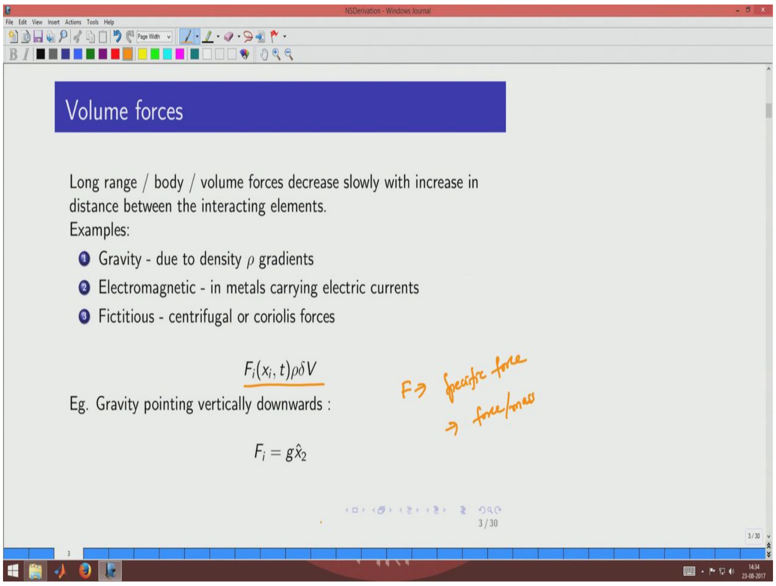

So, there are two types of forces that we are going to consider the long range forces or the body forces or volume forces. These are the ones which decrease slowly with the distance between the interacting element which means that fields that are like gravity. Of course, gravity is a function of distance, but for the kind of distances we talk about in fluid flow then they are not very large, typical about few centimeters some may be in meters. So, therefore, you can assume that the gravity field does not change from one part of the domain to another part. So, such things are basically the volumetric forces that we are talking about.

So, most of the time we are only going to look at the gravity, but there are situations where the electromagnetic flow can also be described and therefore, those also can be brought into this. Sometimes we also have fictitious forces that will be coming in fictitious as in they do not have a separate physics that is causing them, but they are coming because we have chosen a different coordinate system like for example, centrifugal or coriolis forces. So, those also can be brought in as volumetric forces like this.

So, anything that is volumetric should be then described in this manner. So, that you have the $\rho\, dV$ coming as mass. So, F is basically the specific force. So, this basically F is the specific force which is basically force over mass. So, you know that the gravitational force is mg and if you divide with m and only g is remaining and therefore, one example for a specific course is just g, so it will have the units of acceleration. And normally we think of y axes going upwards. So, if you want to write in a vectorial fashion you can write it like this $F = g\widehat{x_2}$.

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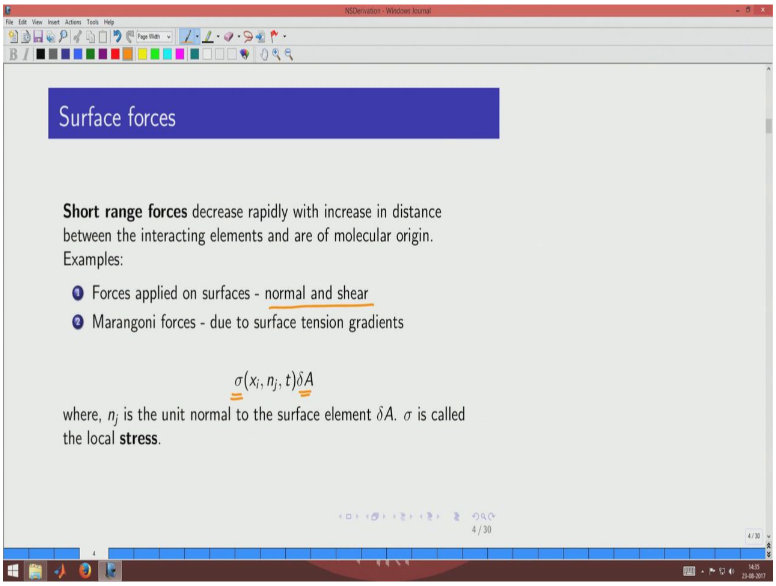

Now, the second type of forces that we are going to use are basically short range forces that is forces that are actually felt only on the surface and not beyond. So, for example, the stresses what we normally talk about are all surface forces because the stress is applied on the surface and then it is going to act on the body and just the force is not actually penetrating into the bulk of the body. So, these stresses are basically both normal and shear together and therefore, we all sum it up and say that it is a stress.

Now, there are also other things that will come in. So, Marangoni forces for example, due to surface tension gradient these are also basically stress that are acting only on the surface and they can also be clubbed up along with stresses like this. So, we have a very generic way of describing them σ. So, it is a tensor of order two, σ into the area is then the force that we are talking about. So, the force is two types. So, one is the body force like the mg and the other one is the forces because of stresses that is basically σ into the area. So, we have got those.

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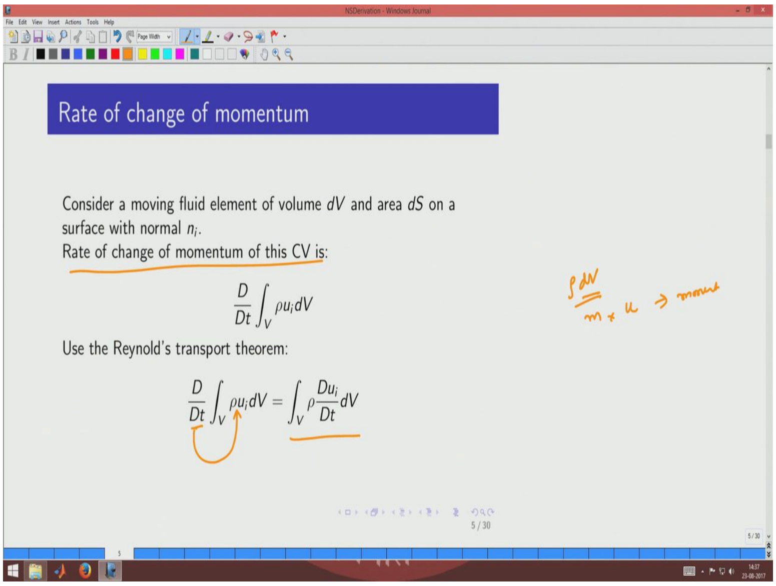

Now, look at what is it that we are going to change the momentum is changing because forces are acting on it this is the summary of the Newton second law. Now, we want to write the momentum as an integral. So, what you are going to is like this. So, we see here $\rho\, dV$ is basically the mass and then you multiply with the velocity and you have got the momentum, which means that this quantity what you have written is nothing but the rate of change of momentum because momentum alone is in this integral and then $\frac{D}{Dt}$ is giving you the rate of change in the Eulerians specification. So, the statement is basically the rate of change of momentum is caused by the forces. So, now, is where the utility of the Reynolds transport theorem is coming in.

So, we could see that the rate of change of momentum when you write like this and you can then take the $\frac{D}{Dt}$ in like this we can by using the Reynolds transport theorem and then you now have rate of change of momentum written in a slightly different fashion where the acceleration term is coming inside the integral acceleration as depend in the Eulerian specification.

So, this is basically now going to be useful because we see that it is over an integral and as long as every other term is over the same integration of over the dV then we can take the integrands. So, that is the strategy why we want these $\frac{D}{Dt}$ is to go inside the integral.

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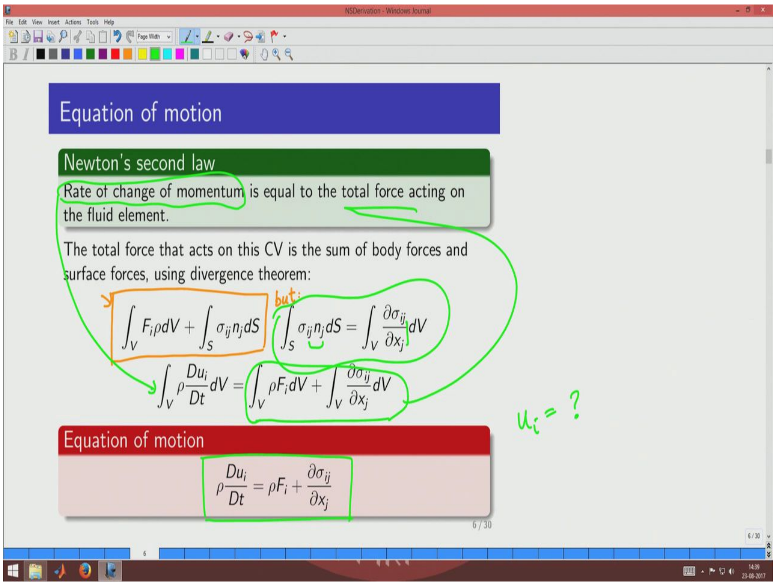

So, we have here the equation of motion. So, the equation of motion can be written in English like this. The rate of change of momentum is equal to the total forces that is, acting on that, fluid element or body whichever bodies undergoing that change of momentum. So, in our case we are taking the control volume. So, the rate of change of the momentum of the control volume is equal to the total force that is acting. We already had written the forces in two ways. So, we have seen that here. So, we have written as two terms the volumetric force the $F_i\rho\, dV$ and the surface forces which is a $\sigma_{ij}n_j dS$. So, we have got those two terms.

Now, we see that the second term we can write it in a slightly different manner we can use the divergence theorem. We already saw that the divergence theorem when we derive the continuity equation we saw it for a vector quantity, but at that time we also mention that you could generalize it. So, we are generalizing this fashion. So, here is the generalized divergence theorem where the index that is matching is what is used for derivation here. So, which is nothing, but basically the stress dot n j is basically $\delta.\sigma$.

Now, we do not want to use those quantities like del dot because it is already implied by the matching of subscripts. So, if you are already familiar with subscript notation you already see that on the right hand side j is matching and therefore, it must be a dot product over that particular index.

So, now you introduce this quantity inside here. So, that the total force is now described in this manner. So, you could see that the left hand side is rate of change of momentum. So, rate of change of momentum is this fellow and the total force is on the right hand side that is the total force. Now you can see that this equation is written over the same control volume everywhere, which means that the integrands also should be the same and that is exactly what we are doing. So, we have written that the equation of motion can now be written without the integrations which means that this is now valid at every location in the domain. So, whichever domain we will choose at every location this particular part is true.

Now, this is not going to be very useful equation of motion looks good it is very brief and conveys the kind of sense that we know from the Newton second law, but it is not useful the sense, on the right hand side we have got σ which we just do not know what to do with because we are actually writing this equation to determine u. So, we want to find out u and therefore, everything on the right hand side should be something that we know, but σ is something that we do not know. So, we need to do something about it and we are going to do that.

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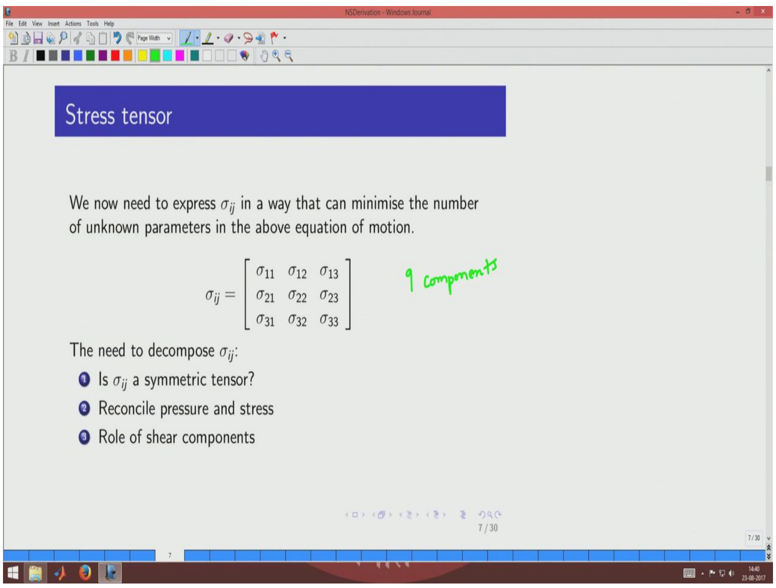

So, what you are going to do is σ is basically a stress tensor and if you expand it because it is a tensor of order two you will have it as 9 components. So, you could see here 9 components here because ij both indices go from 1 to 3. So, we have got 9 components. So, I have intentionally written σ_{12} here and σ_{21} here because it is a most general form.

Now, we can say that how do I reduce this 9 components, I do not want 9 unknowns from the right hand side. So, I want to reduce the number of components the way to do it is to see step by step. First step is σ symmetric tensor if it is symmetric then we know that the 9 will become 6. So, you have less number of unknowns and then if it is. So, then we want then take out some quantity and we already know that the stress and pressure are having similar convertation in planning (Refer Time: 11:10) we say that something is pressurized or something stressed we mean the same thing. So, how do we bring out those kind of quantities and the sense of what we mean by pressure as a that is something to compress a particular body is then preserved, so we will do that. And in the process what we are introduce is on the right hand side wherever σ is we want to change it something that we know or can relate to the real life ok.

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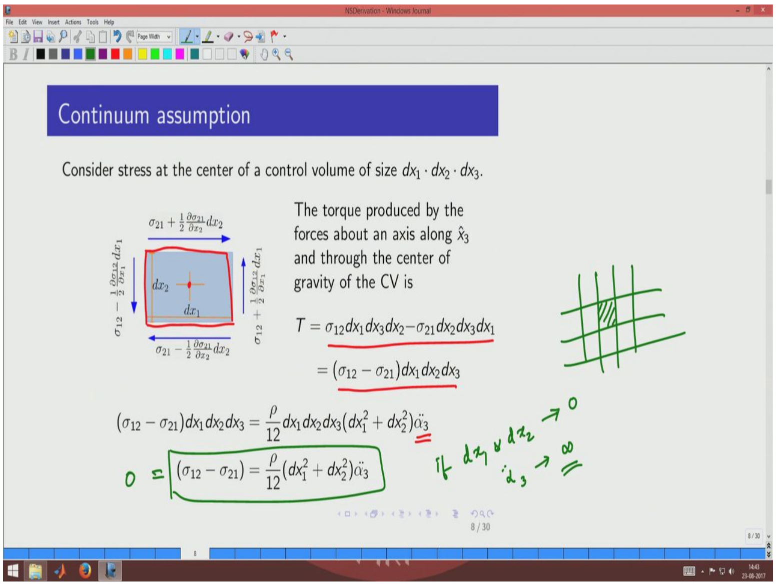

So, here I want to argue that stressed tensor is symmetric tensor. So, you could just simply state that stress is defined for continuum and therefore, it must be a symmetric tensor, but we can see why. So, let us take a control volume that is drawn here. So, the control volume is drawn here in this manner, and at this point we inspect how the forces are going to act if they are going to be caused by the stresses and we have written σ_{21} and σ_{12}, they can also be called as τ_{21} and τ_{12} or τ_{xy} τ_{yz} yx etcetera.

So, it is up to us which symbol we use now the torque which is then acting on at this point is governed by basically the force imbalance which is rotating this control volume about the control over clockwise direction as well as the clockwise directions. So, each of these forces are then going to be summed up and we see that the area element multiplied by the distance you see that the torque is basically the difference in the stresses σ_{12} and σ_{21}. So, if there is actually an imbalance between the two components σ_{12} and σ_{21} it means that on the control volume is the torque that is acting.

Now, what is a consequence of the torque is acting? So, the torque is acting then you can relate it to the angular rotation in this particular manner and you could look up one of these in strength of materials books to write this for a rigid body and blindly apply it for a control volume though it is not a rigid body, but they inspects what happens if we did that. If we did that what happens is that as when you cancel this three volume elements you will see that you have a problem here what happens is that if you choose a smaller and smaller control volume if dx_1 and dx_2, tend to 0 and if the σ_{12} - σ_{21} is finite then what happens is that this will tend to ∞.

So, now, that is a problem because if you have any difference between the two of diagonal terms of the stress tensor, it will lead to a very large or tending towards infinity rotation of this control volume about the z axes which is not allowed because of continuum assumption. By that what we mean is this control volume is stuck to other control elements around it. So, we are define domain like this. So, if this is a control volume of our interest then it is stuck to all the control volumes around it and therefore, it cannot rotate by itself and definitely not at you know such angular rotations speeds and therefore, it means that the only way to avoid this kind of an observed situation is to say that this must be 0 which means that $\sigma_{12} = \sigma_{21}$.

In other words the σ is a symmetric tensor. So, let us then assume that it is true and then go ahead and use that concept.

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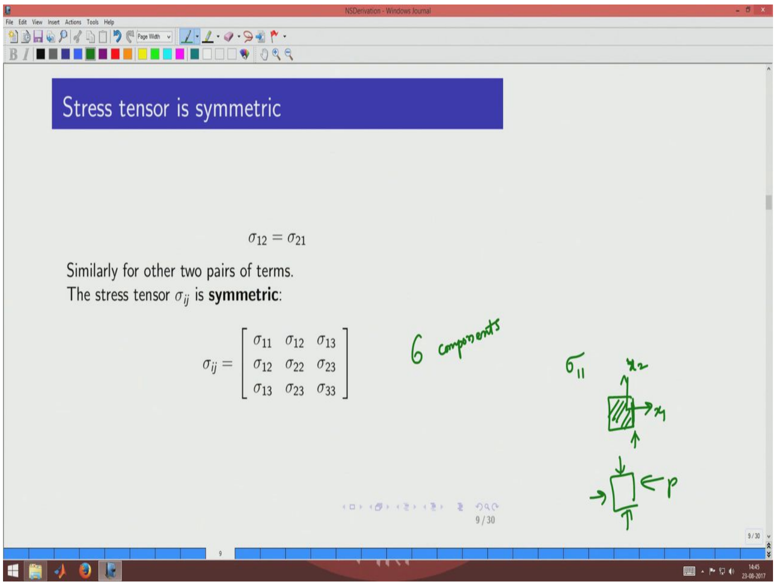

So, we say that if σ is a symmetric tensor we now have written here σ_{12} σ_{13} both ways and σ_{23}, both ways which means that we now have 6 components that are to be determined. So, from 9 we have got 6. So, there is some progress.

Now, let us go further and see what we can do about this? Now, here is where the see the sense of σ and sense of pressure. So, let us consider σ_{11}. So, what does σ_{11} mean? Which means that if you take x_1, x_2 direction which means that the force acting on a plane x_1 in the one direction which means that on this plane force acting with this same x direction which means that it is basically the opposite of pressure because pressure is trying to compress this body. So, we understand pressure in this manner and σ_{11} is basically trying to expand it in the x_1 direction. So, there is a mix up of these two concepts. So, we want to then separate them out. So, that is exactly what we are going to attempt.

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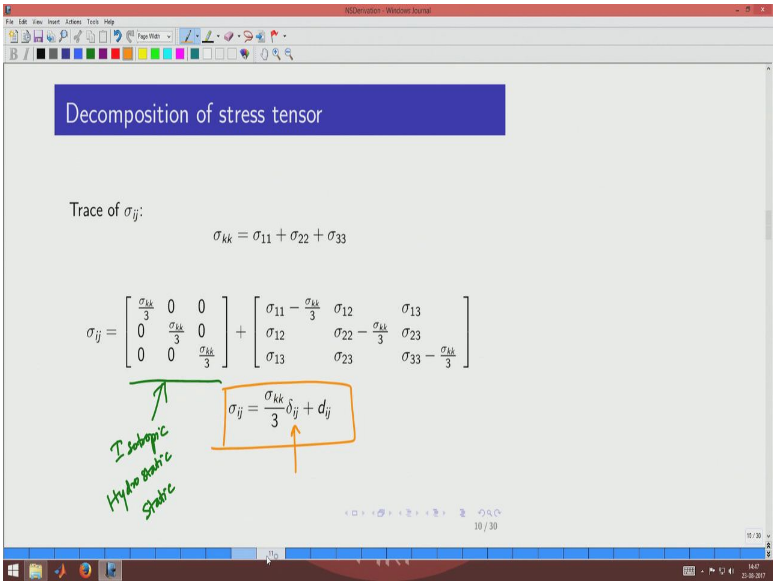

So, what we are going to do is that the symmetric tensor σ is going to be written in two parts, the first part is basically what is going to be isotropic in nature or you could also call it as hydrostatic if the fluid that is in question is water otherwise you just simply called it as static component, and that gives the sense of pressure and rest of it. So, we want to give them names and we can call the first one as related to pressure and the second one as related to something that will make the control volume distort and therefore, we would like to call it as deviatoric stress.

Now, you could see that the quantity that I have used is σ_{kk} which is nothing, but the trace of the stress tensor. Now the reason why we used trace is also for another purpose trace does not change with coordinate system rotations. So, it is a scalar. So, therefore, when you use this expression, the expressions become valid in any coordinate system. So, therefore, we put that here. So, one by three is done. So, that the first part has a trace of exactly σ_{kk} and the second part will have a 0 trace because when you do this summation you see $\sigma_{11} + \sigma_{22} + \sigma_{33}$ minus of these. So, you get them cancelled out and you see that it should you know not have any trace.

So, you now see that you could do the decomposition in a subscript notation form like this. So, what was written in matrix form is now written in the subscript notation because you can see the first part is nothing, but when you take this $\frac{\sigma_{kk}}{3}$ out it is 1 0 0 0 1 0 0 0 1 and that is nothing, but the δ here that is written here. So, you just put it in there. So, we have now decomposed the stress tensor into two parts, one part which has the hydrostatic or isotropic component acting in all directions equally, equally because you could see that all the three components are equal. So, equal stress acting in all directions that is a first component. Second part is whichever is having the shear stress and other components that are there.

Now, d_{ij} will also have 9 elements, but still we will have some way to relate with what we know.

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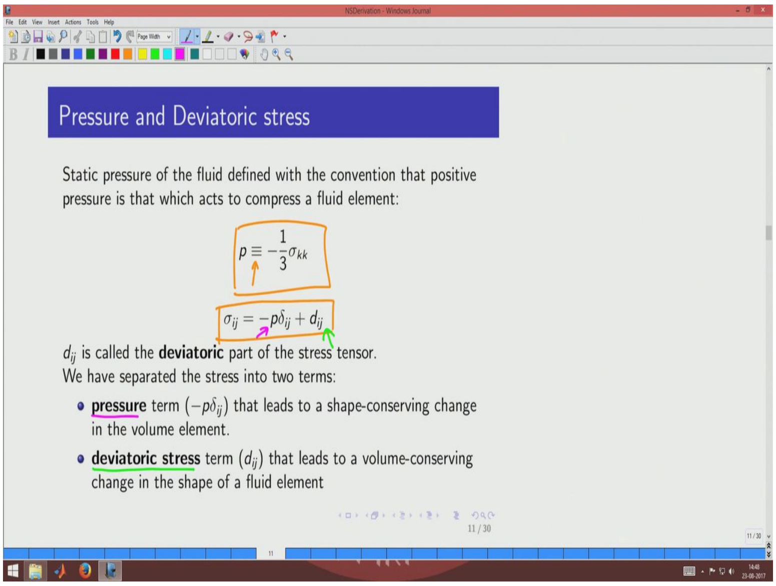

So, once we have done this separation we can then now formally define. So, we are now here formally defining pressure. So, notice that we have got the triple equal which means we define the quantity p. So, we say that the pressure is what is defined as a minus of one-third of the trace of the stress. So, it is basically the isotropic component of the stress tensor with a minus sign in front. Why is minus sign because pressure is always thought as something that compresses whereas, σ_{11} is going in the opposite direction. So, therefore, to make the same meaning come we just put a minus sign there.

So, therefore, we can now write the stress tensor as $-p\delta_{ij} + d_{ij}$ where δ_{ij} is the δ. So, we have separated the stress into pressure term and the deviatoric stress terms. So, deviatoric stress is this fellow, this fellow is a deviatotic fellow and pressure term is this fellow you can see that the stress is now composed of two components. This is going to be useful for us in a moment.

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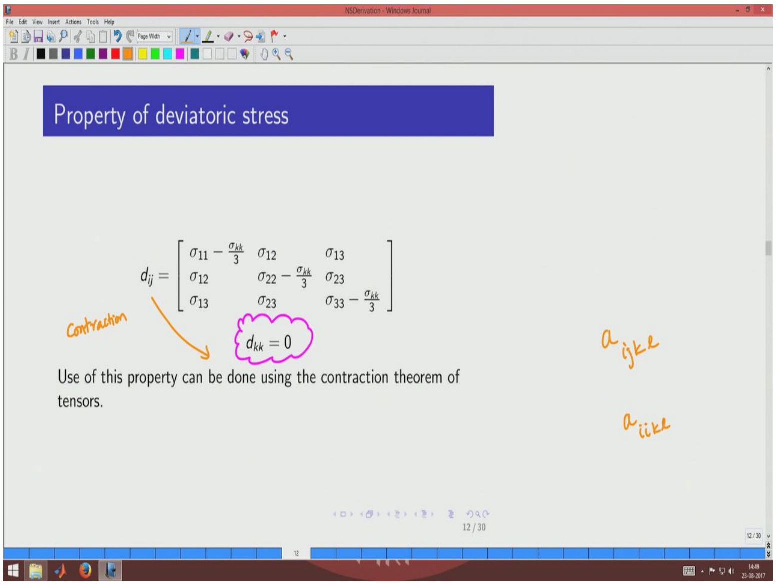

Now, there is some property of the deviatoric stress, you could see that the trace is 0. So, you could sum up the diagonal term and you get a 0. Now this is going to be very very useful because later on when we are doing some derivation we can actually see that this can be put to use.

Now, when d_{ij} is specified in any manner and you are asking what is d_{kk} what you are actually doing is repeating the subscript. So, from here to here you are going basically through a process in tensors that is called as contraction. So, you have an arbitrary tensor a_{ijkl} and if you were to ask what should be a_{iikl} then the contraction theorem says that it is not a tensor, but a order of 2 because only 2 subscripts are free this it as it order of 4. So, you can actually have tensors of order n - 2 obtained and they will also be tensors. So, here we see the same thing that is being used and therefore, we just simply change the subscript and then go ahead and see what happens and those expressions are also valid. So, we let us those this kind of a things useful and note down and keep them aside.

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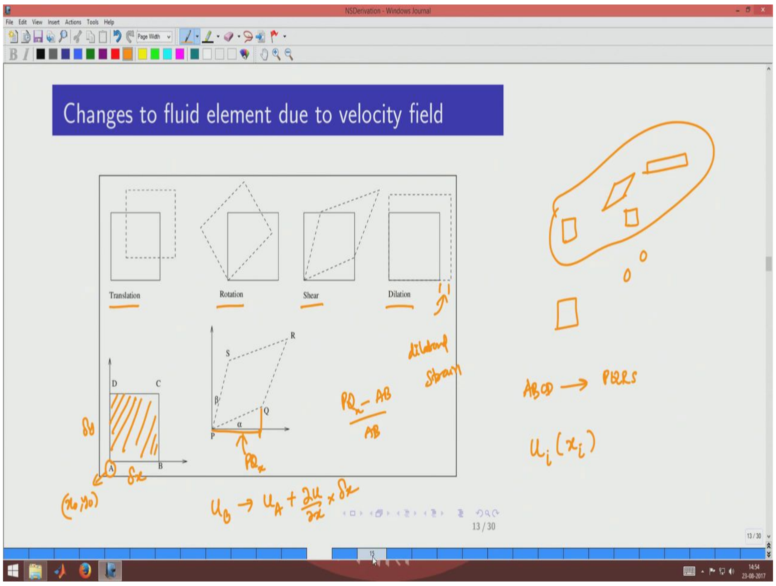

So, what happens when a fluid is undergoing any motion is as follows. So, if we have a fluid element which is described by ABCD then we see that in general it will undergo any arbitrary kind of a transformation. So, ABCD is going to PQRS. So, the transformation can be any arbitrary shape. We do not have things like this, you know you do not have a control volume like that going and becoming like that. So, we do not want that because that is not a well behaved fluid flow. So, you have things like this you know, you could just have them stretched around or just simply moved around or located.

So, it should be only things that are actually going to be simple operations. So, such operations can then be decomposed into four different manners, one is a translation just relocation, we have got translation as the relocation of the control volume, after that rotation pure rotation and then a pure shear and then pure dilation. So, then can prove that such arbitrary shape changes can be described as a mix of all this four ways of changing the control volume. And now you can see that these are also coming as a consequence of the velocity field. So, the velocity field which is a function of the distance will give you this, the reason is very simple for constant velocities it should have a translation of the control volume. In the case of velocity that are having gradients it should lead to dilation we have seen that in just a moment. But you see that shortly for off diagonal terms you see that the average of them will give you the shear and the difference of them will give you the rotation.

So, you could see that these kind of transformations are coming mainly because of the velocity gradients.

So, let us just look at what are those.

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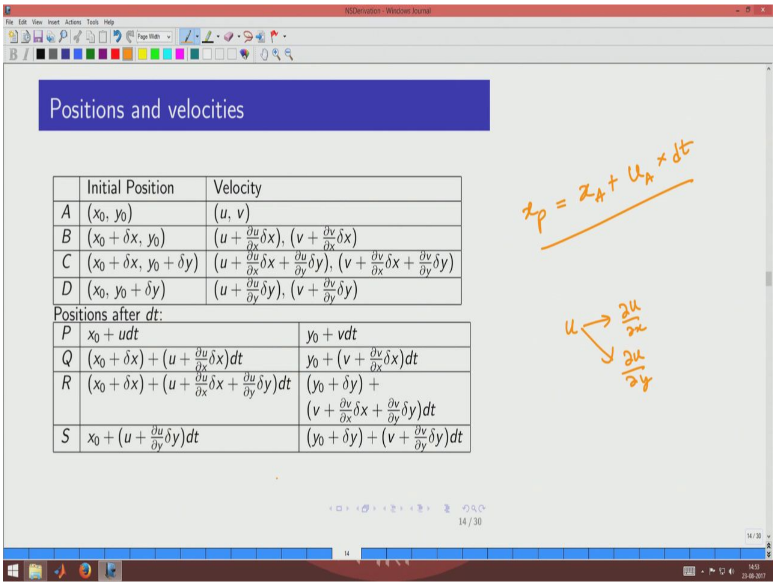

So, what we do is basically ABCD the locations we write the coordinates x_o y_o and the velocities are u and v. And in general there are these gradients that are present, so u has the gradient in both you know x direction as well as in y direction. So, let us see how the velocities are specified.

So, at A B C D the locations are specified as $(x_0 y_0)$, $(x_0 + \delta x\ y_0)$ so on and, which means that the width is δx here and δy here and A is basically $x_{0,} y_0$ and you can see that B is nothing but $(x_0 + \delta x,\ y_0)$, D is $(x_0, y_0 + \delta y)$ and C is $(x_0 + \delta x,\ y_0 + \delta x)$. So, that is exactly what you have written. Similarly the velocities also can be done and by using the first order approximation is in taylor series expansion you can actually see that the velocity at any location. For example, the velocity at B is nothing, but the velocity at A plus the gradient into that particular distance. So, that is what we are going to use and write the velocities.

Now, once you have these velocities and positions what we are going to do is how the position P Q R S is coming from A B C D due to the velocity times to the δt . So, which means that a position of dt is nothing but initial position plus velocity time dt. So, once we do that we could say that the position P. So, x_P is nothing, but $x_A + u_A \, x \, dt$. So, this is how we can arrive at all the positions new positions and we have written them.

Now once we write these positions the advantages basically we now have a way to define the different components of these dilations and shear in rotation. So, we now do the dilation, the amount of dilation is nothing but difference in the lengths divided by the original length that will be give you the amount of strain. So, such quantities are now going to be defined. So, we are going to define the dilation strain as follows $\frac{PQ_x - AB}{AB}$ which means that, if you see this PQ and x component we drop a line this is PQ_x this part is PQ_x component. This divided by AB. So, $\frac{PQ_x - AB}{AB}$ it shows you how much of expansion or contraction is happening along the x direction. So, it gives you this kind of a strain.

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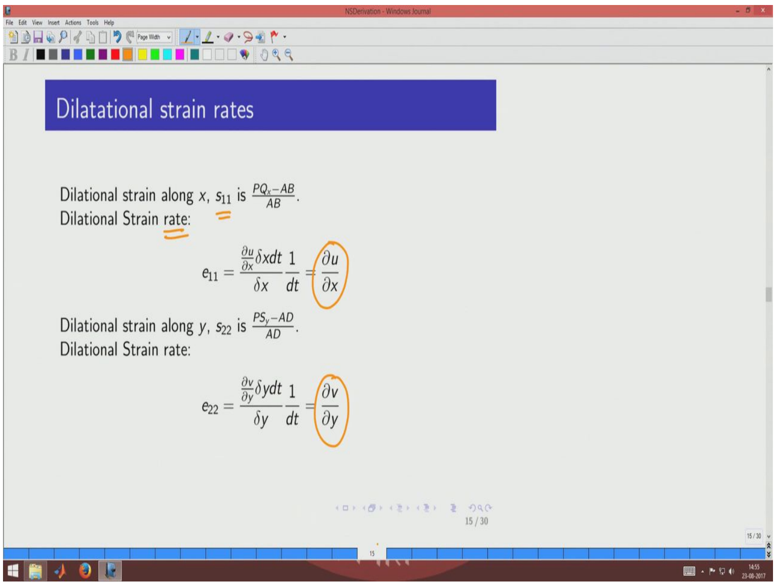

So, dilational strain and that is actually defined as s_{11}. So, s_{11} is nothing, but $\frac{PQ_x - AB}{AB}$.

Now, we have the values for the locations PQ as well as AB. So, we substitute them PQ means basically the position Q minus position P. So, now, we do that and when we do that we get this very simple expression that the dilation is given in this manner rate of dilation. So, that is the rate of dilation because $\frac{1}{dt}$ is already there. So, you could see that the velocity gradient in x direction is giving you the dilational rate in that direction which is already familiar to us. Similarly we can also do it for y direction and you see that the dilatational strain rate of deformation along the y direction is given by $\frac{dv}{dy}$.

So, similarly we can also define the shear stress. So, shear is basically defined in this following manner. So, what you do is that how much of this divided by how much of this that is the shear. So, what we do is with define the same way.

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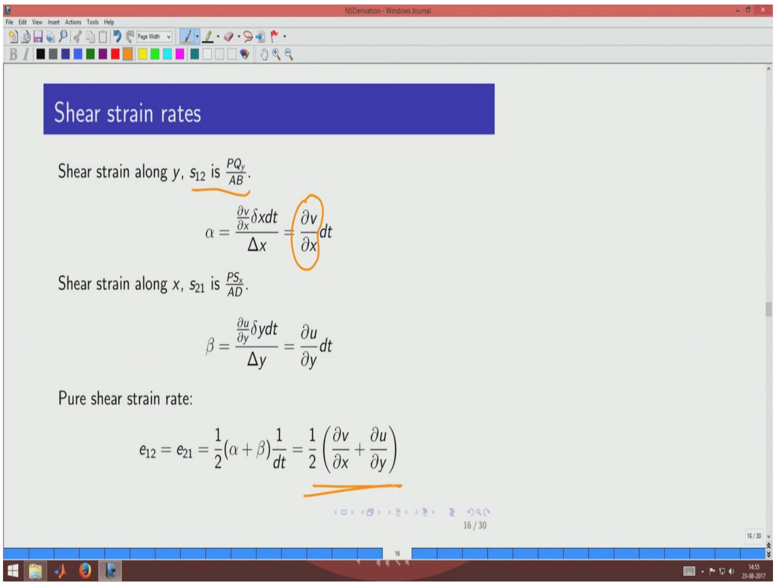

So, the shear strain is defined in this manner $\frac{PQ_y}{AB}$ and then if you then look at the locations and derive you can see that the shear strain rate is given by this you can see the cross terms v and x up in two different directions are coming together. So, shear strain is coming like that. And then the pure shear strain is written as an average of those two angles and then you see that this is how it is coming up.

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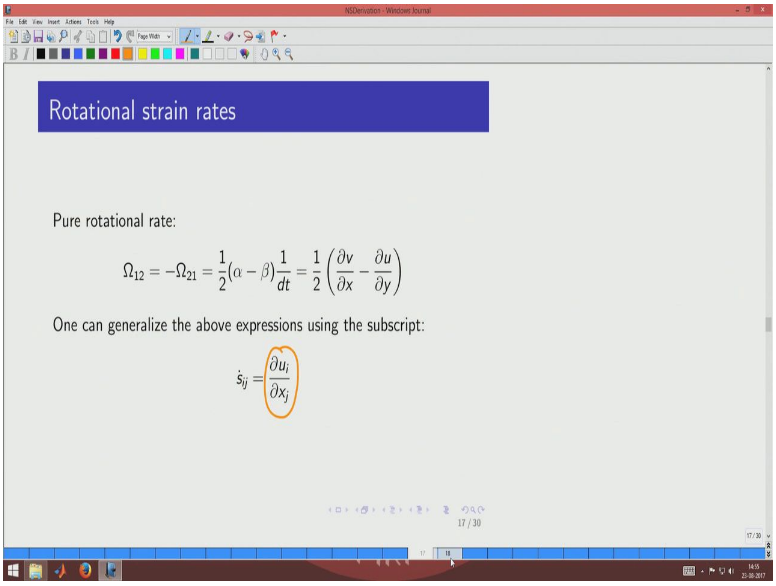

And similarly the rotation also can be written the different of those angles that gives the rotation. So, you could see that this quantity is very generic quantity it is a strain rate tensor and it has basically the diagonal terms off diagonal terms and they seem to have some meaning.

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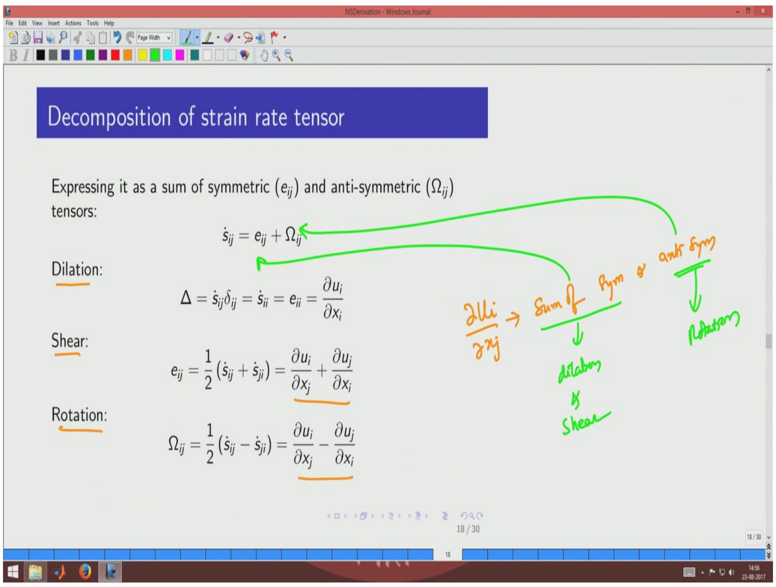

So, we see that these terms the diagonal terms alone are related to the dilation the sum of the off diagonal terms are related to the shear the difference of the off diagonal terms is related to the rotation and the velocities themselves in absolute way are related to the translation which we are not bothered at this moment we are only looking at the differentiations.

So, now, we see that we can actually write the generic quantity $\frac{\partial u_i}{\partial x_j}$ as a sum of symmetric and antisymmetric components stresses and those are actually what we are trying to. So, here the symmetric part we say it is e_{ij} and the antisymmetric is Ω_{ij} and we see that the antisymmetric part is related to the rotation and the symmetric part is related to the dilation and shear. So, this is how the strain rate tensor is going to be decomposed.

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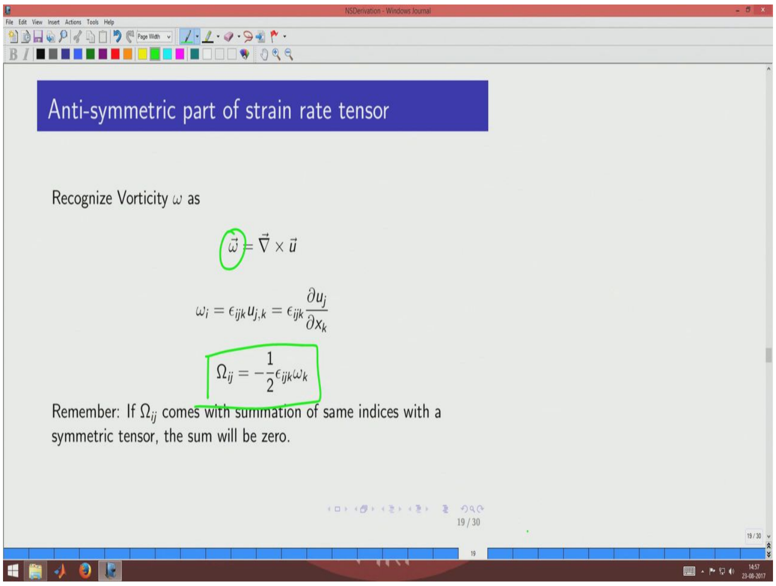

So, once you do then we also can see how the rotation part can be related to something else. So, we see that the rotation part can be related to the vorticity which we introduced as what are the planar flows session. So, you could see that the vorticity is related to the Ω in this manner and because you have got only three different components in omega because they are antisymmetric tensor and those three can be the components of the □, *Ω* and □ related. So, they can be also called as duals. So, which we see that it is an antisymmetric component of the strain rate tensor.

And the reason why we want to do it also because whenever it comes in summation then we know that symmetric and antisymmetric tensor coming in summation we give to 0. So, that is we are actually separating the strain rate tensor into two components. So, let us do that.

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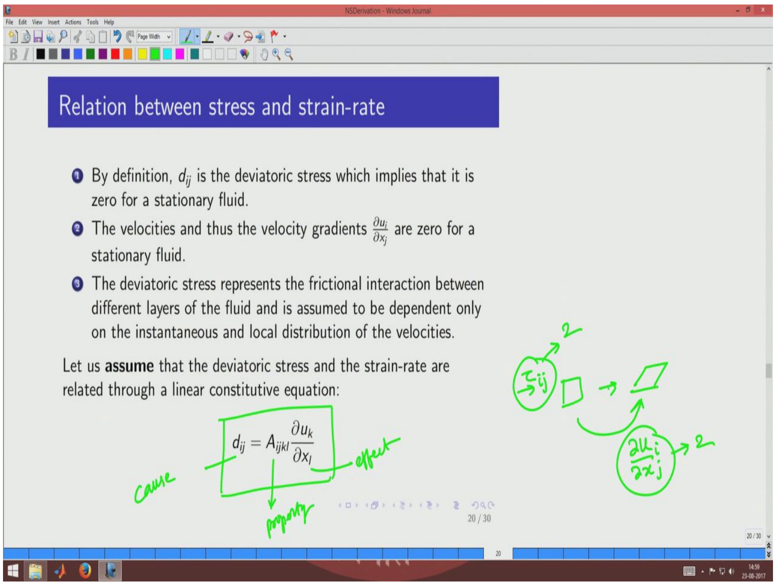

Now, what is the linear relationship that we want to derive? The idea is as follows. The shear stresses are going to cause deformations of a control volume in this manner, so the shear stresses τ that is applied over will going to leave this, this actually this kind of a change of control volume is described by the velocity gradients, which means that you could relate the shear stresses with velocity gradients and because shear stress is a tensor of order 2 and this also as a tensor of order 2. Then the relation should be through a linear relationship with the property having a most generic tensor order which is 4, 2 + 2. So, that is exactly what we are writing here.

So, here is a proposal. So, we are proposing that the cause which is basically the shear stress is leading to an effect which is basically the velocity gradients. So, we are proposing that they are related linearly. So, they may not be related linearly in some situations, but let us hope that this fellow is going to be helpful to us. So, now you can see that on the right hand side we have got σ we separated it into the pressure and d_{ij} and we then change the d_{ij} to the velocity gradients and then now you see that the velocity gradients are appearing on the right hand side which is good because that is an unknown and we can always handle that in some way.

So, here is where the Neumann principle is going to be off use. So, A is basically property you see that if this is a cause and this is the effect and what is related them both it must be a property. And according to Neumann principle any property should have the same symmetry as that of the material for which we are describing that property. And what is a material here? It is a fluid now fluids are isotropic, which means that the tensor that should be used to describe a must be a fourth order tensor which is an isotropic tensor which we have already come across in the introduction to tensors the most generic way of writing is in this manner where there are actually 3 independent quantities which are basically μ_1, μ_2, μ_3 .

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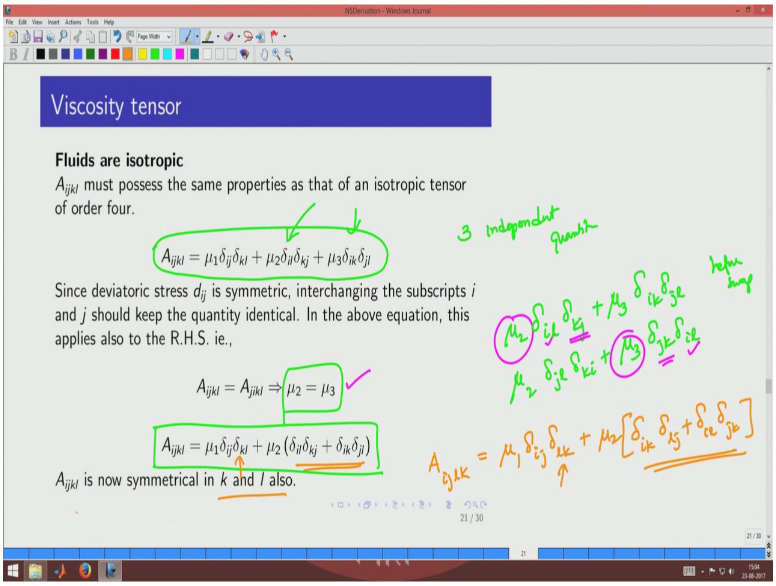

So, if you take a very generic tensor A_{ijkl} which is a fourth order tensor and you must have 81 components, but in this case A_{ijkl} is a property related to liquid which is isotropic. So, we can say that this kind of a expression can be used which means that there are only three independent quantity. So, what we do is that we then put it in and then see what happens.

So, you see that on the left hand side the equation. So, you see that the left hand side equation d_{ij}, now d_{ij} is has a symmetry over the indices i and j the reason because σ is actually symmetric tensor therefore, the deviatoric stress also should be a symmetric tensor. On the right hand side the ij indices are appearing only for A, which means that when you swap the indices on the left hand side then this equation is valid they should also not have any effect on the right hand side. So, when I swap the indices on the left hand side nothing happens because D is a symmetric tensor.

So, therefore, when I swap the indices on the right hand side also the same thing should be prevailing which means that if you take the swap the indices for this quantity and change i to j and j to i then the quantity should not change and that is only possible when $\mu_2 = \mu_3$ because you see these two quantities. So, I want to have. So, $\mu_2\delta_{il}\delta_{kj} + \mu_3\delta_{ik}\delta_{jl}$, this is before swap and after swapping the indices i and j we see that it is jl and $ki + \mu_3\delta_{ik}\delta_{il}$. Now you see that these two are same actually $\delta_{jk}\ kj$ because δ is actually a symmetric tensor is δ is a symmetric tensor. So, these two are same and these two are same which means that this and this must be the same. So, that is exactly what we are written here. So, saying that we actually do not need three independent quantities for the property A just two is enough. So, that is how we have reduced.

So, now that we have that mu 2 is equal to mu 3 let us choose that value to be mu 2 itself and then write it as $A_{ijkl} = \mu_1\delta_{ij}\delta_{kl} + \mu_2\delta_{ik}\delta_{jl}$. So, now, we have got a little less number of unknowns here.

So, now, once you look up this expression it appears as if even if you swap the indices k and l nothing will change. So, this is an outcome of the expression that we have written. So, look at this expression and I will just swap the indices k and l. So, A_{ijk} and l I am swapping and on the right hand side what is coming $\mu_1\delta_{ij}\delta_{kl}$. So, I write l k plus $\mu_2\delta_{ik}\delta_{lj} + \mu_2\delta_{il}\delta_{jk}$. Now, you see that this expression and this expression are actually one and the same it just that the terms are actually swapped. So, i k j l is coming second here it is coming first here and i l k j is coming first here its coming second here. So, constants are same. And this and this are also same because δ, δ is also symmetric over the indices k and l which means that the right hand side does not change when you swap the indices k and l which means that left hand side also should not change which means that a is also symmetrical in the indices k and l. So, we now discovered that imposing these symmetry of the deviatoric stress we are actually obtaining the properties of A which is basically saying that is a symmetric over ij as well as kl.

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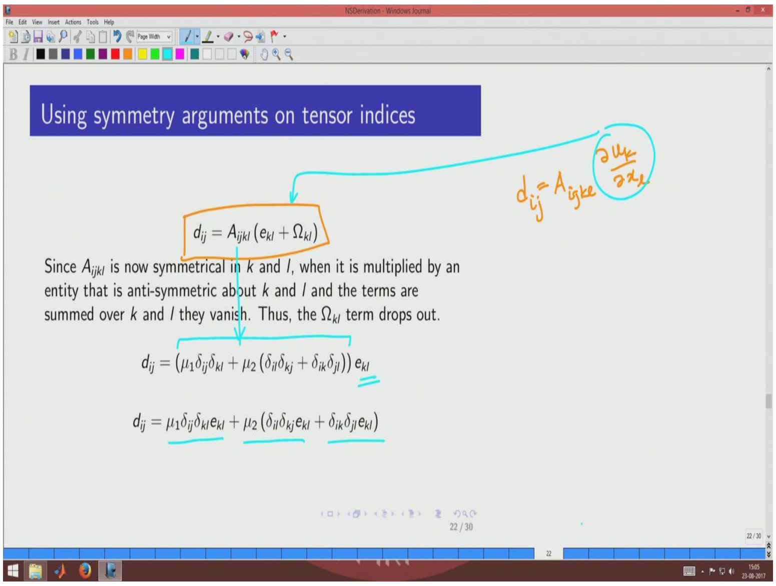

So, now you are writing the expression once again and the expression we have written the linear constitutive relation. We actually wrote linear constitutive equation as follows and we are writing this part as 2; because we can actually separate the velocity gradient as the symmetric and antisymmetric parts we are writing there now we expand A and we write here. So, we write here this part is nothing but the expansion of A.

So, when you do this now what happens is that you could see immediately that there is symmetry of k and l indices for the first term and you are doing a summation over capital omega k l and capita omega actually antisymmetric over the same indices k and l. So, when you sum up that term will gone and therefore, we can just drop this term off. So, that is why you see here I do not have capital Ω_{kl} because the summation will not survive, which means that we can now use this ϵ_{kl} and then multiply with everything and then see what happens. So, when you were doing that we could see that the part by part we can multiply, so $\mu_1 \delta_{ij} kl \; x \; e_{k1} kl$, it is coming here and then we multiply. So, we are doing part by part. So, why are we doing part by part because the δ has a specific property it can use it can be used to swap the indices and we are going to do that now.

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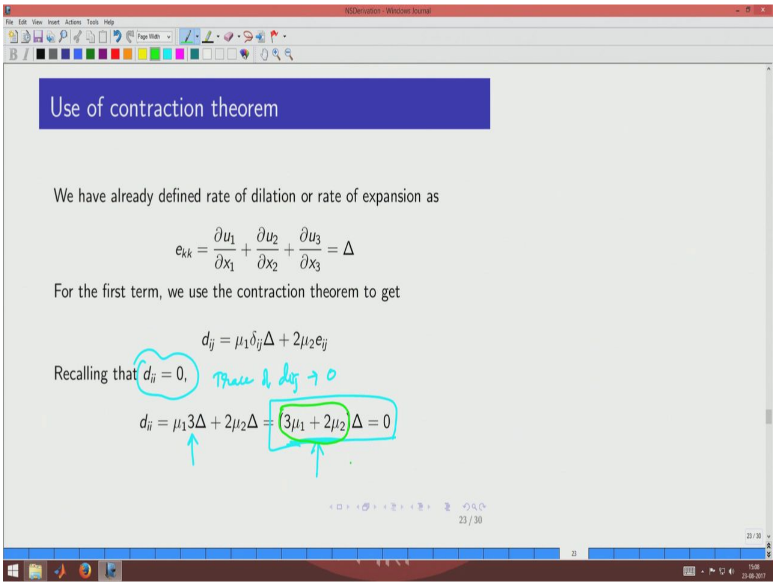

So we are going to use the swapping of indices. Now, the e_{kl} itself we already know the definition. So, that is nothing, but the definition showing that the e is basically the symmetric part and we can also see that d_{ij} as the summation 0 here for trace, the trace of d_{ij} is 0 because we already removed the diagonal elements from it by defining the pressure and that we can now use by seeing what happens if we do a contraction operation.

So, what we do here is that ij you just do a contraction operation write d kk. So, instead of ij you just put kk and then see what happens. So, d_{kk} the first one is like this d_{kk} and then this term δ_{kk} which means by summation over the three diagonal elements of a δ that gives you 3 and the remaining ones also going to be there. So, that is how we actually expand and then we see that we write in this manner and you get a 3 there and here the same term is coming so there must be 2 here. So, which means that we see that as consequences of the trace of the deviatoric stress being 0 we get an equation like this.

Now, this equation is very interesting because this gives us a relationship between μ_1 and μ_2 in other words we do not actually need two independent constant we actually get only one of them. Of course in the case of incompressible fluids the μ_1 and μ_2 can be different and this equation still be valid because actually the δ is 0, but if you want this equation to be valid even for not incompressible fluids then it is better that this part is alone going to 0 which means that μ_1 and μ_2 are related and this has a name this is called the stokes assumption.

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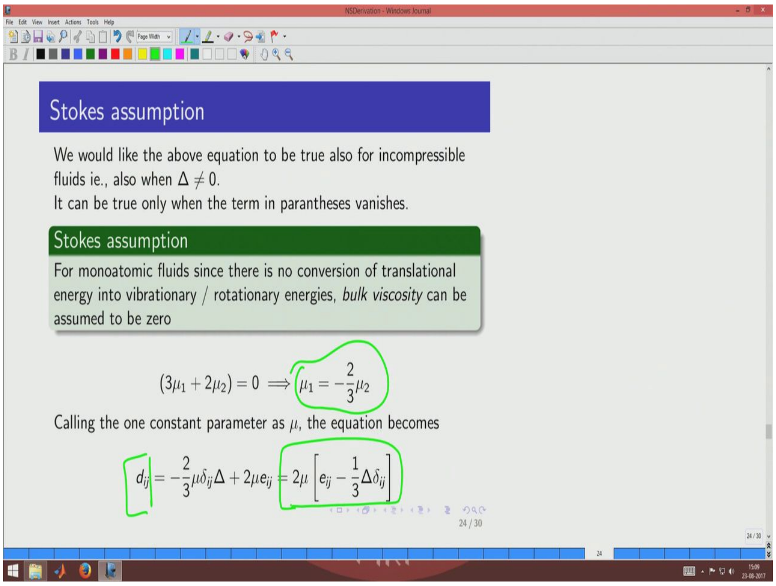

So, stokes assumption, we say that if you want this equation to be 0 even when the rate of dilation $\Delta \neq 0$ then it can only happen when the quantity in the parenthesis namely this quantity, this quantity in the parenthesis should not survive and which means that we can actually relate μ_1 and μ_2. So, this is how we are relating μ_1 and μ_2 and it goes by stokes assumption which is valid for fluids that are consigned containing molecule that are too long etcetera.

So, it is also called as bulk viscosity is taken as 0. So, we will now use this relationship to go back to d_{ij} and write the constants μ_1 μ_2 as just one constant mu and that is we are writing here. So, we instead of μ_1 we write $-\frac{2}{3}\mu_2$ and instead of μ_2 we just say μ, so $-\frac{2}{3}\mu$ δ_{ij} $\Delta +$ this. So, we write this expression.

We now see that this expression; is coming quite in it. So, now, you see that on the right hand side you have got only the symmetric part of that strain rate tensor and we have got the rate of dilation and just one quantity viscosity μ. So, now, we see that this μ the symbol is chosen very you know carefully, this μ is nothing but viscosity and the property tensor A is the most general way of writing it, but with all this manipulations we come to conclusion there is nothing but the viscosity itself we are talking about.

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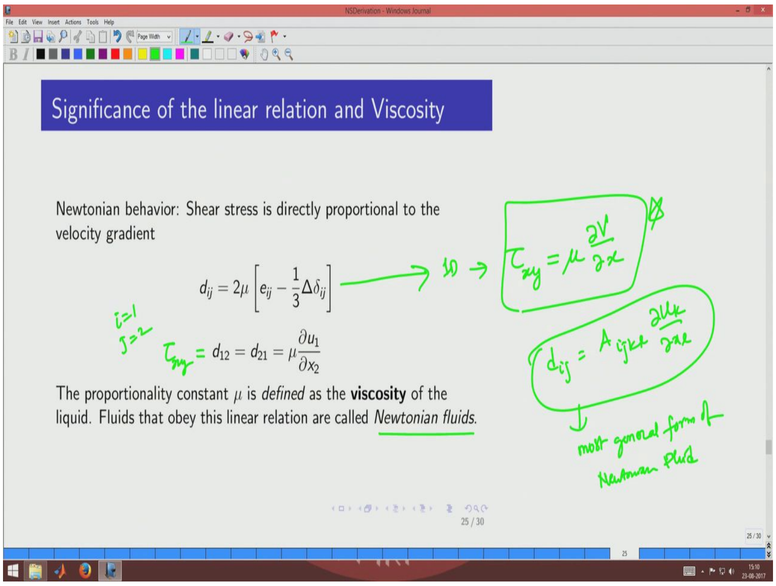

Now, let us take that expression and see what happens. This expression when you take for i and j becoming 1 and 2. So, i=1 and j=2 then d_{12}, d_{12} is nothing, but τ_{xy} . So, we see that this expression for 1D turns out as $\tau_{xy} = \mu\frac{\partial v}{\partial x}$. Now this is nothing, but the statement of Newton's law for fluids which is basically saying that the fluid we are talking about is a Newtonian fluid.

So, in other words when we started of saying that $d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l}$ this is actually saying that it is nothing, but most general form of the Newton's law of viscosity here. And when we apply it for 1D we see the expression which is very popular in the text books and this is what actually we normally use. So, we can see to the tensorial part is actually the starting point and this can be obtained from it straight away using the manipulation that we have done till now.

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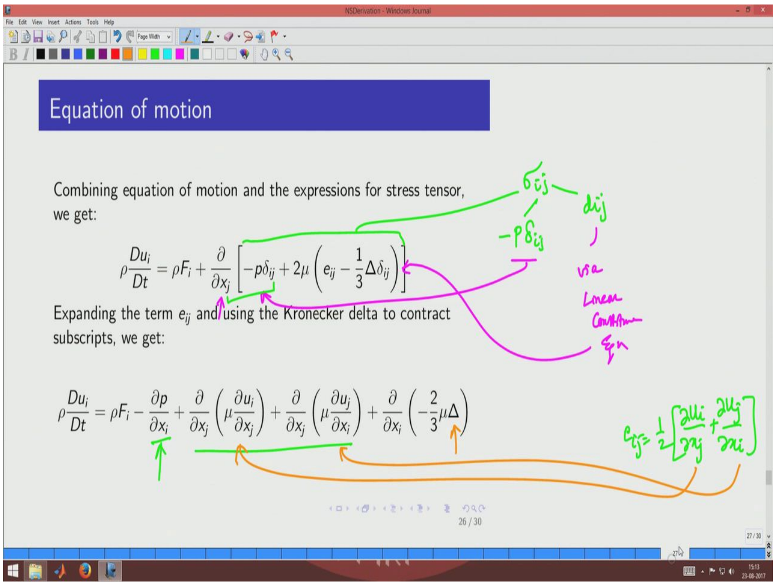

Now, we have the d_{ij} expressed in terms of the velocity gradients. So, let us just go ahead and put it in the equation of motion. So, we saw that this entire thing is basically σ and σ is written as two components the pressure component and d_{ij}. So, the pressure component - p and then it is also written as d_{ij} and then we saw that this is coming straight here in and this is coming via the linear constitutive equation as this quantity. So, we can see that the expression that is written in the equation motion is used as it is except that we have expanded σ on the right hand side.

Now what we do is the term by term we can apply this particular operator and see what happens. Now here is where the property of Δ δ to swap the indices will be useful. So, you could see that when you when you have δ_{ij} and x j then these indices are same. So, the i will be then coming out, so you could see that this will give me this term. So, I have only i index that comes straight away from the subscript notation convention that we have used. So, if you are not convinced you could actually expand and see what happens and you will come to the same conclusion.

Now, the second term the second term you then expand e_{ij}. So, we already know that e_{ij} is nothing but $\frac{1}{2}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]$. So, this is what you are going to substitute in and then you see that two and this half will cancel and we got the rest of the expressions. So, this is coming because you have got this first term. So, the first term is coming here and the second term is coming here. So, you could see that the two and this half is getting canceled the μ is sitting there undisturbed and this is the term.

The last term is kept as it is. So, the 2 is going in. So, $-\frac{2}{3}\Delta\delta$ and then the δ_{ij} and j, I am just making this ∂x_i. So, the subscript notation usage is very much there here for us to make this simplification. So, what we have written now here in the bottom is basically equation of motion for a generally any fluid whether it is compressible or incompressible because we are just leaving it as it is here.

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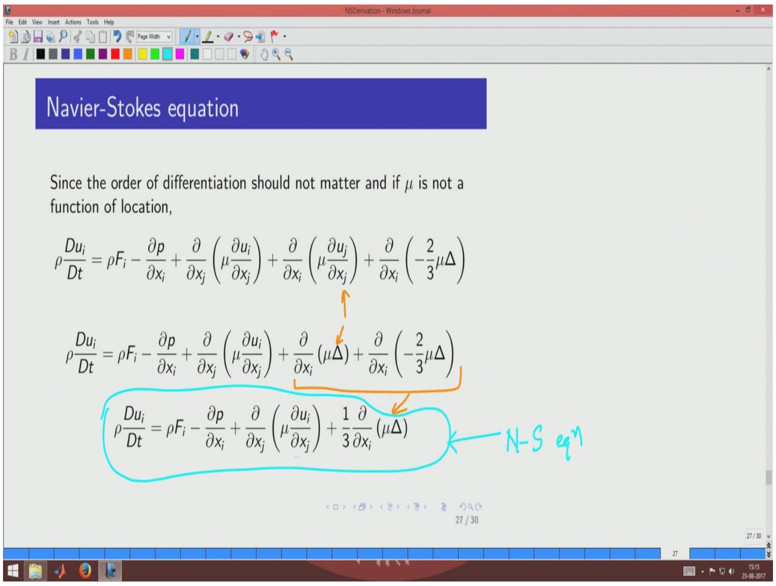

So, at this moment what we do is that we do further manipulations of this expression. So, all the terms are kept as it is and then we expect to term by term what happens. So, what happens is that here you could see that the equation here you have got the indices the same here and they are not the same here. So, what we do is that can we swap these two indices because you can see that the order of differentiation should not matter if the quantity u is well behaved function and μ is not location dependent. So, then you can swap those, so we do that and when we do that you see that here we get δ coming in. So, when we swap the indices. The first term is left as it is.

So, then when you take this term and the second term then you can see that here it is $\mu\Delta$ and here it is $-\frac{2}{3}\mu\Delta$. So, it must be $\frac{1}{3}\mu\Delta$. So, these two together are giving this term. So, that is about it. Remaining terms on the left hand side and the first three terms are untouched.

So, now, you have got this equation which basically is the typical form that you would see in text books which we also call as Navier Stokes equation for any fluid which can be either incompressible or compressible. So, very often one would wonder where is this 1 by 3 coming and we now have a origin for that. The $\frac{1}{3}$ coming because of the stokes assumption and the stress of the Δ being 3 etcetera. So, that is how this term is coming and this statement is nothing but the Newton's second law applied to control volume. So, this equation is what is worth remembering. So, this is refer to as the N-S equation.

And if you want to now look at the Navier Stokes equation for limited situations like an incompressible fluid then we can actually knock off one of the terms. You can see the last term has the delta there we know that, but an incompressible fluid the delta is 0. So, we can just knock it off and then write the Navier Stokes equation for incompressible fluids by knocking of the last term. Now, when we are knocking of the last term we see that there is this term which we want to expand and then we want to expand it by showing that you could write it as $\frac{\partial^2}{\partial x^2}$.

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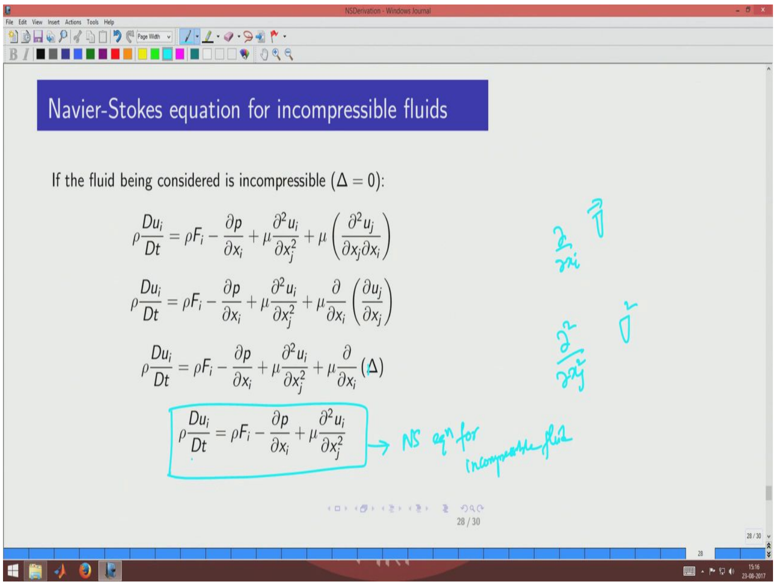

So, here is basically the Navier Stokes equation for incompressible fluid and this operator $\frac{\partial^2}{\partial x_j^2}$ is nothing but the laplacian operator. So, you could actually write that also and this operator is nothing, but the gradient operator. So, you could actually now write the Navier Stokes equation for an incompressible fluid using the vectorial operators as well. That is what we do here.

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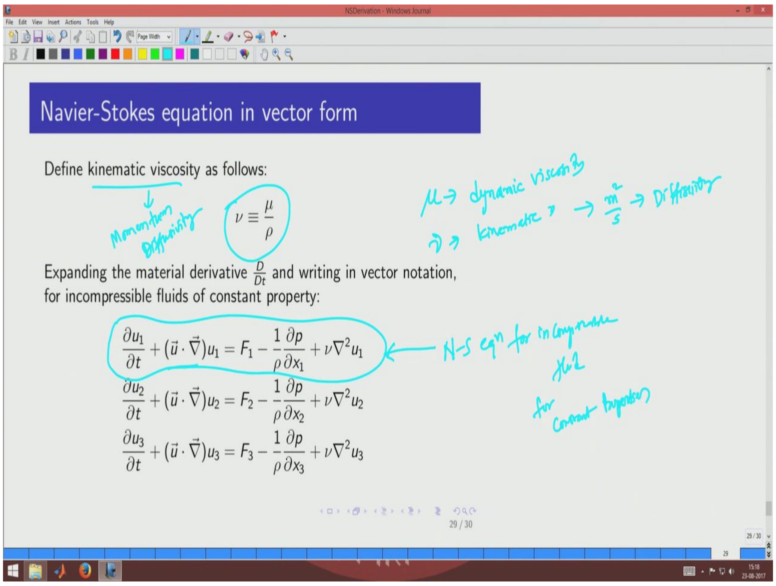

So, we use the vectorial operators and we have the gradient here, we have the see the left hand side as $\frac{d}{dt}$ the material derivative we are expanding the material derivative on the left hand side and on the right hand side we have got the gradient operators and the Laplacian operators we use. So, you could see that this now becomes the in vectorial notation the Navier Stokes equation for incompressible fluid.

We are defining a quantity called kinematic viscosity. So, if μ is called the dynamic viscosity, then ϑ becomes the kinematic viscosity and the units of ϑ happens to be m^2/sec and that will be the same in units as the diffusivity which means that you could think of ϑ as momentum diffusivity. So, ϑ can be thought of as momentum diffusivity. So, you can then plug that in and therefore, this is also one of the popular ways by which the Navier Stokes equations are appearing for an incompressible fluid and you also notice that the ∇^2 operator also implies that there is no location dependency of the viscosity which also means that for constant viscosity constant properties. So, we can also say for constant properties. So, this is how a derivation of the Navier Stokes equation has come about.

So, I hope you now are clear how this equation came about starting from the Newton second law all the way through all the manipulations with respect to the control volume, with respect to the usage of the tensor properties, the Neumann principle, the material derivative and so on. So, you can see that this is a combination of all the things that we have been discussing for the last 4 or 5 sessions and it is a very good idea to brush up all those concepts before we go through this particular derivation. What we now do is we use this equation as a starting point for all the problem solution from now on that is because we can avoid having to do the momentum balance for every problem it does not make sense because once you have done momentum balance as a part of the Navier Stokes equation. Then you are done with θ and we can just simply use that equation as it is and then knock off terms that you do not need and then solve the problems. So, we want to do that in the following sessions.

So, you can check the website for the practice assignments and practice the derivation by changing some of the terms in between to make sure that you have understood the derivation.