Transport Phenomena in Materials
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Lecture - 08
Transport Theorem

Welcome to the session on Transport Theorem as part of NPTEL MOOC on Transport Phenomena in Materials. This theorem is popularly known as Reynolds transport theorem. We will need the outcome of this theorem as one of the inputs to the derivation of Navier Stokes equation. Those of you who are not interested in the mathematical aspects need not go through this session; however, the end result will be one of the important inputs we will need while derive in Navier Stokes equation which will be in the next session.

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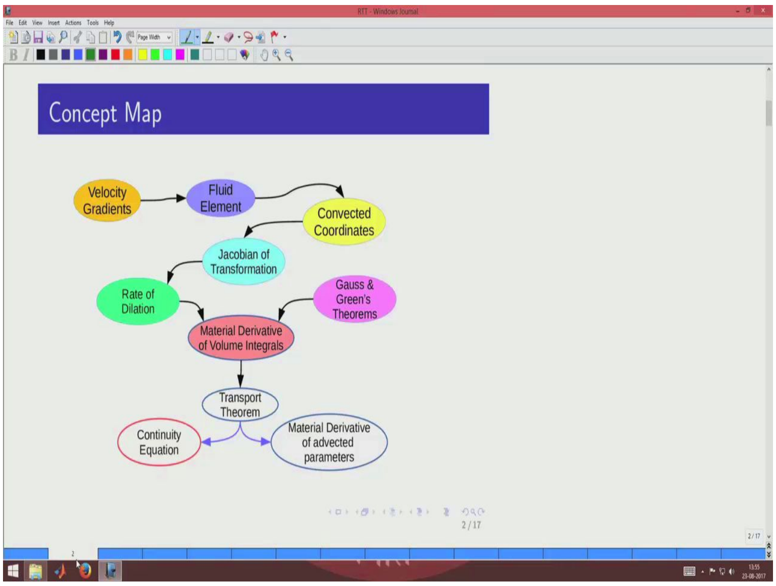

So, the concept map behind the theorem is given here. We could see that when we have a velocity field which is varying with the spatial coordinates we do have velocity gradients and these velocity gradients are going to change the shape of any fluid element they would be stretching the fluid element in directions where the velocity gradients are positive, shrinking them in the directions where they are negative and so on.

So, naturally we can see that velocity gradients are going to affect the fluid element. And what we are going to do in this session is introduce what are called as the convected coordinates. So, we are going to fix a coordinate system to the fluid element as it is moving around. And when we do that then the translation of any quantity from the spatial coordinates we use normally for Eulerian specification and the convected coordinates which we are fixing to the moving volume element will be done through a transformation and there we come across a quantity called the Jacobian of transformation. And from there we derive what would be called as the rate of dilation which we have defined earlier while deriving the continuity equations. We will see an alternative definition of the same quantity.

We will put this in along with the theorems that will allow us to convert the divergence and dot products into the volume integrals. So, we are going to do that together and then have a material derivative of a volume integral derived and that basically gives you the transport theorem. And as a special case of a transport theorem where the function that is being introduced can be giving us the continuity equation when that function happens to be just density and if that function happens to be for example, an advected quantity like momentum then it will also give you any equation that we can readily use in the derivation of the Navier Stokes equation. So, this is the concept map. Each of these elements that I have shown in this ellipses with colors can be derived upon deeply separately, but we need to go through all these to arrive at the derivation.

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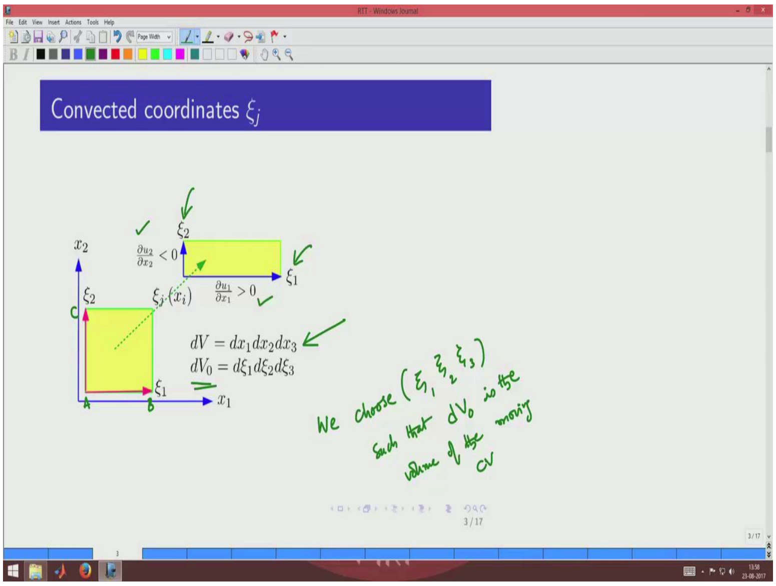

So, here is the concept that we have already studied while coming across the material derivative. So, where we say that if there are points like this if we have a b points and if there is a velocity field as you can see that if the velocity is there in both x and y directions then this element is going to move away. That means, the velocity is positive in both x and y directions x_1 x_2 directions. And if the velocity at b is greater than the velocity of a then the length ab will get stretched as it moves. So, you could already see that the control volume or the volume element has got stretched here.

Now, in the y direction for example, if the velocity gradient is negative which means that its velocity is still positive, but the gradient is negative which means as you go up in the x_2 direction the velocity magnitude is coming down in that case what would happen is that this line segment ac will shrink as the volume element is moving. So, as you can see that this kind of a thing is quite acceptable as far as the incompressible continuity equation is concerned because one gradient is positive another gradient is negative,the sum can be 0 satisfying that such velocity fields are valid ok.

Now, what we are doing is that we want to define the volume of the fluid element using the spatial coordinates x_1, x_2, x_3 like this. So, because the spatial coordinates where always taking them as orthogonal, we can take it as a triple product of the unit vectors dx_1, dx_2, dx_3 and then we can already see that the product is directly giving us the volume of a volume element in the spatial coordinates.

Now, what we are doing is we are going to fix the coordinate system ψ_1, ψ_2. These are all the new coordinate systems we have introduced. So, we are going to fix the ψ_1, ψ_2, ψ_3 coordinates to the volume element and in terms of those coordinates we choose. So, I stress we choose the coordinate systems ψ_1, ψ_2, ψ_3 such that such that this fellow the volume of the element in the new coordinate system convected coordinate system will be constant and it will be the same as the initial volume. So, dV_0 is the volume of the moving element moving CV. So, the coordinates will then be fixed to the moving element which also means that the length unit length of ψ_1 will not be the same as x_1. It has to change because as the control volume keeps very stretched and compressed and rotated, so ψ_1 ψ_2 ψ_3 cannot be the same measure as x_1 x_2 x_3. So, this transformation between these two is what we introduce through a transformation of coordinates ok.

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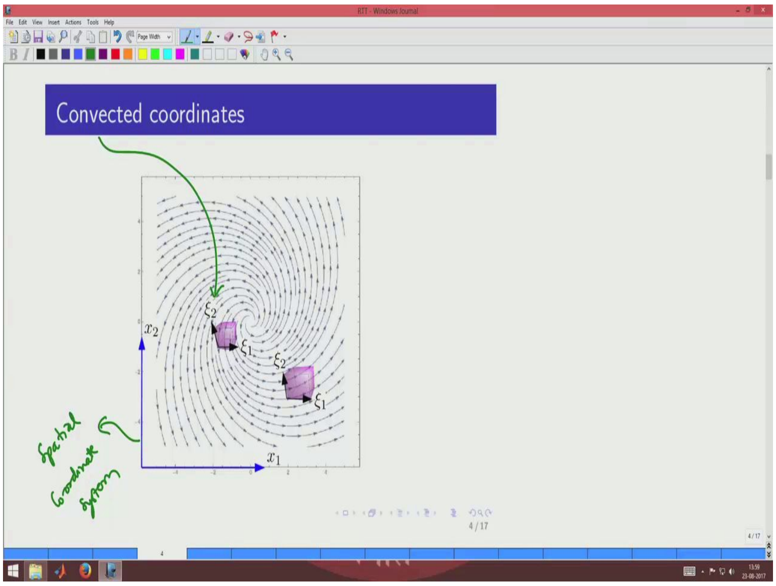

So, a little bit more about this. So, why are we calling these, ψ_1, ψ_2, ψ_3 as the convected coordinates. So, the reason is that they are fix to the volume element and they would expand and shrink as the need arises so that the volume as defined in that coordinate system will be always same. Whereas, the volume of the control volume which is moving in the spatial coordinate system namely x_1, x_2, x_3 can change. So, this is why the necessity for both of the coordinate system to be there.

So, convected coordinates are basically these. And what are these? These are basically the spatial coordinate systems and there is a translation between these two that is possible as transformation of variables. Now, this is a little different from the transformation of coordinate systems we discussed at the beginning of this course where we said that the coordinate system is having a pure rotation. So, here it is not just pure rotation we also have dilation introduced and that is why we cannot use those expressions directly, but there is some parallel that we can see in situations where the volume change of the control volume as it is moving around happens to be 0. So, if a volume change is not there then we could have the same expressions used, but otherwise in general situation we should not use.

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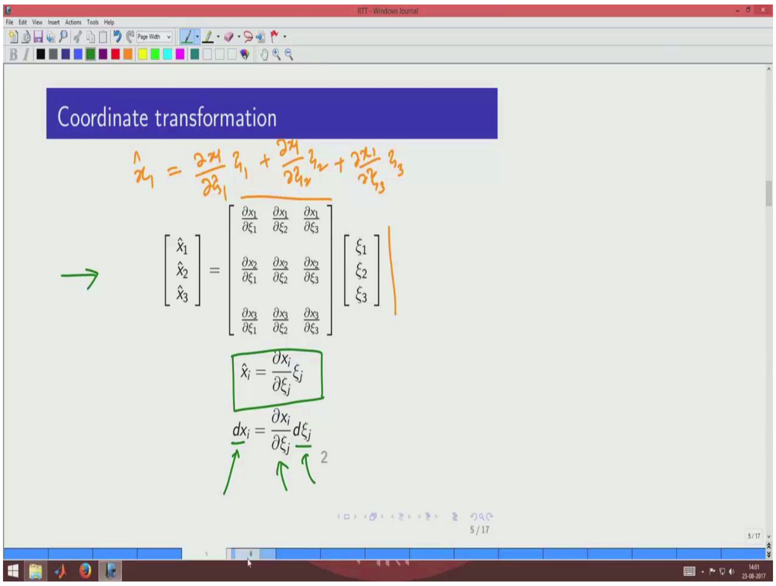

So, the translation between these two sets of coordinates is given by a expression which we write in the same way we wrote for the coordinate transformation where we have the variation of the x_1 in the 3 directions is given and the way we do it is like matrix multiplication. So, we write like this. So, you have got $\widehat{x_1}$ and that you would write it as $\frac{\partial x_1}{\partial \psi_1}\psi_1 + \frac{\partial x_1}{\partial \psi_2}\psi_2 + \frac{\partial x_1}{\partial \psi_3}\psi_3$. So, like this you could actually write all the 3 equations for the transformation of variables and then you have them in a matrix form.

Now, we have gone through the subscript notation precisely to avoid such (Refer Time: 07:54) expressions and that is why we can actually now introduce that and in subscript notation it comes quite briefly here like this. So, x_i that is the spatial coordinate system is then specified in terms of the convected coordinate system ψ_j through a transformation matrix $\frac{\partial x_i}{\partial \psi_j}$. Now, we could also see that this is applicable for elements. So, small volume elements that we want to relate. So, on the left hand side and on the right hand side we write dx_i and d ψ_j as necessary. Now these are what we are going to use as we go along.

So, the triple product of the 3 components of the vector dx_i will then be giving the volume in the spatial coordinate system and similarly the triple product of the 3 elements of the vector d ψ_j will give the volume in the advected coordinate system which is dV_0 which is always remaining the same. So, we see that there something that is coming in between to relate these two and that is where actually we are going to introduce what is called the Jacobian of transformation.

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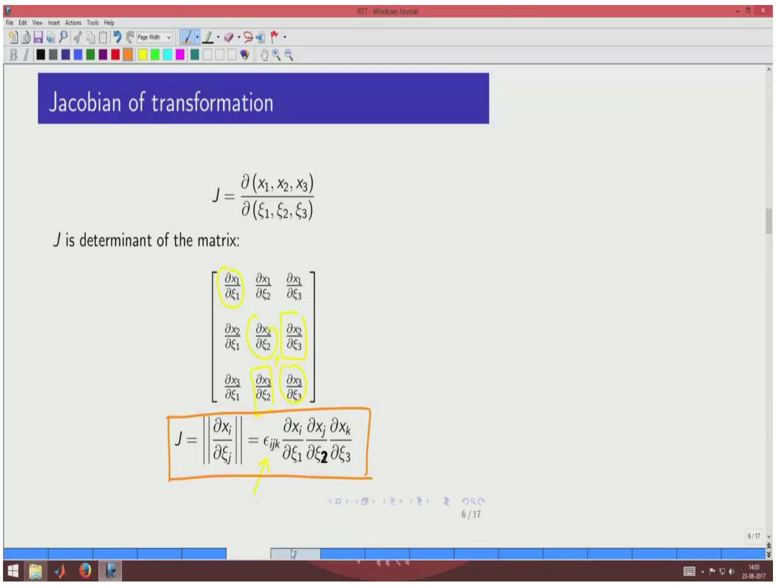

So, Jacobian of transformation is nothing, but the determinants of the transformation matrix that is about it there is nothing, but more to it. So, we already see here the transformation matrix is given here. So, we just take the determinant just like we take determinant of any matrix for that matter. So, basically we take for example, this times this into this minus this into this like that we have got 9 terms and then we can expand.

So, we can now refresh our memory with respect to why this quantity the so called (Refer Time: 09:35) with a tensor or permutation matrix ϵ has been introduced, we remember that it was introduced with multiple purposes. One of them is being able to write the triple product and being able to give the determinant with very small number of terms. So, here we have that application straight away coming and we can see that the determinant of the transformation matrix which is called the Jacobian can just be written with this expression here.

You can already see that there are no free indices on the right hand side all the 3 indices are dummy indices and therefore, you get a number out. This number happens to be unity one when we have the coordinate system as only pure rotation that is not the case in this situation. That is why the Jacobian has to be there it could be taking any value depending upon the where that transformation is happening here and it depends upon the velocity gradients.

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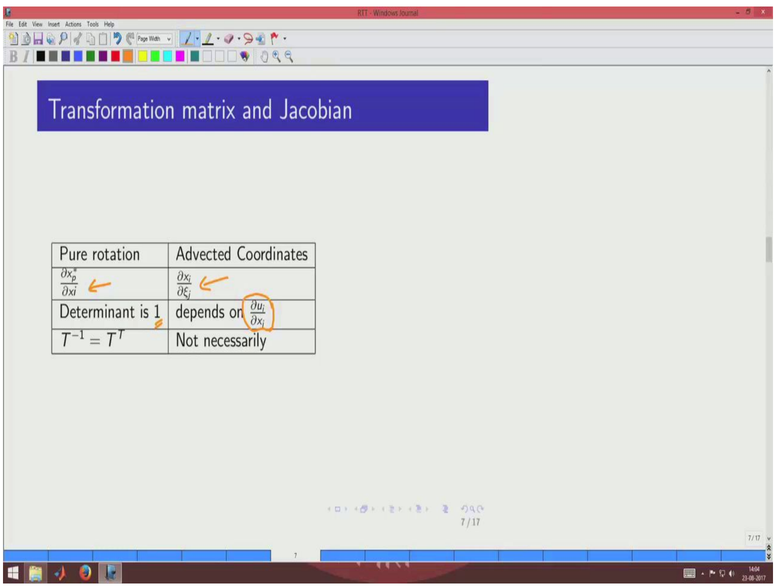

So, in summary the difference between the coordinate transformation which we did for pure rotations earlier and the Jacobian which we have introduced now the summary is here we wrote this both in the same manner, only thing is that the determinant of the transformation matrix for pure rotation is 1. Whereas, here it depends upon the velocity gradients and because the determinant is 1 in the case of transformation of coordinates where this only pure rotation then we saw that the determinant being 1 gives you the formula that inverse of the transformation matrix is same as the transpose which has actually helped us in some simplification of subscript notations.

But this is definitely not directly applicable for us here because the determinant is not 1. So, this expression cannot be used which means that when we want to transform the spatial coordinates to the advected coordinates and vice versa then we need to watch out because determinant is not 1 ok.

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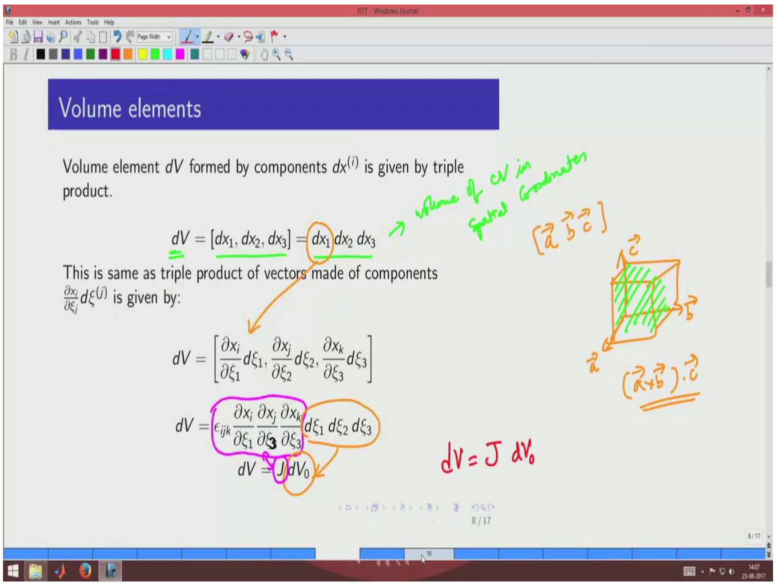

Now, let us look at the volume element change. So, that is where we started off. So, we said that the triple product which is indicated with the square brackets. So, we said that if you wanted to write the triple product of 3 vectors what we mean is that. So, the 3 are in this directions, $(\vec{a}\, x\, \vec{b}).\vec{c}$ that is a triple product which you define with the square brackets there and that gives basically this you can see that it gives you the volume it gives you the volume of this entire control volume that we see which is bounded by the 3 vectors a b c. So, here dV is nothing but the volume bounded by the 3 vectors components dx_1, dx_2, dx_3. And we just write it straight away by multiplication all the 3 because it is a orthonormal coordinate system which means that this is the volume of the control volume which is specified in the spatial coordinates which will change as a function of time because the control volume is moving around and it would be expanding and shrinking with respect to the external observer which is what we are doing here as a spatial coordinates it is like an external observer. So, the dV would change. So, very often this is given as a function of time.

Now, when we now expand each of this quantities which we expand each of these things in terms of the unit displacements along the advected coordinate system then we can use the same expression we have derived earlier. So, here we are having this expression. So, we use this expression to translate and we look at how this terms are written. So, we can see that the triple product of the same terms will give you the same volume of the volume element of the fluid that is moving around and now you can see that the triple product can then be written with ϵ and that we write here. And then we collate the terms we can move the terms back and forth because we are using the subscript conventions. So, the position of the number does not matter the commutativity allows us. So, we then arrange them and when we arrange them we notice that these 3 are coming together and this is nothing but dV_0, and then the remaining ones are also coming together and you could see that you have got something else coming here and this we just now saw that it is nothing but the Jacobian, so j. So, that is what we write here this is nothing, but the j.

So, we can see clearly now that we have this expression giving you $dV = JdV_0$. So, we now have one more meaning for the Jacobian earlier we said Jacobian is the determinant of the coordinate transformation from the spatial to the advected coordinate system and we are also now saying that it is the factor by which the volume element in the spatial coordinate system is related to the moving one which is actually defined as constant with respect to time. So, this relationship is useful so we are going to then use it later on for changing the coordinate system as we need.

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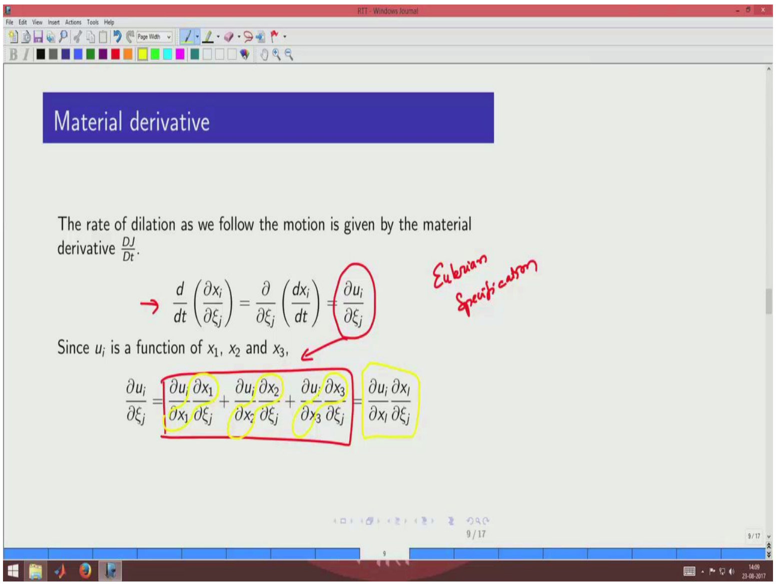

And we can now see what would happen when we take a material derivative that is we take complete derivative in the Eulerian specification and see how this quantities are going to change. So, we need a two terms for our derivations. So, first term and second term are enlisted here. So, consider this first term. So, $\frac{d}{dt}$ the D is capital which means it is for the advected material derivatives. So, which we see is that if you look at this we could just take this in and then you see that $\frac{dx_i}{dt}$ is nothing, but u_i. So, this quantity can be written as this.

Now, we want to then expand this here and we see that you can expand it by this formula because we see that u the velocity is a function of all the 3 spatial coordinates because velocity is specified with Eulerian specification. So, this is because of the choice Eulerian specification we have taken. So, this becomes a function of the 3 variables x_1, x_2, x_3, or x y z for that matter and which means that when we are looking at this derivative partial derivative $\frac{\partial u_i}{\partial \psi_j}$ we can then use the chain role and then express this as summation of 3 terms and that is why the x_1, x_2, x_3, these 3 of them are coming. So, this cross terms show that you are using the chain role.

Now, once you have the chain role then you can also see that there is the summation there and which means that we can use subscript notation to simplify. So, we choose an index which we did not use in other context. So, dummy index l. So, with the dummy index l we then write the expression very simply here $\frac{\partial u_i}{\partial x_l}\frac{\partial x_l}{\partial \psi_j}$. So, it is a matrix of 9 elements varying with i and j taking values from 1 to 3 and then l is telling the summation. So, we have the summation coming here. So, this expression is then going to be used as follows.

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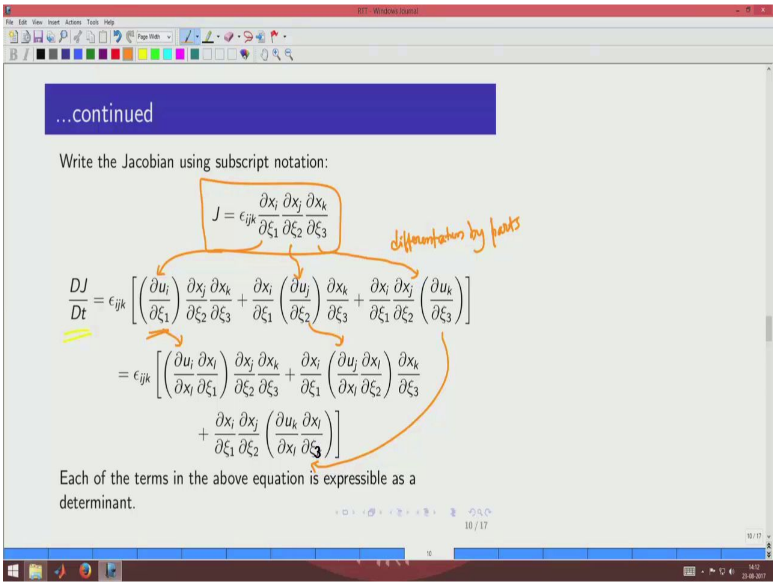

So, we want to now look at the Jacobian and see what does it mean by taking a derivative of the Jacobian. So, if you take a material derivative of Jacobian that is we write here. So, what we do is that look at the definition of Jacobian and when we want to take a material derivative we then have to apply $\frac{D}{Dt}$ to each of these terms. So, we can do it by parts. So, we can say what we are doing is differentiation by parts.

So, when we do it by parts. So, you take the first one and then we immediately see that this is what is coming. So, you can verify here. So, you see that $\frac{d}{dt}(\frac{\partial x_i}{\partial \psi_j})$ is nothing but $\frac{\partial u_i}{\partial \psi_j}$, which means that is what we have put in the first term. Now, when we take the second one then that comes here, the third one that comes here. So, by differentiation part by part we are actually now expanding.

Now, we already saw that this quantity can be then expressed as a function of x_1, x_2, x_3 and then we used l as the dummy index and therefore, this term can then be written here in this manner the same thing is true for other quantities also. Now, once we write this now you can see that we have got 3 expressions with an ϵ. Now, take term by term and see what will happen. Now if you see the first term you have got the indices ϵ_{ijk} , then i here j and k. So, for what values of l will this term survive? So, l can, l is a dummy index. So, it can take values from 1 to 3. So, what happens is that if l takes a value matching that of i then you will have $\frac{\partial x_i}{\partial \psi_1}\frac{\partial x_j}{\partial \psi_2}\frac{\partial x_k}{\partial \psi_1}$ with ϵ_{ijk} which means that there we can get these 3 out and then write that as for example, the quantity that we just now saw. So, which means that the one value will take only the matching value with I for the first term, it will take j for the second term, with k for the third term.

If it takes a value that is matching with any other things then what happens is that these two terms will be coplanar and then the values will be 0. So, technically we have actually you know 3 + 3 + 3 that many terms, but actually we will have only 1 + 1 + 1 because the remaining two in each combination will go to 0. So, that is the reason why we write like that and then we gather the terms. When we gather the terms we see that this and this together we already know what that is, that is a Jacobian.

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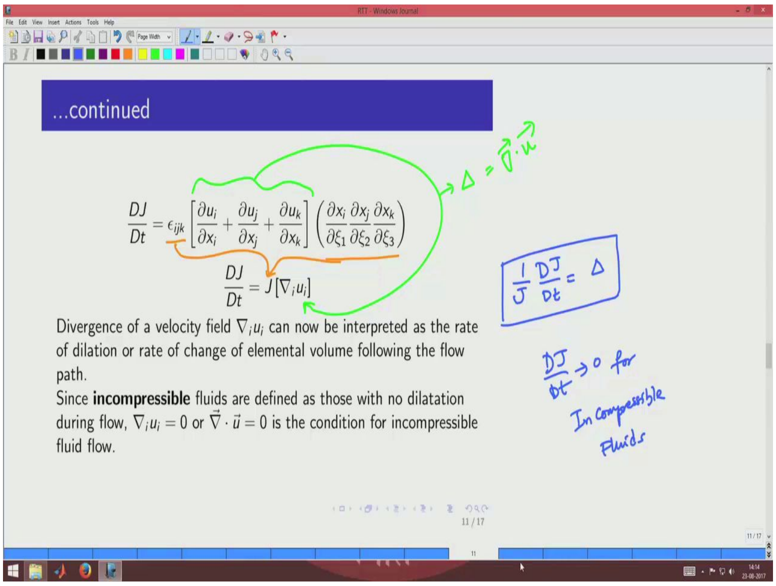

And we already have this terms separately we look it up earlier and this is nothing, but the rate of dilation which is ∇ So, this is nothing but the ∇ definition which we wrote it as $\nabla .u$

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So, clearly we can see that this term this expression by inspecting what values of l will allow the quantity to be nonzero allows us to go ahead and then make those manipulations and we now see that it is coming with nice gathering of the terms showing that the material derivative of Jacobian is nothing but Jacobian times at the rate of dilation, which we can write it in a normalized manner like this. We can write it as $\frac{1}{J}\frac{DJ}{Dt} = \Delta$. So, which means that we have a another way of describing incompressible fluids.

We earlier described or introduced the concept of incompressible fluids say in that those of the fluids for which the material derivative of density is 0 or density of a control volume as it is moving along the flow should not change. And here we can see that there is other way describing it if ∇ is 0 it also means that $\frac{Dj}{Dt}$ is 0 for the incompressible fluids. So, which means that we have one more definition the Jacobian of transformation from spatial to advected coordinates does not change with the time then that is an incompressible fluid. So, there are multiple ways of defining the same things. However, basically we use this expression for another term that come shortly. So, this is only one of the steps in the derivation of the transport theorem.

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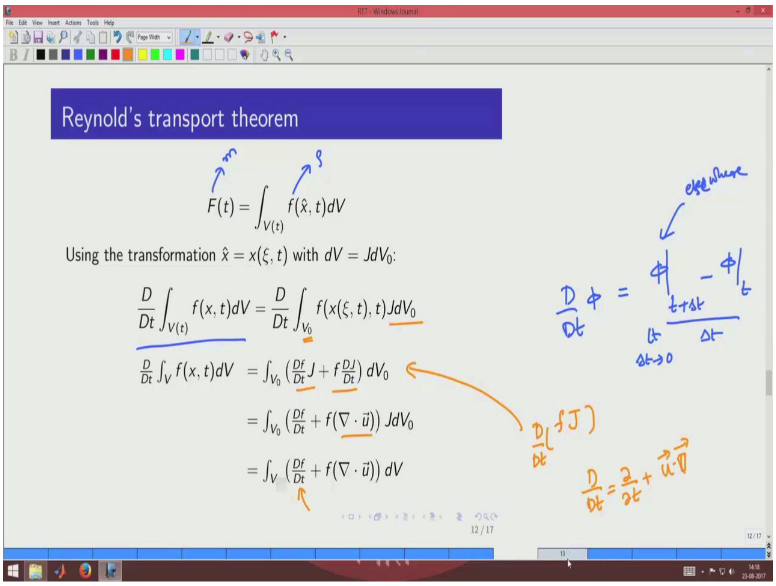

Now, the transport theorem derivation is very simple just four lines they are all in front of you and the idea is as follows. So, if you take any quantity that is expressed with the density then if you integrated over the control volume then you get the quantity out. So, you take the example like this small f, if this small f happens to be like you know rho then the capital F would be like the mass and in the case of small f being the concentration the capital F would then be the amount of solute for that particular quantity. So, like this you know you can actually think of any quantity that is specified as small f then respectively you have capital F also defined. And because we do not want our theorems or derivations to be specified by the geometry of the domain that we are looking at, it is always a good idea to write all the expressions as integrals and then look at what is there. So, so capital F is written as an $\int f dV$, where f small f is a density form of the capital F and in the case of mass it becomes the density itself and capital f becomes the mass.

So, now, what we are doing is that we see that if you were to inspect what would happen to the material derivative of the left hand side then you would see that we are trying to do the material derivative of a quantity which is actually changing its position. So, you could see that $\frac{D}{Dt}$ of any quantity. So, any quantity you would like that to be $\lim_{\Delta t \to 0} \frac{\Phi|_{t+\Delta t} - \Phi|_t}{\Delta t}$. So, which means that if at $t + \Delta t$ it is elsewhere then you have basically a problem you cannot then differentiate straight away. So, for us to do this differentiation we need to ensure that the integral e is actually over the same control volume of same volume.

So that is why we need to do the transformation. So, here what we have is that we have dV. So, we are then transforming the integration from over the volume which is defined in the spatial coordinates to the volume which is defined in the advected coordinates. So, we now go over the integration with V_0 and we already saw that dV is actually equal to JV_0. So, this, the transformation we already discussed so that we are introducing. Now, that the integration is over v not then we can safely take $\frac{D}{Dt}$ in. So, we take it in and when we take it in we know that there are two quantities that are occurring f and J. So, then we into differentiate by parts. So, we did that by parts. So, there are two parts, f and J. So, you have got f and you have got J and you then want to take this guy and we then can see that it is coming with two terms. So, differentiate f first and then J next.

So, we now can see that the integration is split into two parts and then we can separate them out. So, the first part then will be separated out. So, $\int_{V_0} \frac{Df}{Dt} J V_0$ and in the second part you can see that it is $f\frac{DJ}{Dt}$, and $\frac{DJ}{Dt}$ is just now saw that it is equal to the rate of dilation. So, we introduce that here. So, we could see that the expression we derived here is immediately applicable here. So, we introduce that. And then we see that the terms can then the collated and what we do is we expand this term. So, we know that $\frac{Df}{Dt}$ is defined as material derivate. So, we expand it. So, we already know that this operator is nothing but this operator. So, we use that expansion and then we write the terms there.

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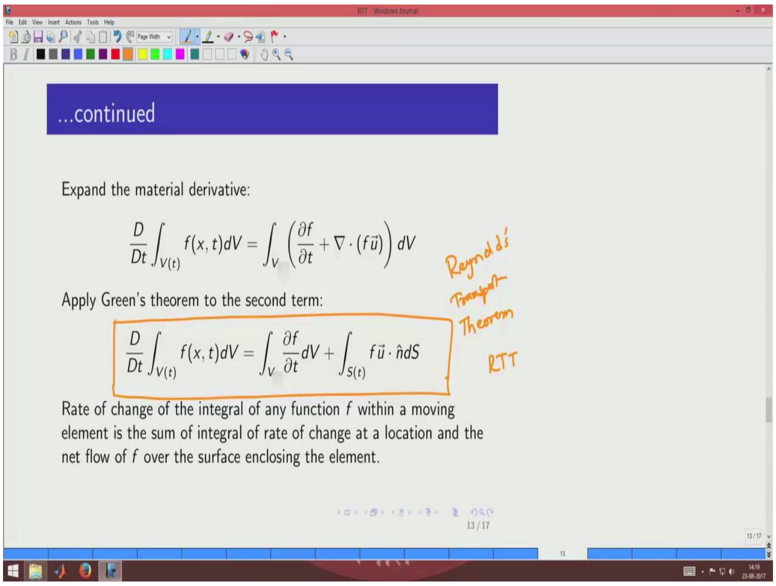

And you could see that on the right hand side when you the expansion you then have the combination of the terms. So, the f can go along with the u inside and we write this. So, you have now a dot product and with a $\int_{V_0}$ then we can change it also to a surface integral and that is what is done here.

So, what we have arrived at basically is a theorem that tells us that if there is a quantity a density of a quantity small f that is being advected and we want to know what is the material derivative of it and then we have it possible, and the way it is defined as follows the rate of change of integral of any function f within a moving element is a sum of two terms, the integral of the rate of change at one location and the net flow of that particular quantity over the surface enclosing that element. So, you could see that it is defined with two terms here and this basically is the transport theorem and we call often it as Reynolds Transport

Theorem, so RTT. So, very often people refer to that as RTT. So, RTT is this particular expression. So, you could then substitute various quantities for small f and see how the expressions turn out to be and those are the applications of the RTT. So, let us just take two applications because we need them immediately for our next few sessions.

So, one application you can take is what happens if that small f is nothing but the density itself. So, small f is a density. So, you see that here we put a ρ and here you put a ρ. So, the same expression is written you put a ρ here you put a ρ here.

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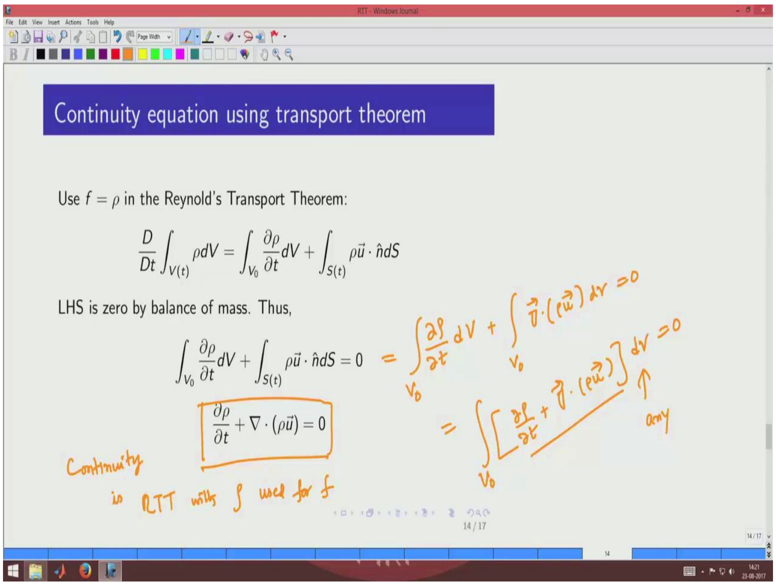

Now, you can see that the left hand side is giving you the material derivative of the mass of the particular control volume which is moving and that is expressed as part two parts the change of the density at particular location and the flux of mass around the control volume at which we are evaluating this. So, which we then expand and then we can see that immediately on the right hand side you could see that the term gives you the equation that we already know. So, the left hand side has to be 0 because you cannot have mass change and therefore, the right hand has to be 0 and immediately we see that the continuity equation has come out of it.

So, this has to be changed to be the volume integral and then it is both terms are over the same volume and therefore, we can write like this. So, the intermediate step for this would be like this. So, let us just write it off, so that we. So, the sum of the two integrands is over the same integral. So, you can do the sum inside and therefore, you could write it is like this. So, you could write like this. So, and if this were to be true for any control volume at any location then; that means, that the integrand has to be 0 and that nothing, but is the statement of continuity. So, you can see that continuity equation is an outcome of RTT with ρ used for f. So, if you substitute the function f with the density then Reynolds Transport Theorem reduces to be continuity equation. So, therefore, it is a much more generally applicable equation we can say.

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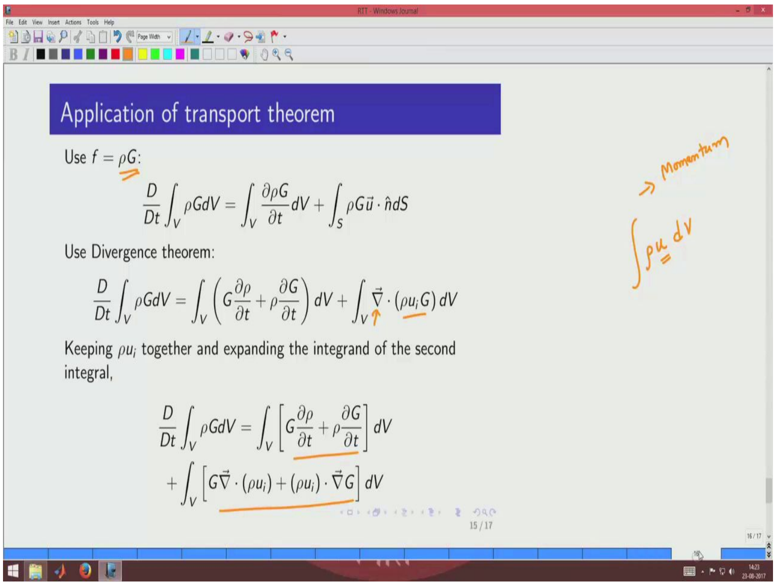

So, now, we can also then use it for quantities that can be expressed as advected quantities for example, let us take momentum. So, you would like to write momentum as follows. So, instead of writing mass x velocity what we do is that mass$x \rho x u$. So, this would be the momentum. Now is basically mass into velocity.

Now, we can see that here we have a situation where a rho is coming with a particular function which gets advected. So, we want to now think what will happen in a general case. So, we write it as rho G. So, in a G is basically a quantity that is getting advected and we want to just look at how the expression turns out to be. So, what we do is that we substitute small f is equal to ρG and then just expand it and we see that on the left hand side the material derivative of ρG dV and on the right hand side we can then expand it. And we see that you have this two terms and then when we separate the terms by applying this operator on each of these quantities separately then you will see that you have got a quantity here and here, and there is a small amount of algebra we can go through that because steps are quite straight forward you can just do differentiation by parts. So, there is nothing more to that.

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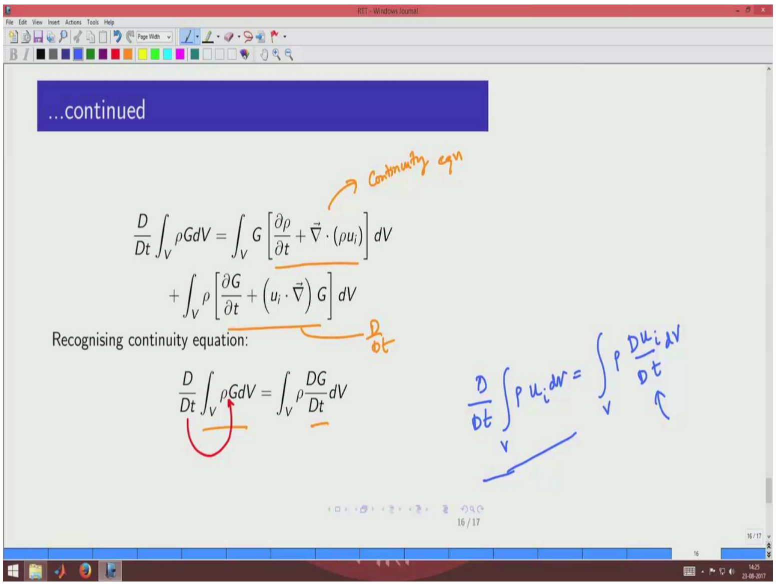

And then when we proceed further we see that while gathering that terms you have the continuity equations sitting in there. The moment you have continuity equation sitting in there we can apply it and then knock that off and then we knock that off and then you notice that this is nothing, but operator $\frac{D}{Dt}$ operator and this is nothing but the continuity equation. So, what happens is that left hand side $\frac{D}{Dt}\int_V \rho G dV = \int_V \rho \frac{DG}{Dt} dV$ which means that whenever you have got a material derivative with integration where one of the quantities is ρ then when you take that in then you should go like that. So, it goes in here. So, the ρ is coming out.

The reason is that the continuity equation is actually knocking off the term that correspond to the differentiation with respect to ρ. So, now, this is very valuable because this gives us a hint that when we have a term like this, when we have terms like this for example. So, we can then use the Reynolds transport theorem to write it in this manner and we can also apply this for vectorial quantities because you could do it for each component at a time and therefore, which means that you could also do it in these situations.

So, now, this is valuable because this is nothing, but momentum and this together gives you the rate of change of momentum and therefore, we can now see that rate of change of momentum can be related to acceleration. So, you can see that this term is nothing, but acceleration in Eulerian specification. So, we have a way by which we can take this $\frac{D}{Dt}$ inside the integral and that is going to be useful for us in a moment in the derivation of Navier Stokes equation.

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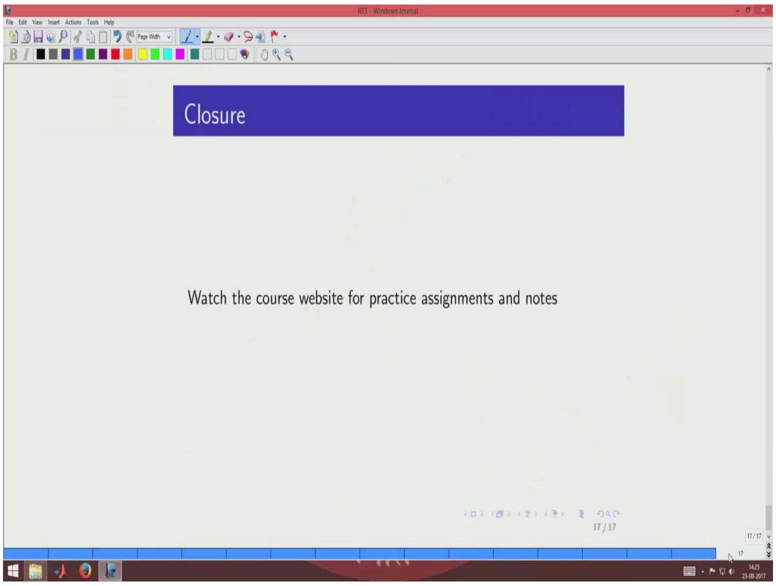

With that we close this session. We can actually use this session essentially to find inputs for the derivation, but otherwise if you are not very thrilled by mathematical jugglery then you can skip this and that still will not affect the way you learn this transport phenomena course.