Transport Phenomena in Materials

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Lecture - 07
Planar Flows

So, welcome to the session on Planar Flows as part of the NPTEL MOOC on Transport Phenomena and Materials. By planar flow we mean flow field where the velocity vector has only 2 components and we can also call them as 2D flows.

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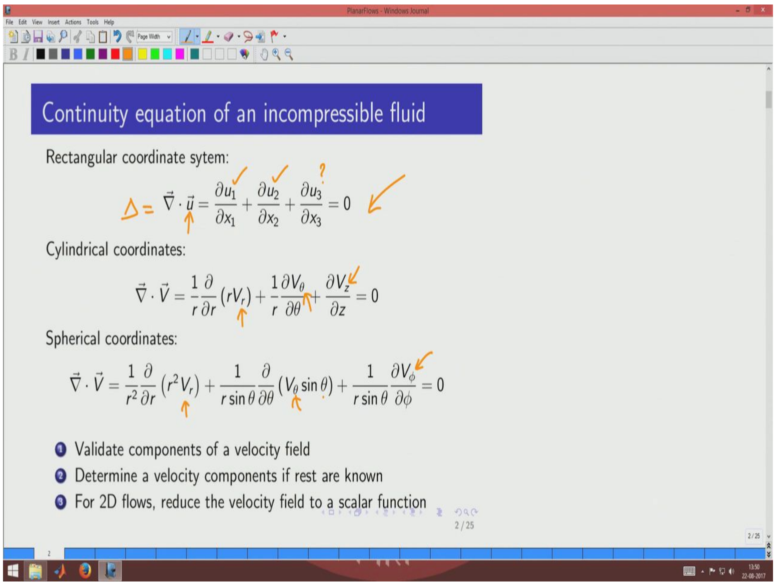

So, a quick recap of the continuity equation which we derived and used in the previous session. So, we are writing here the continuity equation for an incompressible fluid and if you take the general 3 dimensional flow field u which is given by the $\vec{u}$ here then it has 3 components u_1, u_2, u_3. And therefore, the continuity equation or the mass balance for an incompressible fluid will become the rate of dilation which is given by the symbol $\Delta = \vec{\nabla}.\vec{u}$ is equal to 0 and we have got this equation.

Now, this equation can also be written in other coordinate systems and we have done that earlier and introduced. So, in cylindrical coordinate system the velocity will have the radial component, the θ component and the z component and the continuity equation would look like this. And in the spherical coordinate system you would have the radial velocity the θ component and the ϕ component. As we (Refer Time: 01:32) and then the continuity equation would look like this.

And the application of continuity equation can be done in various phase. So, we have basically a possibility to validate the velocity field has being valid or not which means that is the velocity field allowing the mass conservation to be possible and if it passes the test then it means that the valid velocity field which we can use. And we can also use this continuity equation to determine unknown components from a velocity field. So, in this case for example, if you have 2 components u_1 and u_2 available and we want to find out what is the u_3 components then readily we can plug in u_1 and u_2 integrate the expression with respect to x_3 and then we have got the u_3 component made available.

And in the case of 2D flows for example, we can also reduce the number of unknowns from 2 components of the velocity to 1 unknown using the continuity equation because it can be treated as an additional constraint and that is exactly what we are going to do in this session.

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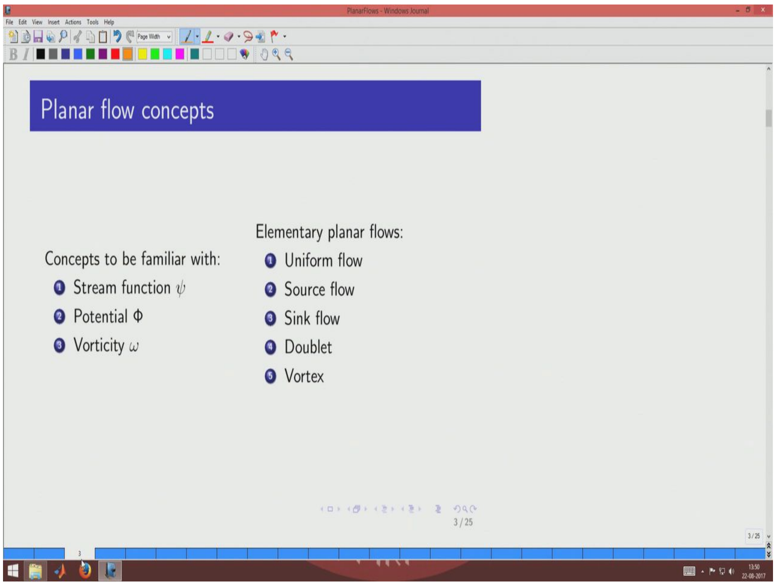

So, in this session we are also going to introduce some new terminology stream function ψ, potential ϕ and vorticity ω. We are going to define these things as part of this session. So, that we can also use then to describe 2D flow fields.

We will also be getting use to the terminology use to describe the flow field in 2D when we want to call something as the uniform flow or source flow, sink flow or doublet flow vortex etcetera. Generally we can use combinations of these to determine how the flow would be called in any general situation. So, we want to find out what is the expression that can be used for each of this elementary planar flow fields.

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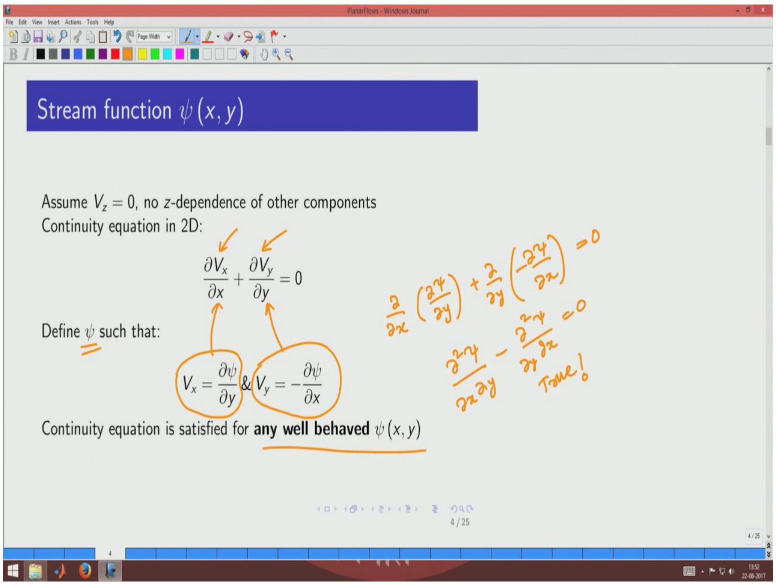

So, we are now going to reduce the number of unknowns in a 2D field. So, you have basically 2 components of a velocity field. Here we are having them as V_x and V_y as the components in 2D, which means that we now have the continuity equation having only 2 terms and that is written here and we can use this equation to reduce the number of unknowns from 2 to 1. So, the way we do it as follows.

We will define a function ψ in such a way that this is how the velocity components are defined. So, you define ψ, in such a way that the velocity components are given by the differentiation of ψ, with respect to x and y as here. So, what happens when we do that is when we then plug in to the continuity equation then you could see that the first term would become $\frac{\partial}{\partial x}$ and then the V_x actually is $\frac{\partial \psi}{\partial y}$ and then the second term is $\frac{\partial}{\partial y}$ and V by is actually minus of $\frac{\partial \psi}{\partial x}$. And this implies that it is $\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0$, but this is true; true for any well behaved function ψ. So, if ψ is well behaved function then this is always true which means that if we can actually come up with a function ψ which is well behaved and we can take its derivative with respect to x and y and then call those 2 expressions as velocity components because they would readily satisfy the continuity equation. So, because of this utility we actually would like to give a name for this function and we call it as stream function.

So, the reason why it is called as a stream function will be evident in a moment, but at this moment we would just call ψ as a function that allows us to generate the velocity components. So, we could have defined for example, these expressions we could have defined with minus sign either or further V_x component or V_y component it could be both options are possible and the continuity equation would satisfy either way.

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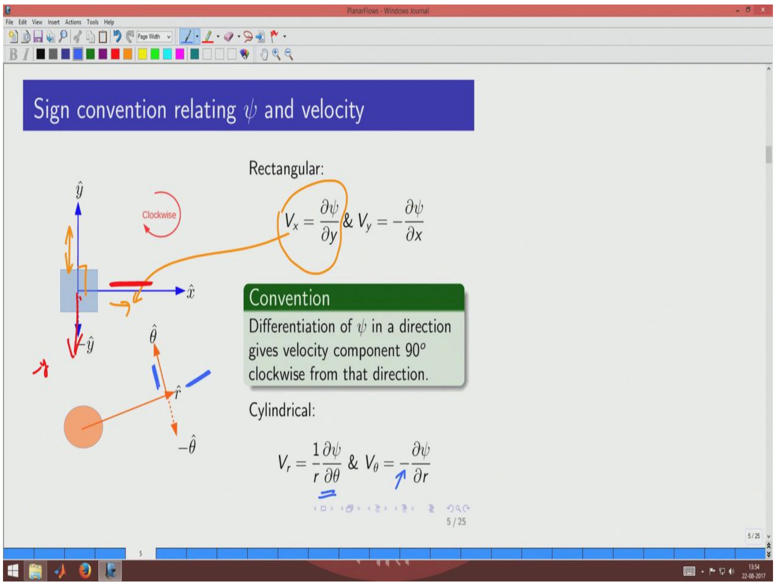

But there is a convention by which we define and that is the convention is what is being introduced. The convention is as follows the derivative of ψ if you take along 1 direction then 90° away in clockwise is the velocity component being given. So, which means that if you look at this first expression you can see that $\frac{\partial \psi}{\partial y}$ is being taken in this direction and then 90° away is this direction which is x which means that is what is being given as a component.

And you can see the second expression again the same thing is valid you are differentiating with respect to the x which means that you are differentiating in this direction and then 90° away will be this direction which is actually the minus y direction and this will be giving velocity in the -y direction and that is why there is a minus symbol here. So, this is the convention that is used, so that in any other coordinate system if you want to define the velocity components and if you follow this convention then the sense of ψ that we use will be the same in all the coordinate system. So, for example, if you take the cylindrical coordinate system the $\vec{r}$ is shown here, which means that when we differentiate with respect to r then 90° away will be -θ direction and that is why we have a minus sign here and then if you differentiate along θ direction then 90° away is r direction and therefore, we got V_r there. So, the convention is being used in all coordinate system same way and the convention is basically differentiation of ψ in a direction gives velocity component 90° clockwise from that direction.

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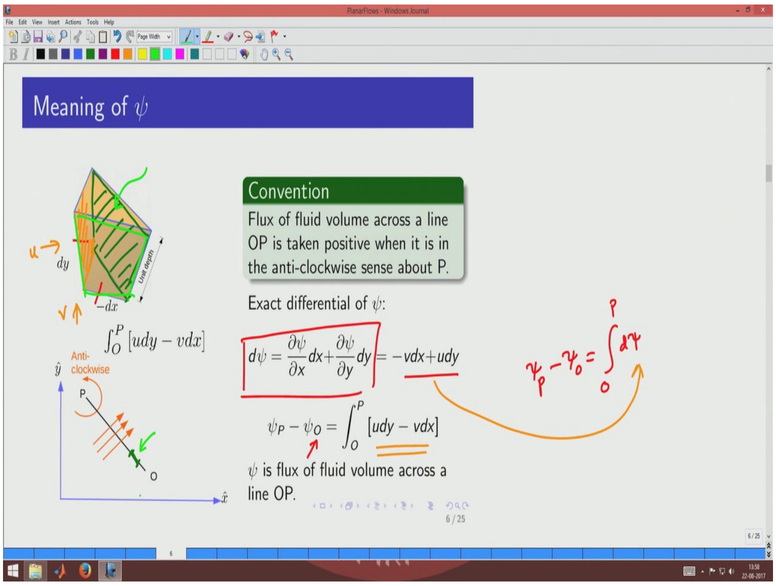

Now, what is the meaning of this function stream function we have mentioned that it is useful it will definitely reduce a number of components from 2 to 1, but then does it have any sense in the what we can actually call it as. So, we have that being defined here. So, consider a small volume element and remember that we are using it for 2D flows. So, the area element in 2D will become a line element. So, we want to consider a line element as follows. So, I take a line elements like this.

So, this small line element between O to P will then be given in terms of its components dx and dy and in 2 perpendicular directions now in the vertical direction if you have unit depth which means that we are talking about flow through this plane. So, we are talking about the flow through this plane and you could see that the flow through that plane can be thought as flow through the 2 planes that we are indicating here, one is through this and another is through this. So, you can actually split the component in this manner and the convention that is used to sell to tell whether ψ is actually positive or negative is as follows.

About P the flow we are giving is actually anti clockwise. So, I have given this direction here, so about P, which means that if you are taking the value with reference as the O point then $\phi_p - \phi_O$ is given as the flow through the line segment in the anti clockwise direction. And then now why are we saying that it is actually the volume flow of fluid through that segment that will be evident like this. So, you see that the ψ is actually a well behaved function, so you could write the exact differential of ψ in this manner. So, it is now a function of x and y. So, you could write it like that and when you write like this and you can already see that the 2 differentials can be expanded and you now have this expression, which means that because this is an exact differential, you could actually say that $\phi_p - \phi_O$ is nothing, but $\int_o^p d\psi$.

Now dψ is actually already available to us. So, that is what we plug in here and that is what we have written here. Now, you could see that a term by term what does it mean if you if you take the first term udy, udy is basically velocity in the u direction through the dy element. So, this is the dy element and the velocity is in x direction, which means that the first term correspond to flow through this are element.

Similarly, the second term. So, you could see that the distance from O to P is actually -dx, which means that the second term is be velocity through this plane through this plane and which means that together when we summed up in this manner it would mean that we are talking about the flow through that plane which is nothing but this element with a unit depth in the z direction. So, we can see that the meaning of this quantity ψ is nothing but in relation with respect to the ψ value at another location reference value at a location O it gives you the volume of fluid going through the area that is formed by the line segment OP and the unit depth in the z direction. And if you forget the z direction for a moment it means that the volume flow across the line segment PO. So, that is, what is the meaning of a ψ here?

It also means that when you have the ψ field plotted as contours then the distance between the 2 contours will actually give you the measure of the fluid flow in that particular direction.

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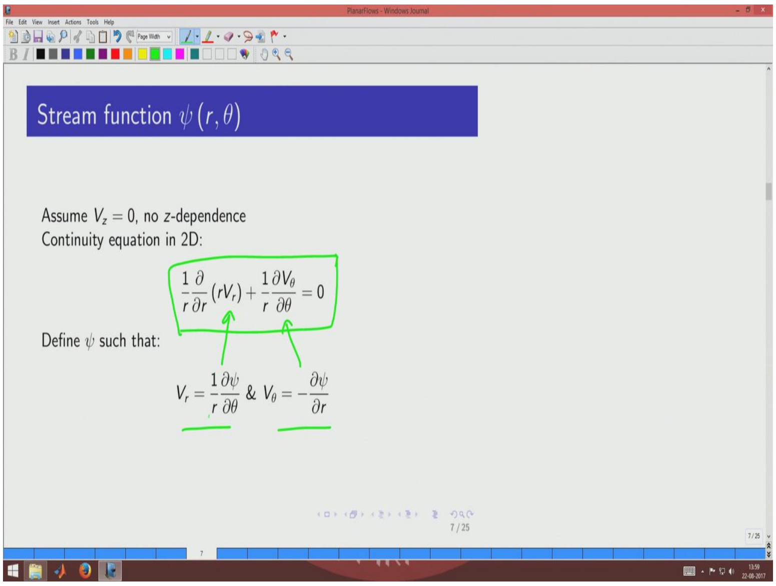

So, we could actually now that we have this function ψ we could actually define it for other coordinate systems. So, we are defining it for example here cylindrical coordinate system where r and θ are the 2 directions and we assume that z direction there is nothing happening there is no velocity in the z direction and therefore, we can write the continuity equation with only 2 terms and that is what is done here. And then we define the ψ the same manner with the same convention that we have done earlier and we could see that when we substitute these then the continuity equation is quite satisfied. So, you could already see that the r gets cancelled and we have got each term giving an $\frac{\partial^2 \psi}{\partial r\, \partial \theta}$ with plus sign in the first term and minus sign in the second term . So, therefore, automatically it is satisfied.

So, if you want to generate the velocity components in cylindrical coordinate system then we come up with any function ψ which is a function of r and θ and then take the differentiations like this and then they will become the component of velocities along r and θ directions because this already takes care of the continuity equation to be valid.

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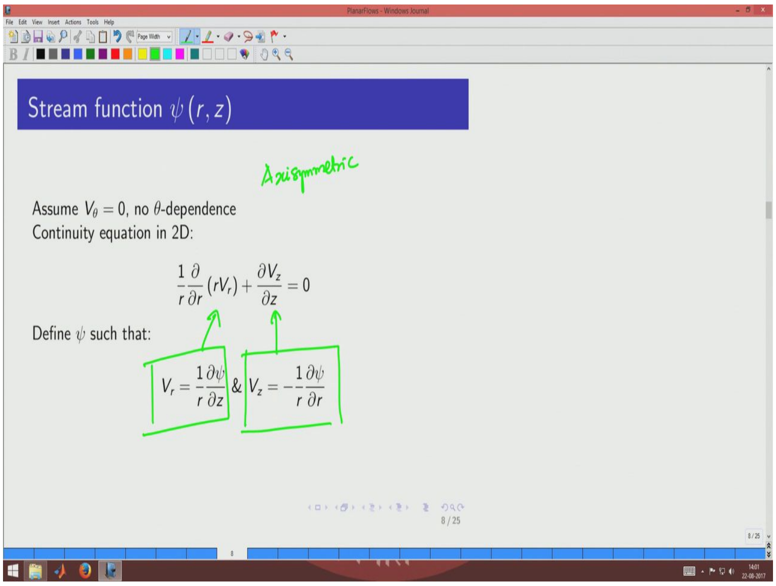

So, we could also define the stream function in cylindrical coordinate system in another manner. So, in the first case for example, we are taking it in the form of a r θ which that within the plane and here we are actually taking for axisymmetric, axisymmetric flows because in the θ variation is now being dropped this is also 2D flow. And we could then see that the terms that are remaining in the continuity equation are different pair and we could also define the stream function in the cylindrical coordinate system for axisymmetric flow in this manner and we could readily see that when we substitute we will see that the continuity equation is satisfied. So, this way we can actually define the stream function for 2D flows in cylindrical coordinate system in 2 different manner.

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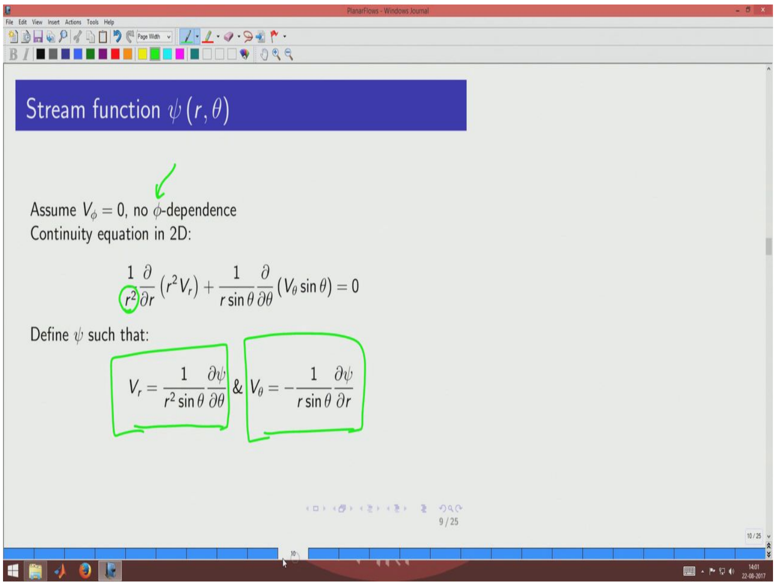

And we could also do that in spherical coordinate system and the difference between spherical and cylindrical will be evident here. So, watch out the power of r. So, you could see that here it is $\frac{1}{r}$ and here just $\frac{1}{r^2}$.

So, even if you did not label the expression correctly sometimes we can actually detect it by looking at these terms carefully and the continuity equation in spherical coordinate system where the ϕ dependence is lost then we can actually see it as a 2D flow and the continuity equation is given with two terms. So, which means that the stream function in spherical coordinate system can also be defined in this manner and these two are expressions that we use to generate the fluid flow components in r and θ directions in a spherical coordinate systems.

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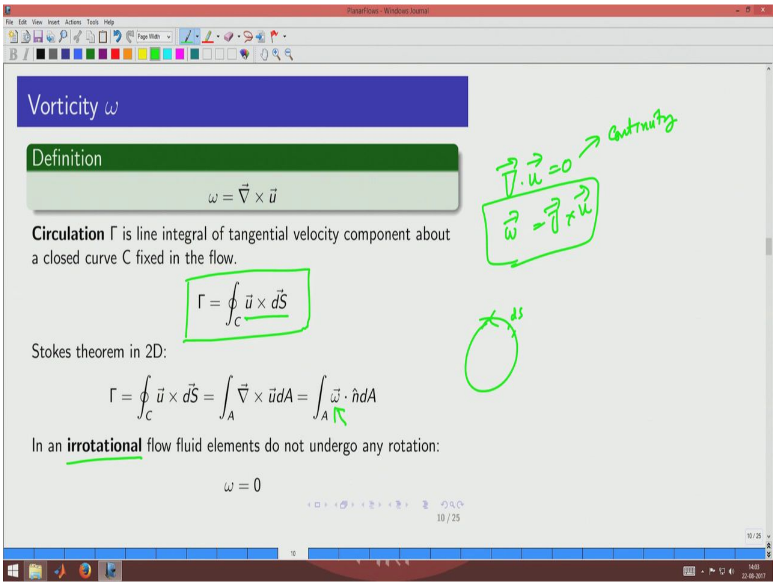

So, now that we have defined it we have also other quantities that we need to make ourselves familiar. So, we introduce a quantity called vorticity ω. So, vorticity actually is nothing but the $\nabla\ x\ \vec{u}$. The reason why we say that this quantity is important will be evident in moment.

So, we have this curl and we already saw that this is applicable and we now want to give a name for this quantity. So, this is applicable because of continuity equation and we want to give a name for this because when that is 0 we want to define it with another name. So, if this is 0, this if what it is 0 then we want to give a name for that kind of a planar flow and we call it as a irrotational flow.

So, the way this ω is defined is such that we have a sense of circulation of the flow. So, the circulation of flow is defined in this manner capital gamma is a line integral around a closed loop given by this expression $\vec{u}\ x\ dS$ where dS is the length element along the loop. So, you are integrate along a close loop and for every segment dS we want to now do this integral. And we could then already see that using this stokes theorem in 2D we can actually convert the cross product into the dot product and we can already see that the circulation is given by ω itself. So, we can say that the vorticity and circulation the sense of how the fluid is actually going around is actually embedded in the definition of vorticity because of this expression what we have written here and in the case that the ω is 0 then we would like to call that flow as irrotational flow.

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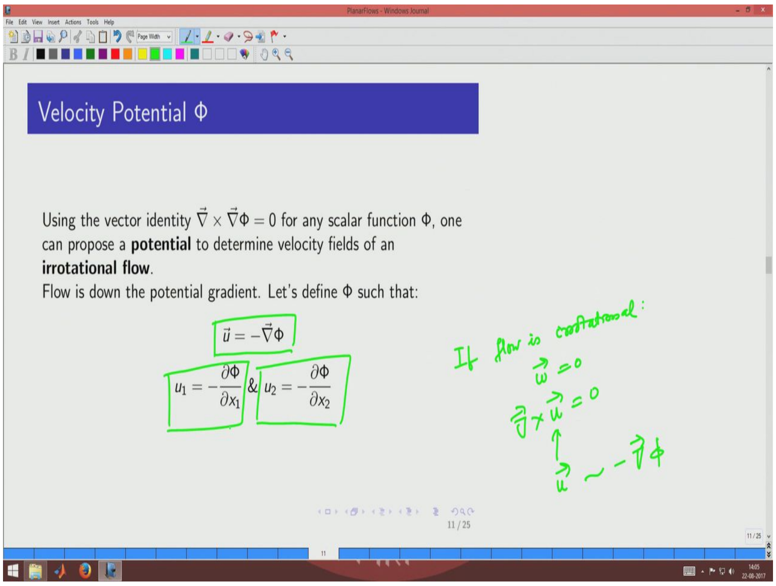

So, the reason why we have actually come up with this particular type of flow is because it allows us to define another quantity which we call as velocity potential. So, that we can actually generate the velocity components for a very specific 2D flow situations. So, we call that as basically velocity potential in the following manner.

So, we saw that if it is a rotational, if the flow is a rotational which means that we say that this is 0 or the $\nabla\, x\, \vec{u} = 0$ then we know from the vector identities is that if this vector happens to be actually gradient of a scalar function then it will be always true, which means that if we were to now think of this as related to some kind of a gradient of a function φ then this is always be true for any well behaved function φ. And we want to define with a minus sign that is because generally it is felt that potential is one where its gradient should lead to the flow. So, if you want to have that then the down the potential gradient is the flow taking place it is true for many many other phenomena as well for example, temperature gradient down the temperature gradient is where the heat is actually flowing and so on. So, in that similar way if you want to define then we will put that minus sign there, which means that we have one more way by which we can generate the velocity components that is by using the velocity potential.

So, you could see that the way velocity potential is used to create the velocity component is given by this expression. You could see that this is different from the stream function expressions you could see that the indices are matching here and in the case of stream function then will be opposite. So, that is how we can just know recognize these expressions when it different and we can now go ahead and see what are the kind of functions ϕ which we can come up with and then plot the velocity vectors for all those functions and see how those flows look like. And then by giving names to such flows then we can actually describe in any arbitrary situation how the flow is actually happening.

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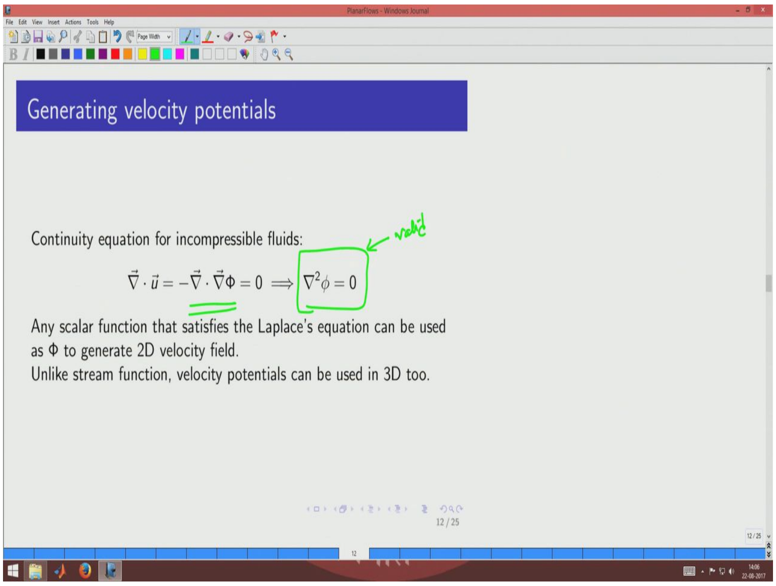

Now, the way we go about creating this potentials as follows. So, if we have created a function ϕ which is like this we already know that it satisfies the continuity equation we see that then this must be true which means that any function ϕ which satisfies the Laplace's equation must be valid for us, which means that we now look for functions of ϕ such that this is valid. Then we automatically know that the continuity equation is valid and we also see that it is going to give us the velocity component. So, this is how we go about generating them.

Technically the stream function gives us flow components in 2D, but the velocity potential has no such restriction it can give us velocity components in 3D as well. So, it is a little more generic concept than stream function.

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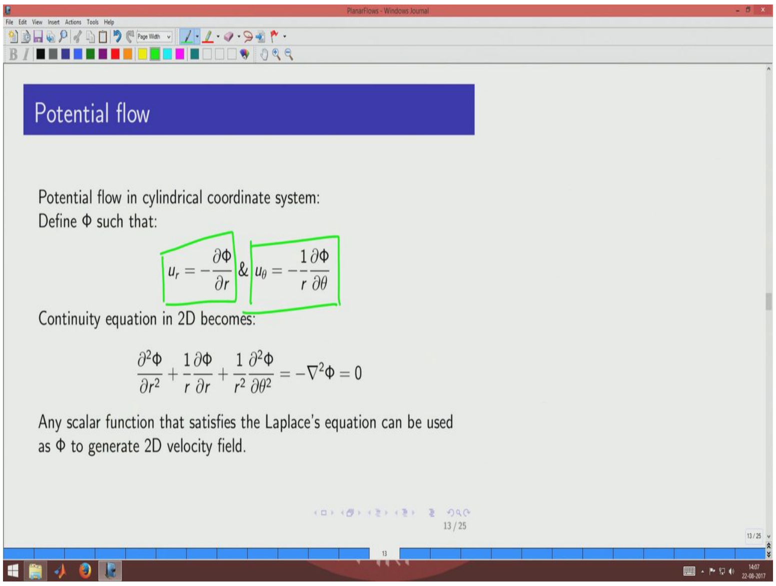

So, the potential flows are basically flows where the flow is irrotational and the flow is also coming with components from the velocity potential function. So, such flows are categorized separately as potential flows and we can then see how they can be arrived at. So, you could also see those potential flows in other coordinate systems like cylindrical coordinate system and we can see that we could define the components in this manner the rθ components of the flow for a potential flow can be defined in this manner and we know that when we substitute into the continuity equation then it will be satisfied.

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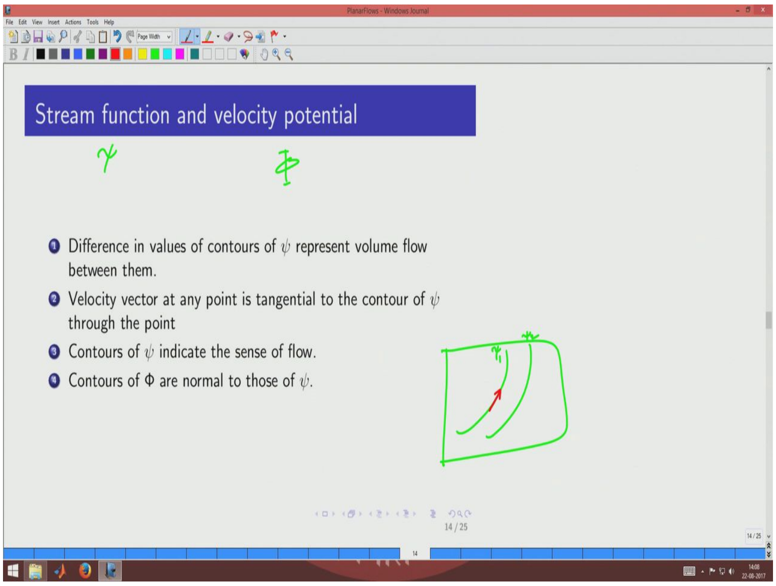

So, what is the relationship between these two things that we have just now introduced? We have introduced ψ which is the stream function, we have also introduced velocity potential ϕ so how are these 2 related. So, we know that the ψ has meaning that difference in the values of the contours of ψ represent the volumes flow between those 2 locations and velocity vector at any point is tangential to the contour of ψ to that point. So, if which means that if you draw contours of ψ in any domain for example, if it were to look like that, so ψ_1 and ψ_2.

Now what it implies is that when we take a tangential vector then this vector is actually also going to give us the velocity components at that particular location. So, that is how the sense of flow is captured by the contours of ψ that is one utility of the stream function. And we also know that if you look at the components that the contours of ϕ will be normal to that of ψ. So, this we will also practice with 1 or 2 examples shortly and one can also take it as homework to prove the same the idea is basically take the exact differential of ϕ and exact differential of ψ and then inspect the terms and then that would be evident to you.

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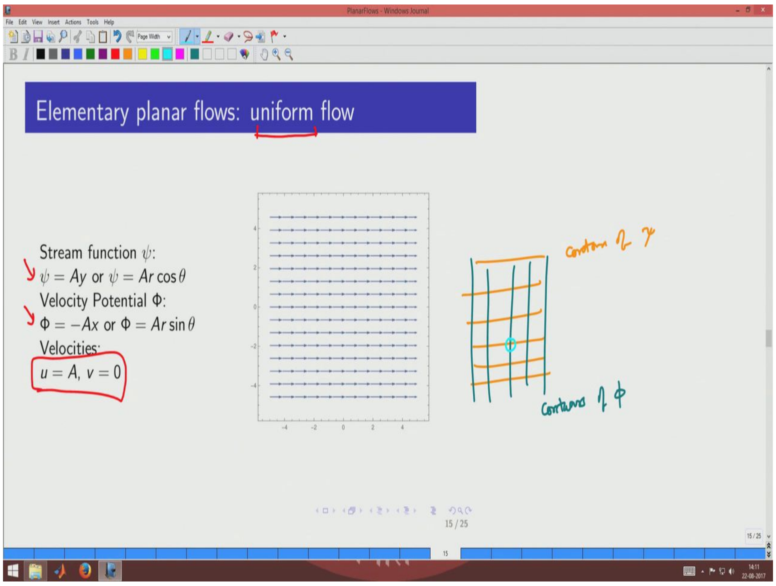

So, we have some examples now. So, we are going to introduce elementary planar flows here. So, we have those names coming up and those names and those flow patterns that is a kind of relationship that we want to now use to describe flow in any arbitrary situation. So, uniform flow as a word in immediately implies it means that the velocity is constant and in only 1 direction and you could actually see that the uniform flow can be described in this manner you could have the φ or ψ which is a y or -Ax. So, which means that it is a potential flow and we could actually generate them either through the stream function or through the velocity potential and the velocity itself is given in this manner here. So, in all these examples in the next several minutes we are going to give the velocity components and also from where they are generated.

So, you should use the expression that are given in this session on how to generate the velocity components from stream function as well as the velocity potential and verify that all these expressions are valid. So, when we take this field and then plot it in software for example, this particular plot is made using mathematica. So, you can plot and see how the flow field would look like. Obviously, this can be expanded to any other direction. So, you simply have to change this x and y to be y and x respectively and you have the uniform flow in another direction that is coming up.

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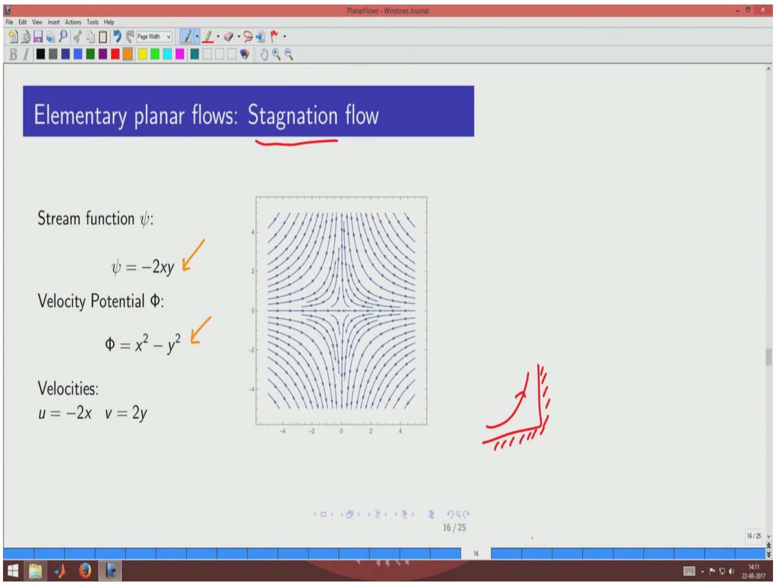

So, the stagnation flow is basically something that we have discussed earlier namely flow near a corner. So, if you have a corner and how the flow would look like near the corner is described by a type of flow that is called as stagnation flow and the stagnation flow has the ϕ function given here and you could also see here that the stream function and the velocity potential are given here. And when you plot the contours you could already make out, so 2xy you see that this must be hyperbole and you can already see that the stagnation flow will give you hyperbole.

So, just a moment to recollect what we discussed earlier about the contours of stream function and velocity potential being normal to each other. So, let us just look at this what would be the contour of stream function it will be at different values of y which means that those contours are going to look like that. So, these are the contours of ψ in this particular situation and we can see that the contours of ϕ will be different values of x so those would be looking like that these are the. As you can see that at every point where they are intersecting the contour of ψ and contour of ψ are normal to each other.

So, this can be validated for all the flow fields when we go ahead. So, that is what I was telling earlier and here you could see that the contours of a ψ will be given by hyperbole and then the contour of ϕ should be normal to the wherever they are intersecting and the velocity components are given by the components that are here -2x and 2y. So, the flow field near corner can be described by stagnation flow.

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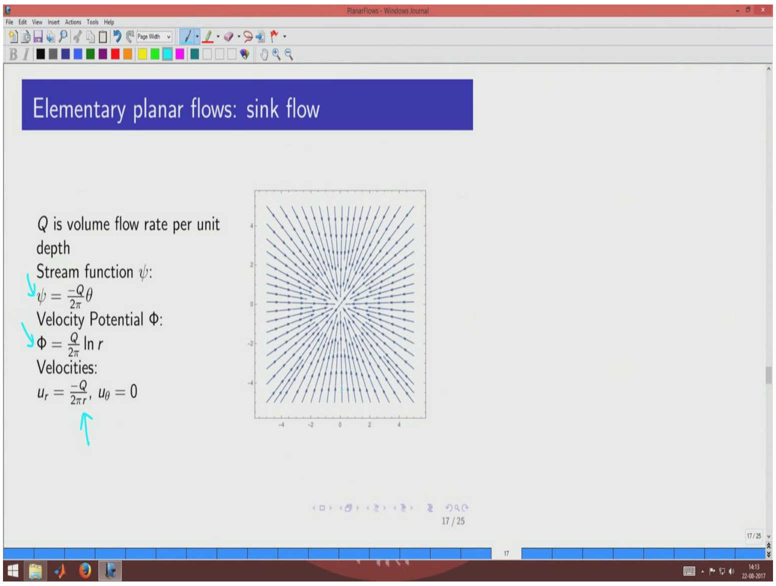

Sink flow is a situation where you have got a singularity a location at the center here in this case its 0 0 location where the flow is actually converging and you have a problem there actually, but you could actually imagine that this is a flow field where in a tank there is a small hole at the center and then the liquid is going through that to the bottom. So, that kind of a flow field can be described using sink flow. And the stream functions and velocity potential for sink flow are given here you can see that the logarithm r is what is giving you the velocity potential and if you take differential with respect to r is $\frac{1}{r}$ and you can immediately see that the velocity is unidirectional along r direction as $\frac{1}{r}$.

So, in the case of cylindrical coordinate system we already know that in a single component velocity as you go away from the origin the strength of the flow field has to come down as $\frac{1}{r}$ and that is already used here. So, the inside that we have from continuity equation can be readily applied here. So, the way we plotted here it is not evident, but the magnitude of this vectors should die down as $\frac{1}{r}$ when you go down center to the distance away from the center.

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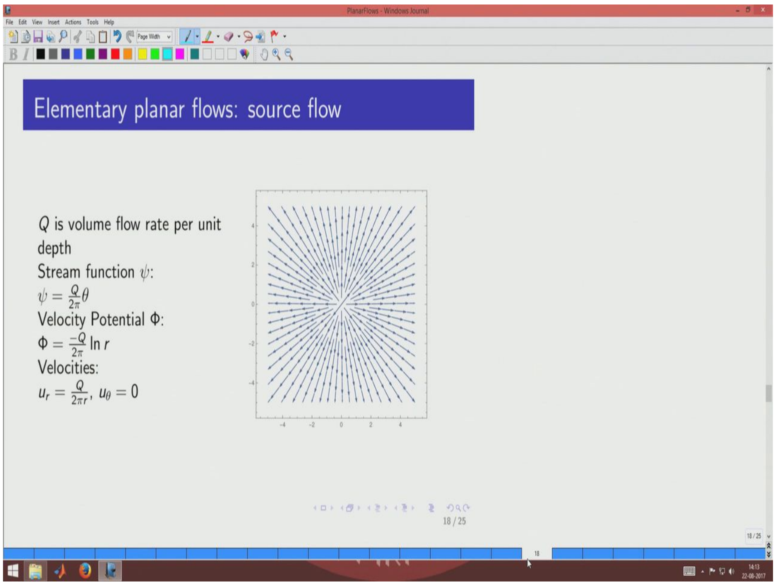

So, the source flow is the exact opposite of sink flow. Here basically the directions of the arrows are reversed and you could see that it can be given by the function that is just with a minus sign. So, you could also see from here the velocity components are just plus of $\frac{Q}{2r}$. So, the quantity Q in both the sink flow and source flow basically indicate the volume flow rate per unit depth the z direction and that can be used to calibrate the velocity components in the r direction.

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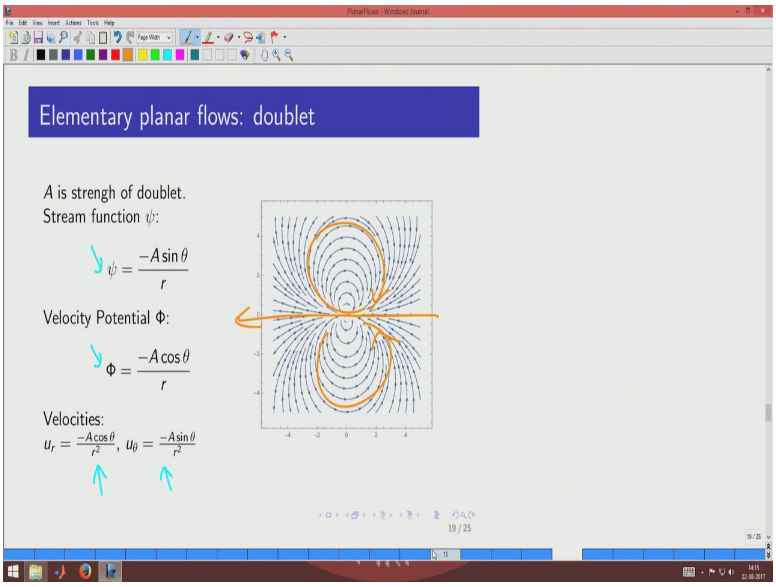

So, there is another flow field elementary flow field called doublet and doublet is described by stream function and velocity potential that are given here. So, you have got a sin θ and cos θ there and now here actually it is not really a unidirectional flow it has 2 directions both r and θ direction components are available here. And then we plot them you can see that the flow looks like here. We can imagine that the flow is basically in this manner and you have got one rotation this way and another rotation that way. So, the doublet flow can be coupled with other flows to generate complicated you know real life flow situations.

The idea why we are introducing the elementary planar flows is because in situations where we need to have some idea of how the flow field would look like we can mix and match these things an calibrate and arrive at a velocity field without even solving any equation. So, that is the one of the application of using elementary flow fields.

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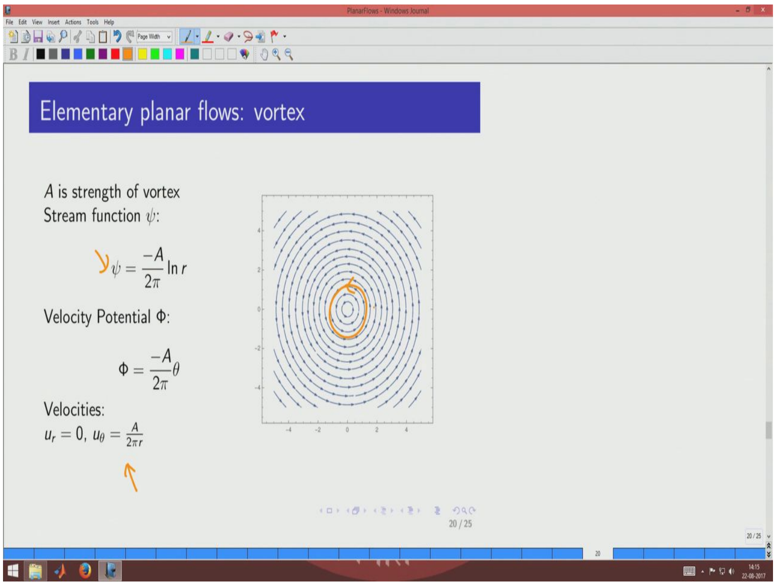

The last one is vortex flow. So, vortex flow as a name indicates should have a rotatory motion of the liquid. So, we can see that vortex flow would look like that, where the liquid is going around and you have basically that as only the θ component being present r component is not there. And you can already see that the magnitude of the velocity should go down as $\frac{1}{r}$ cylindrical coordinate system and that is also satisfied. So, you could plot the contours of ψ and you could see that they go in circles concentric circles to give a sense of flow that is going around the origin.

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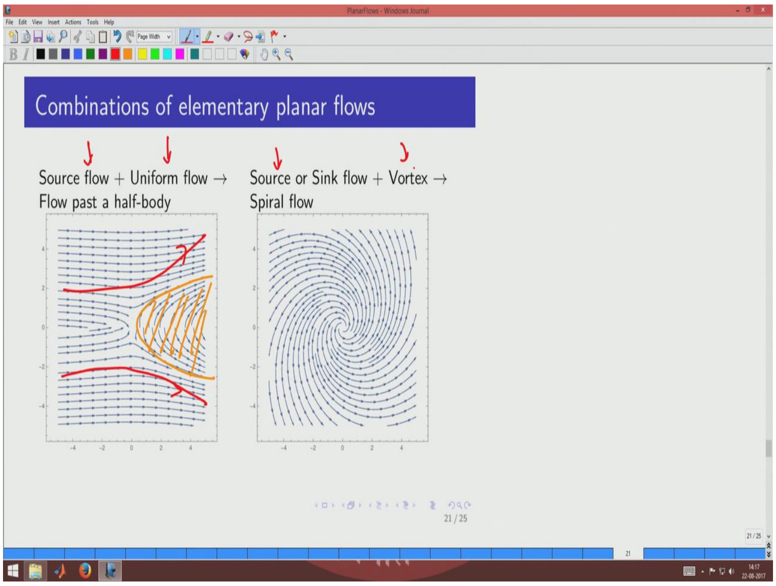

Now, the application of elementary flow fields is to describe situations that are closed to real life situation and be able to combine the elementary flow fields to arrive at a flow field that looks quite realistic. So, you could already see that in this case flow past the half bodies, where is that half body? You could think that the half body is like that. So, you have, how does the flow happen around that, that is what actually is now being shown.

So, you can see that flow passed half body can be described using a source flow and uniform flow together. So, you can take those two functions simply add them up and then plot and then you get this kind of a flow field. And when you add you can also have calibration factors to both this terms so that you can make this strength of this you know velocity gradients different as the requirement comes up. And if you take a source or a sink and then add it to a vortex so then you would have a spiral flow. So, you could see that here there is a spiral flow that is coming up and the direction of spiral can be reversed if you change from source to sink that is being added to a vortex.

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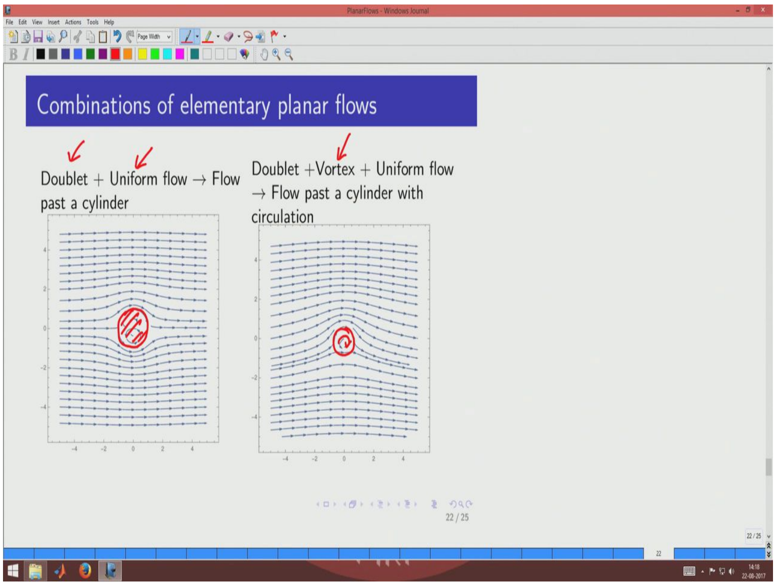

So, doublet and uniform flow then be combined to see how the flow would look like when it is going passed a cylinder. So, the cylinder is here. So, the velocity passed a cylinder would require it to go up and down like this and we can already see that the combination of these two functions already gives us that kind of a sense of the flow. So, without solving any equation we can come up with the velocities field description for flow passed a cylinder directly in this manner. So, there is one of the uses of elementary planar flows.

So, if you then have one more element that is being added vortex being added to doublet and uniform then it would help us to see that it is for flow field around a cylinder which is having a rotation. So, these two are very similar situation only thing is one the left hand side we have got the cylinder stationery and the right hand side it is rotating and the flow field around such cylindrical body can be given by a combination of elementary planar flows.

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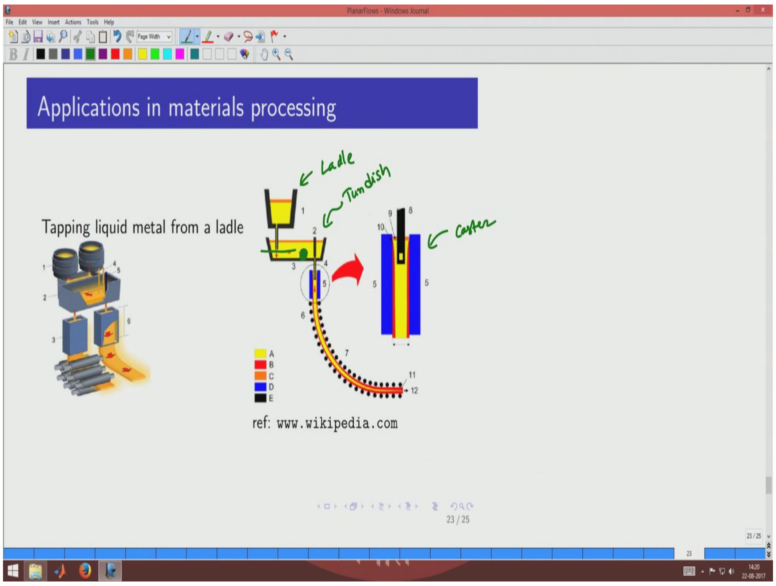

So, here are some real life situations encountered in metallurgical industry where these elementary planar flows can be used. For example, here this is what we call as a Ladle and this is what is called as a Tundish and the process that we are seeing here is continuous casting process and this is the Caster. So, liquid is coming into the caster and then it is becoming solid. So, by the time it comes out everything is ready here which means which is fully solid, the yellow is the liquid metal.

So, if you now see that way liquid metal is coming out of the ladle you could see that the ladle has a small hole at the bottom and through which the liquid is coming down, which means that you could describe the flow of the liquid metal at the bottom of the ladle using elementary planar flow field which is basically a sink flow. So, a sink flow is adequate to describe the flow in this region you could also describe in the case where the flow is not very turbulent then you could actually see that in this plane you could use source flow to see how the flow is happening. And if you have obstacles that are placed in the tundish so that the inclusion that are coming the draws and inclusion that are coming with the melt do not enter the caster then if there is a cylindrical obstacle that is placed then the flow around that obstacle can then be also described using elementary planar flow fields that we have discussed just now.

And you could also see that the tapping the liquid metal from a ladle also can be described using sink flow. So, you could see that the ladles have a small hole at the bottom through which the liquid is being drawn out and that can be described using sink flow. So, like this in metallurgical industry some of the flow fields can be directly described without having to solve any equations using elementary planar flow fields.

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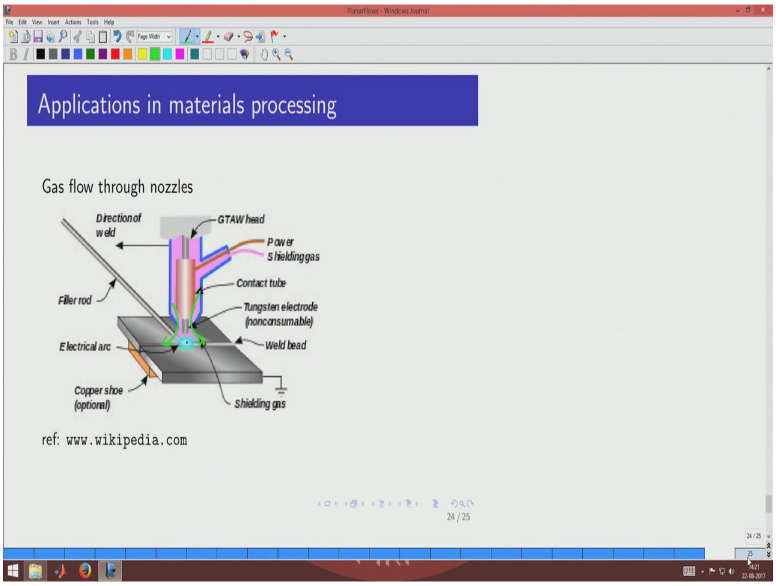

So, here is another metallurgical situation where we have a welding set up. So, here we have a welding torch a plate that is being welded and the torch is going to move from left side to the right hand side and the liquid metal is at the center and to protect the liquid metal from getting oxidized from the ambient air we have gas shroud. So, the gas is actually going through this and it is going to have shroud around the liquid metal to protect. So, the way the gas is actually coming out of this nozzle can also be described using a source flow.

So, in situations where we are unable to solve the equations to arrive at the flow field as a function of x and y distances in situation like this if you want some approximation then you could use elementary planar flow fields to make those approximations. And then go ahead and make some estimates as very first measure of how this flow is happening.

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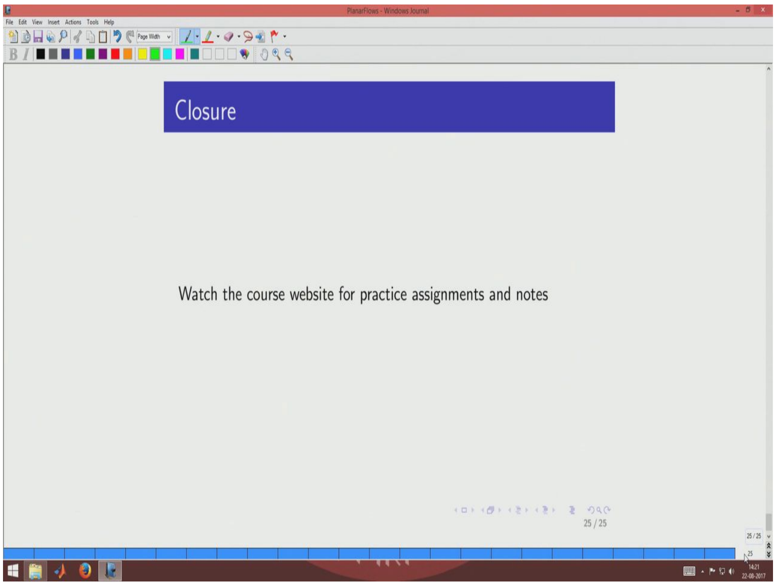

So, with this we close this session on elementary planar flow fields and the 2D flow and we will have some exercises that are possible to check the various components whether we are able to express the velocity components using those two functions that we have come across, the stream function and the velocity potential. So, we will have some practice assignments that will be put up in the course website for that.

The plots are made using mathematical. So, the mathematical note book that is used to make all those plots is also going to be uploaded into the course website. So, you could use it to make some changes and play with those flow fields and then gain in insight into how those functions will determine those flow fields.