Transport Phenomena in Materials
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Lecture – 06
Materials Derivative and Continuity Equation

Welcome to the session on materials derivative and continuity equation as part of the mooc on transport phenomena in materials.

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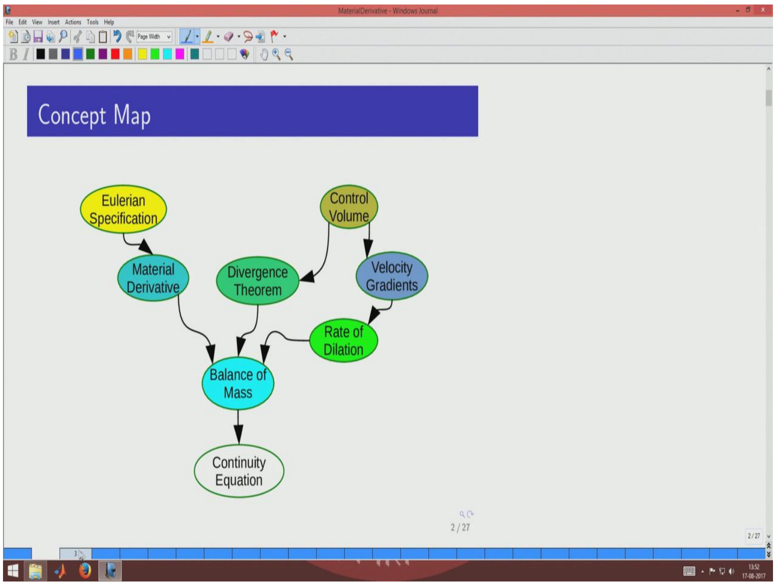

The various concepts that we will be needing at the end of this derivation of continuity equation are given here, we will need to specify what kind of coordinate system we are using. So, we will have what is called Eulerian specification that will be mentioned, and then the material derivative concept will be introduced. We will also need to introduce the concept of control volume, the convention used for the phases of the control volume and then how the velocity gradients will change the control volume and to define the rate dilation based on that. We will also have divergence theorem that will be useful for us and will put all these things together into the balance of mass, and then we will get the continuity equation.

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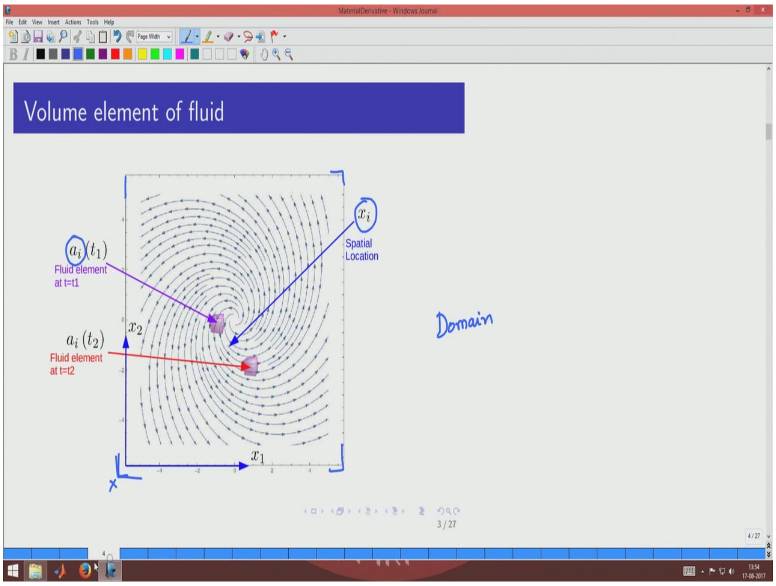

So, let us consider the domain, here the square that we have shown here, starting from here onwards all the way here. So, this is called the domain. So, domain is basically the region of space of interest to us. So, here in this domain we will have the fluid flow shown as a (Refer Time: 01:33). So, consider a small element of fluid a volume element of fluid at this location here. So, you can see that we are specifying that to be at time t_1.

So, at a later time t_2 where t_2 is greater than t_1, you can see from the flow that this small element of volume element of fluid would relocate and will come here. So, if you were to specify the coordinates of this center of mass of this volume element, then you would use a element like this a_i, but we would also use sometimes a location specific symbol for example, x_i. So, we use these 2 in 2 different contexts; take for example, the situation of a car that is going along the traffic, and then a traffic police who is sitting it a location here. So, the way the traffic police would look at the way the cars are moving is different from how the car driver would look at his speedo meter.

So, you could see that there are 2 different ways of specifying the velocities or for that matter any parameter; you could specify it as a function of the location where it is moving the fluid elements is moving, or as a function of the location in terms of the special coordinates.

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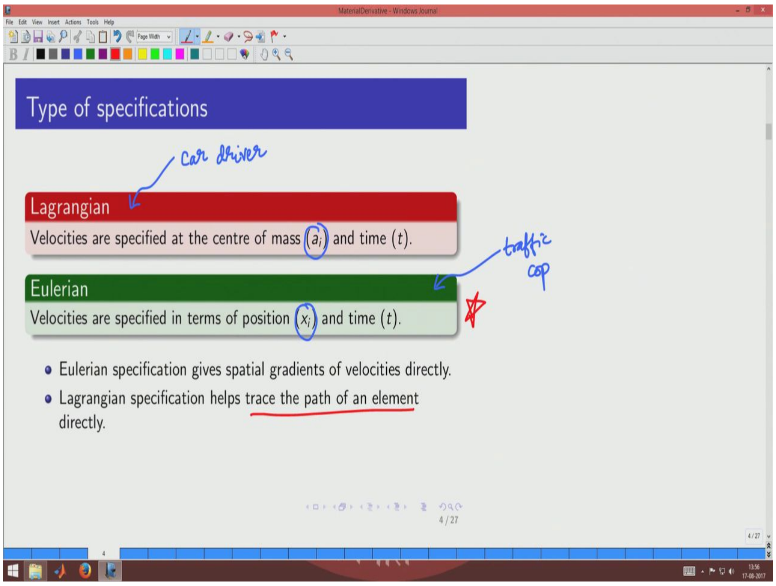

So, here for example, we will see how they are refer to. So, here we have the 2 names being introduced Lagrangian and Eulerian. So, the Lagrangian is what we say the car driver, how the parameters are being seen as defined at the location where the car is in a traffic flow. And Eulerian is by the same example how the traffic cop would see the various parameters like the acceleration or speed of a vehicle, and he would look at these as a function of the location of space. So, here we see that if you have any parameter velocity, acceleration or for that matter any parameter that is being specified at the center of the mass, then you are using what is called the Lagrangian specification and if you specify at a special location then; that means, that you are using Eulerian specification.

And let us just go back and see if you are using for example, Eulerian specification and your location of interest is here for example, which means that at different times when the volume elements that is sitting there is different. And in case the fluid flow is a function of time, it means that at that location the volume element is different and it would have different velocities. So, the Eulerian specification allows us to have the complete information about the velocities as a function of space as well as a function of time. And by default if we are not talking about the specification, then we are referring to the Eulerian specification. So, this is the way we are going to specify in this particular course.

And in what way these are going to be used? So, in metallurgy for example, you would use normally Eulerian specification for most of the situations, but you also need Lagrangian specification for situation where you need to trace the path of a volume element are for that matter a piece of material that is flowing. So, for example, when you are looking at inclusions that are being nucleated and grown in liquid metal during casting, you actually use Eulerian specification to describe the fluid flow and Lagrangian specification to trace the particle, which is nucleated and being adverted in the domain ok.

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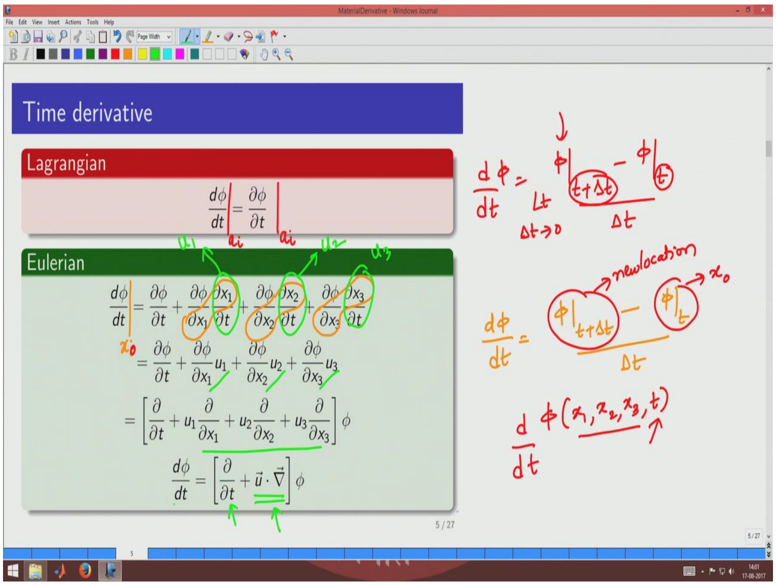

So, let us now define the time derivatives for these 2 specifications. So, because the Lagrangian specification is always at the location where the particle is, then we do not have to worry the way you define the time derivative is always at a_i. So, simply taking a partial differentiation with respect to time, would give you the time derivative and the φ can be any parameter. So, it could be for example, velocity in which case what we are talking about is acceleration. And we know that when we are taking the derivative what we are actually saying is this, this with a limit Δt going to 0. So, what it implies is that at both time t as well as time $t + \Delta t$, the particle is at a_i and therefore, we do not need to worry about the motion of the particle, because we are able to define the parameter φ at the center of particle at all times, but the same thing is not applicable when you come to Eulerian specification.

If you take Eulerian specification you look at this quantity, this quantity is actually at a special location x_i. So, which means that in this situation you would like for example, this way, $\frac{\phi_{t+\Delta t} - \phi_t}{\Delta t}$, now you see that here at $t + \Delta t$ it has gone to another location compare to this. So, if your specifying at x_0 for example, then this is available, but this quantity is at a new location because in the time $t + \Delta t$, they would have the particle would have move to a new location if the velocities are non zero.

So, because of this what we do is we use the chain rule. So, that we can get the differential with respect to time and that is what we are doing here. So, we assume that ϕ is now a function of the special coordinates x_1, x_2, x_3 and also time and therefore, whenever we want the differential with respect to time, then we can take it with respect to time here and in the rest of them you use the chain roll. So, that is why we are now able to write like this. So, you have these terms. So, you can see the cross terms that are coming. So, this is coming from the chain rule ok.

And then we recognize that these terms here are nothing, but the velocities itself the components of velocity. So, you have them coming here. So, therefore, we can then put them in and you can see that $\frac{d\phi}{dt}$ can be given as $\frac{\partial \phi}{\partial x_1}$, and then this u_1 here. So, you can see that we are seeing that the index is matching for all the 3. And when we see this we could also immediately recognize that you could take that as a dot product and you can then write this entire expression as a dot product $u.\nabla$, where ∇ is the differential operator with respect to the space and u is basically the velocity at that particular location.

So, you can see that when you want to look at the time derivative in Eulerian specification, you have got multiple terms the partial derivative with respect to time and also the second term which we normally refer to as advection term. So, that is taking care of the new location being moved. So, we see that in the Eulerian specification the time differential is coming with respect to 2 terms, we have the partial differentiation here and then we have got the advective term. So, this is how the Lagrangian specification and Eulerian specification are going to make a big difference, when you are looking at the differential with respect to time and if ϕ happens to the velocity then we are talking about acceleration.

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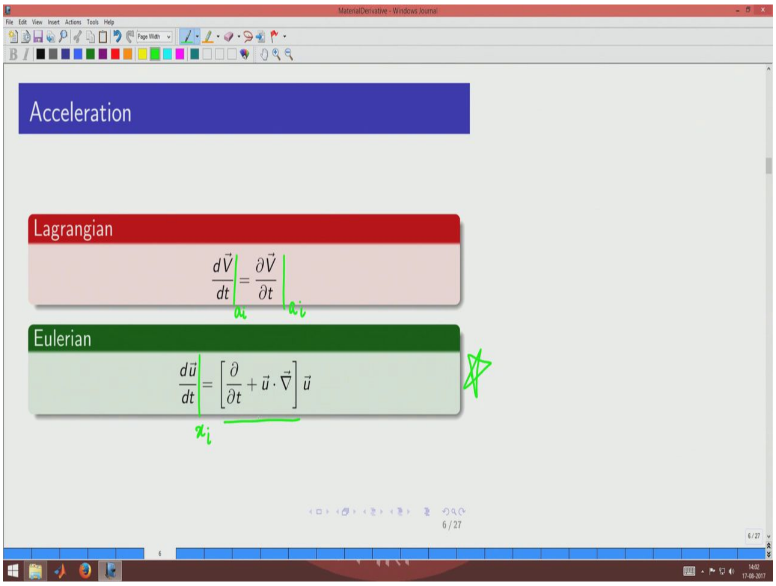

So, how would the acceleration look differently, you could see that the way acceleration is defined is different in Lagrangian Eulerian scheme. In the case of Lagrangian you will show that it is always specified at the center of mass of the particle. So, you directly differentiate with respect time and that is what it is.

Whereas in the case of Eulerian specification, you are actually defining at a special location and because the particle is moving as a function of time, you have the additional term that is coming here and therefore, you have got this as the operator to give you the acceleration. And because we are going to use this as default for all our expressions it may be a good idea to not use the same symbol for the complete differential with respect to time. So, we use a new symbol. So, that we remember this particular additional term every time.

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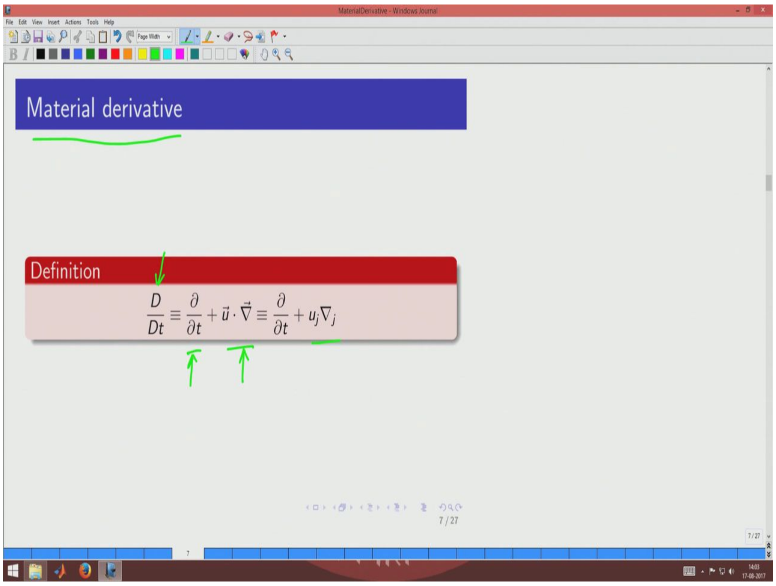

So, that is why we introduce a new symbol here D and what we want to refer to it as material derivative. So, what we want to say is that, material derivative is nothing, but the complete differential with respect to time in Eulerian specification and it will have 2 terms the first one is the partial differential with respect to time, and second is the advective term given by $u.\nabla$. So, its an operator and if we remember from the dot product usage in subscript notation, you could also write it as $u_j \nabla_j$ where J is the dummy index ok.

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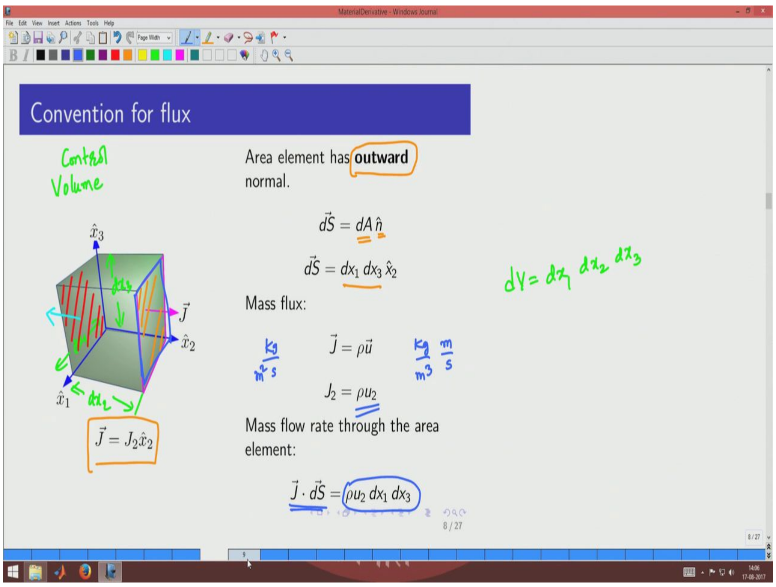

So, now that we have defined the material derivative, we have few more concepts that we need to bring across. So, next concept is about flux. So, flux is defined as the mass flow per unit area per unit time and it has to be across a phase. So, what we have drawn here a cube is, what we referred to earlier as the volume element or control volume. Now the control volume would have a particular volume and we would write that as dx_1 dx_2, dx_3 where dx_1, dx_2, dx_3 are the bits of this cube in the 3 directions. So, you would have this as dx_2 here and this is dx_1 and this is dx_3. So, if you are looking at the flux of mass through a phase, now let us take this phase which is already colored and noted then the flux J through that phase can be defined as J for example with is component.

So, J_2 x_2 is the flux through this phase. Now the phase has to be given some way to define it and we define it by the area of the phase and the normal to the phase. So, the area of the phase is $dA = dx_1\ dx_3$ as you can see here and the normal is defined as the outward normal. So, this is very important because this is going to make a difference to the derivation shortly. So, the outward normal in this case for example, this phase would be along the x_2 ok.

Now, to test the understanding what should be the outward normal for the other phase here which I would just color in a different manner. So, let us look at this phase, for this phase the outward normal would be along $-x_2$. So, you could see that depending upon the phase you must always see what is outward normal and that would be the dS for that particular phase. Now the mass flux is defined as density times the velocity. So, you could see here density you have kg/m^3 and the velocity m/s, and here the flux is kg/m^2s. So, that is a mass flow per unit area per unit time.

And in the case of the phase that we have indicated here this phase, then you see that the flux is given as ρu_2, where u_2 is the velocity of the liquid going through the this particular phase, and the area of that phase is already dx_1 dx_3. So, which means that the mass flow of liquid through this phase of the control volume is given by the expression $\rho u_2 dx_1 dx_3$. So, the convention makes it possible for us to write like this, and the dot product here is telling us that if it is for this phase this will be going outward and if it is for the other phase then it will be going inward because the phase normal is going to the other direction ok.

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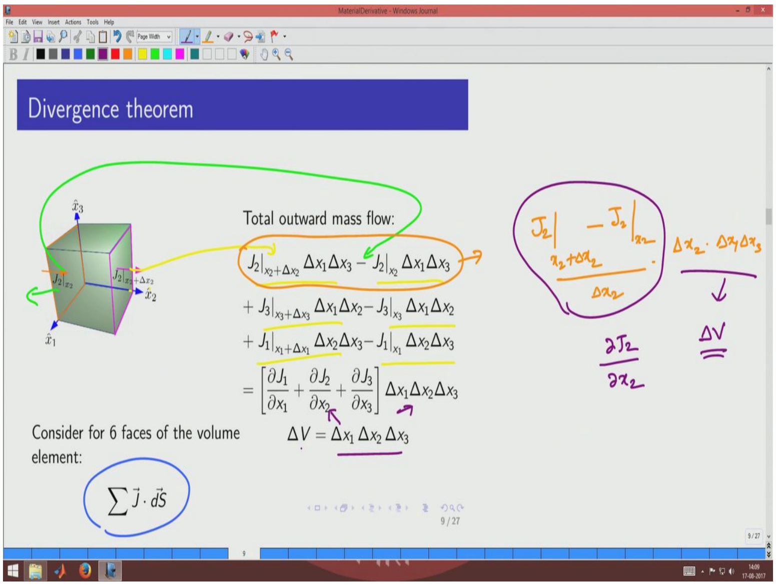

So, whenever we now are looking at the summation of the flux through all the phases, then we comes through the direction of divergence. So, to get the divergence theorem we have only just few steps to get it and what we do is like this. Consider all the 6 phase of the cube, and look at the quantity that is given here $\sum JdS$ what is the total outward flux of mass through all the phases together. So, you take it for one phase at a time. So, you have got 6 terms and the 6 terms are given here. So, 1, 2, 3, 4,5 and 6 terms. So, for all the 6 phase they are given.

And what we are also making a small variation here is the flux is actually a function of location because velocities are actually function of location. So, see that the velocity on this side would be $u|_{x_2+\Delta x}$ whereas, here it will be only u at $u_2|_{x_2}$. So, we are making that small differentiation here $J_2|_{x_2+\Delta x_2}$ and $J_2|_{x_2}$. So, the flux outward from here is positive. So, this fellow is here and this fellow is coming with an - sign, the reason is that the normal is actually pointing outward, but the flux is actually going towards the +x_2 axes. So, that is why there is a - sign. So, similarly the other 2 pairs for the other 2 directions will coming. So, when you then add them up, we have a interesting observation here. You see that Δx_1 Δx_3 are there what we can do is this expression alone, let us just take this expression alone and then modify it. So, what we do is that we write it as $\frac{J_2|_{x_2+\Delta x_2} - J_2|_{x_2}}{\Delta x_2} \Delta x_2 . \Delta x_1 \Delta x_3$. So, which means that we now have a way by which we can combining this and for a very small control volume, this would then look like this and this is ΔV . So, that is what we have written here and similarly for all other 2 directions if you write then you would see that the sum of the flux is given as the terms $[\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3}]\, \Delta V$ that is coming here with the definition.

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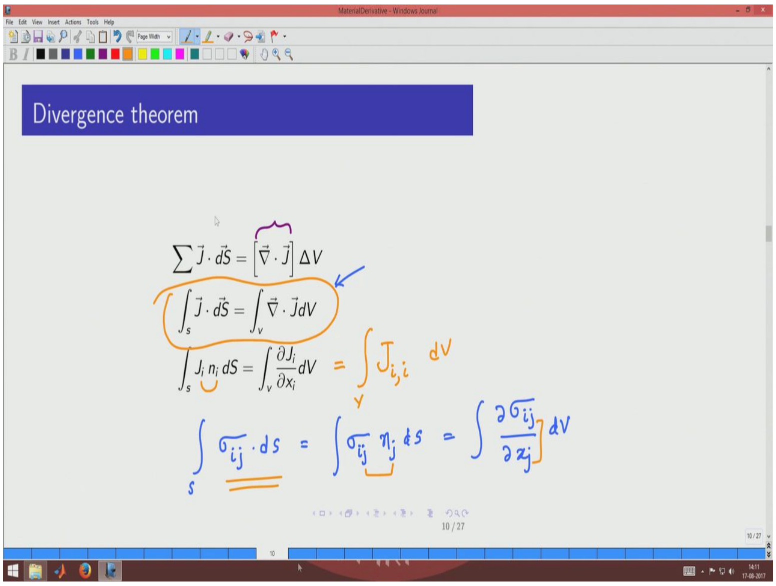

So, which means that by looking at the sum of the flux through all the 6 phases, we are seeing that it comes about a term which here shows its a $[\nabla.J]\, \Delta V$. Now the summation if you are going to do it for smallest possible control volume, then you could and also not restricted the cube shape then you could actually write it as a integral and therefore, you could convert this $\sum$ to integral, and then you would see that on the right hand side also you could do the same thing as integral with respect to the volume.

So, we now have this coming up directly and here you can see that what we have actually written is nothing but the divergence theorem. And the same thing expressed with the subscript notation is showing here with the dot product there. So, you have the same subscript that has to be coming up and $\nabla.J$ would not appear like this and if you want you could also write it in this manner. So, you could write $J_{i,i}dv$ in the subscript notation. So, this is nothing, but the divergence theorem.

Now, the divergence theorem when we write it in subscript notation allows us to expand it to higher order tensors. So, what we have written here is for a tensor of order 1 namely for a vector, vector quantity j. So, you could then also write it for a tensor of order 2 in the following manner. So, let us say you are looking at $\sigma_{ij} ds$ now you would write this as i j and there is a normal to the surface. So, you choose the index for the surface and ds, now you see that the index that is matching has to go here with the comma. So, you would write this as by ∂x_j ok.

And. So, now, what we have written is a divergence theorem applicable to tensors of order 2. So, you could see that we are seeing the matching index is what is going here, and which means that later on when we encounter such requirements where you have got the tensors of order 2 appearing, with a surface integral we could also convert them to volume integral by applying the generalized divergence theorem. So, this is the advantage of it.

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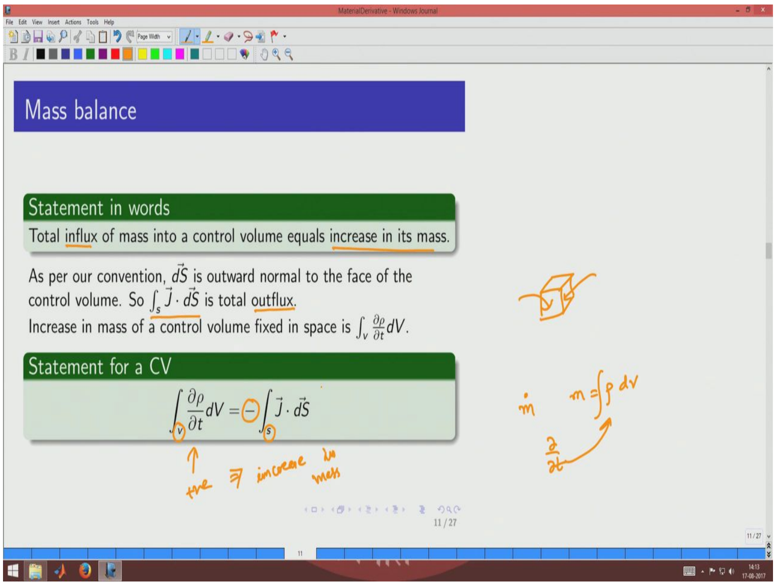

Now let us come to the mass balance. So, what we mean by mass balance is first to be stated in words. So, we are stating it in words. So, what we are saying is that, here into the control volume whatever that has come in the influx the total influx of mass into the control volume equals the increase in its mass. So, that is the balance of mass we are talking about? So, we have written earlier that $J\ dS$ is the total out flux and we are saying that total influx must be equal to the increase in the mass. So, what we are going to do is that to address influx and out flux here we just put a - sign. So, that what is written here is out flux and with a minus sign would become influx and this is equal to the increase in the mass which means that if this is positive, it implies that there is a increase in mass ok.

So, what we are written here is nothing, but the statement of mass balance for a control volume. So, we are going to do this for every other major governing equation in the transport phenomena. So, we must get use to the way of writing here. So, every time we refer to all the phases then we have a surface integral coming in, and whenever we have a volumetric term we have a volume integral coming. And we see that instead of writing mass and then with a dot which is a change in the mass increase in the mass what we are doing is a mass is written as $\int \rho dv$.

So, which means that it is now integrated over the entire control volume and then this change is then give with a $\frac{\partial}{\partial t}$. And we are then writing the expression like this and we are able to take this inside is because we are writing this expression for a control volume fixed in space, we are not looking at the control volume moving with space it fixed in space. So, we can write like this and this is the statement of mass balance in mathematical expression and here it is in English.

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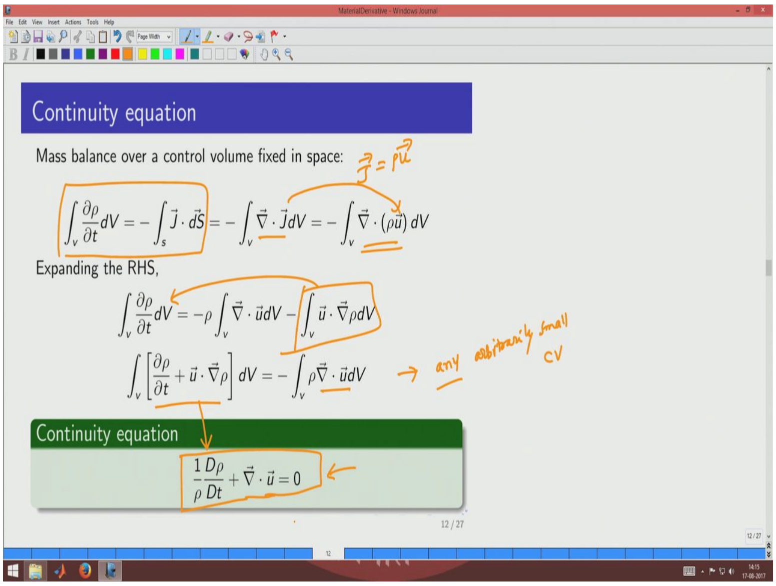

So, what we do is that we just now manipulate the equation on the right has side with what we have already know from the divergence theorem. So, this is what we have written and the right hand side then being expanded. So, the surface integral is being converted to volume integral. So, whatever is J dot something, we then see that it should be the divergence and then J is an we already know that this is nothing, but ρu. So, that is what is being used here. So, we have the expression with the velocities on the right hand side. So, what we do is that we then expand this. So, there are 2 quantities which are going to be operated by the ∇ operator. So, we operate one at a time. So, we have got 2 terms here. So, take ρ out operates on u and take u out operates on ρ. And what we do is that we take this term and to the left hand side and then because the both are integration now the same control volume we are able to then put it inside and we see that on the left hand side we have $\int_v [\frac{\partial \rho}{\partial t} + \vec{u}.\nabla\rho]dV = -\int_v \rho\nabla.\vec{u}\, dV$.

So, now we see that this is written for any arbitrarily small control volume and if this were to be true for any control volume, then the integrand also should be equal. And the integrand on the left hand side is this and integrand on the right and side is this and what is one the left hand side if you notice this is nothing, but what we have just now defined as the material derivative of density and we take the rho to the left hand side as $\frac{1}{\rho}$, and what is on the right hand side if you see it is nothing, but the $\nabla.\vec{u}$ that is the divergence of velocity.

Now, which means that the equation that we get here saying that material derivative of density plus the divergence of velocity is equal to 0 this is nothing, but the statement of continuity. Continuity equation is nothing, but this what we have written here and it is a statement of mass balance this is nothing more to that.

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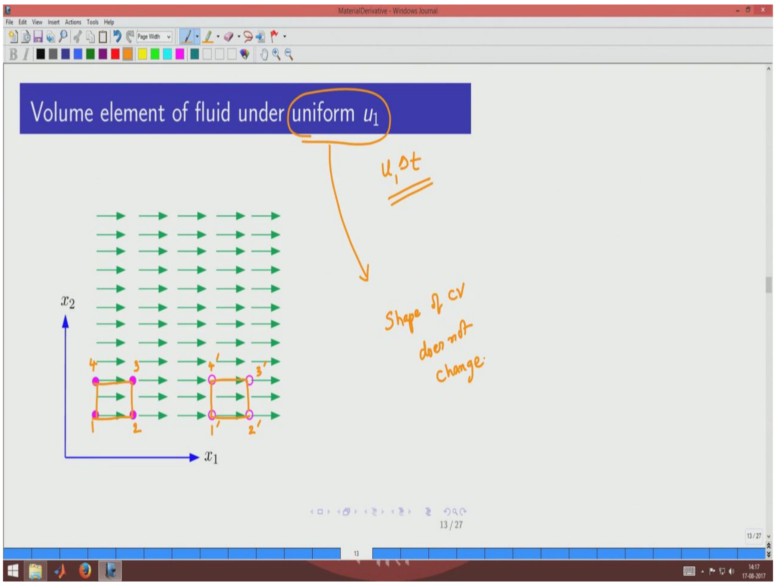

Now, let us see what would happen if we look at 4 locations in a fluid in the domain and what would happen to those 4 locations under the fluid flow. Now you could see that when you have this locations labeled as 1, 2, 3, 4 and if the velocity is uniform velocity along the x direction, you would notice that after a sometime t the position should be gone to the right hand side by an amount $u_1\Delta t$; which means that 1 will go to 1^1 here, 2 will go to 2^1 here, 3 will go to 3^1 here and 4 will go to 4^1 here the distance by which the 4 points are translated in the plus x_1 axes is given constant for all the 4 points that is given by $u_1\Delta t$.

Which means that a volume element with a particular shape is going to only translate along the x direction and the shape does not change; you can see that under the uniform velocities the shape of the c v does not change ok.

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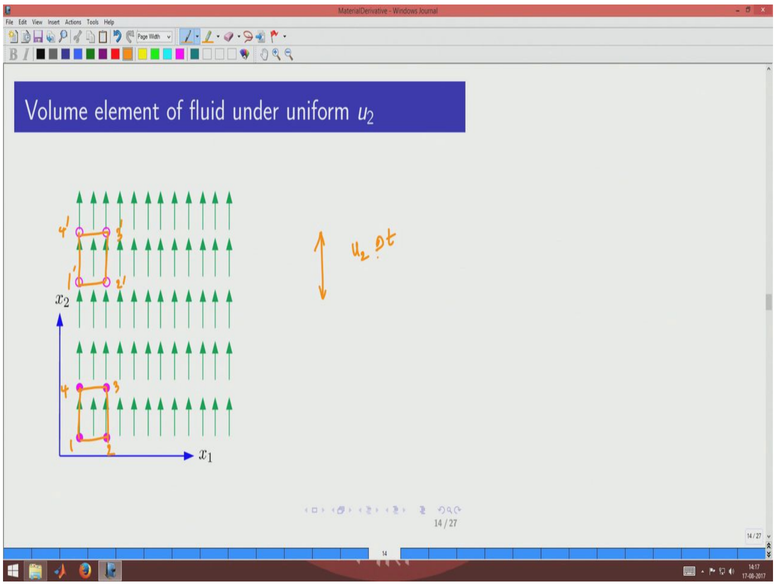

Now, you can also apply it to the velocities along the other directions and its trivial to see that the same thing will be happening here also, and in the vertical shift this distance shift is given by $u_2\Delta t$ that is the amount of time spent from 1 to 1^1 and 2 to 2^1 3 to 3^1 and 4 to 4^1 respectively. So, if you see a volume element of liquid then its shape will again not change, it will only get shifted in the y direction or x_2 direction by a certain amount which is given by this.

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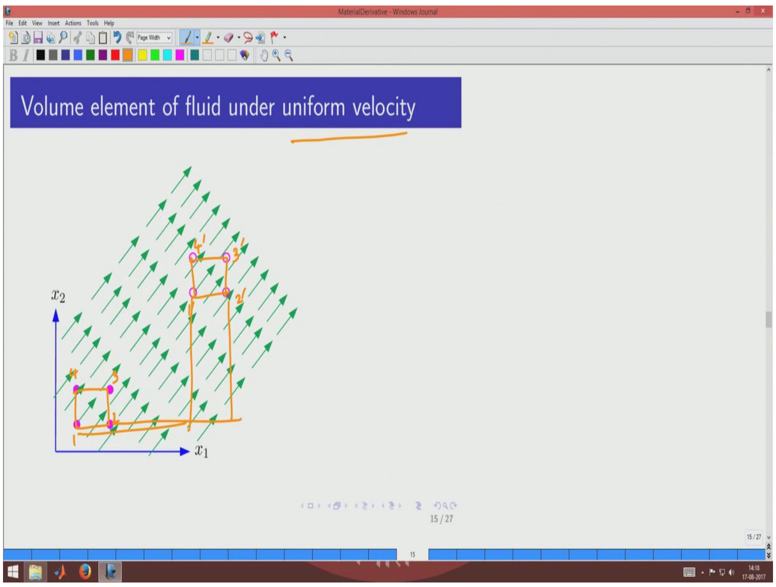

You can also see that is true for any arbitrary velocities as long as all the 3 components are same, then the shape of the control volume will not shift. So, you can see that the $1,1^1$ the horizontal distance and vertical distance should be the same for every point. So, for every point the shifts are same in all the directions, because the velocities uniformed. Now this means that uniform velocities constant velocities do not change the control volume shape as it is moving.

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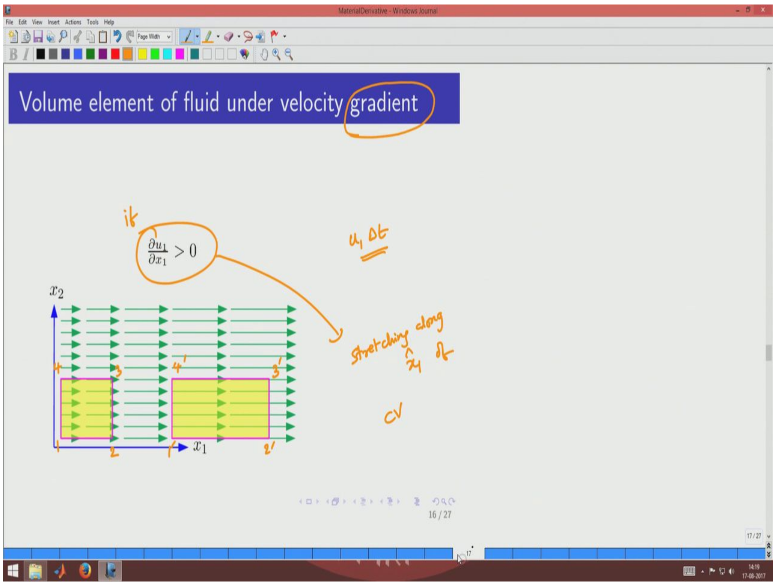

But it does change when there is a gradient. So, that is the reason why we are talking about velocity gradient, the reason is that shape would change you can actually see it here. So, 1 2 3 4 are the 4 points and then they would go to 1^1, 2^1, 3^1 and 4^1.

Now, look at the velocity vectors I have drawn them intentionally in increasing size. So, that this is followed that is the velocity is increasing in the x direction which means that if one would go by a distance given by this amount to 1^1, then at 2 the velocities actually more. So, which would actually means that 2^1 will be further away, and this means that if the velocity is changing such that it is increasing in the x direction, it implies that there is a stretching along x_1 direction of the control volumes. So, the control volume is getting stretched the shape is changing according to the velocity gradient. So, it is now trivial to again apply this to any other direction and say that the velocity gradients in respect to directions will stretch the control volume in the respective directions ok.

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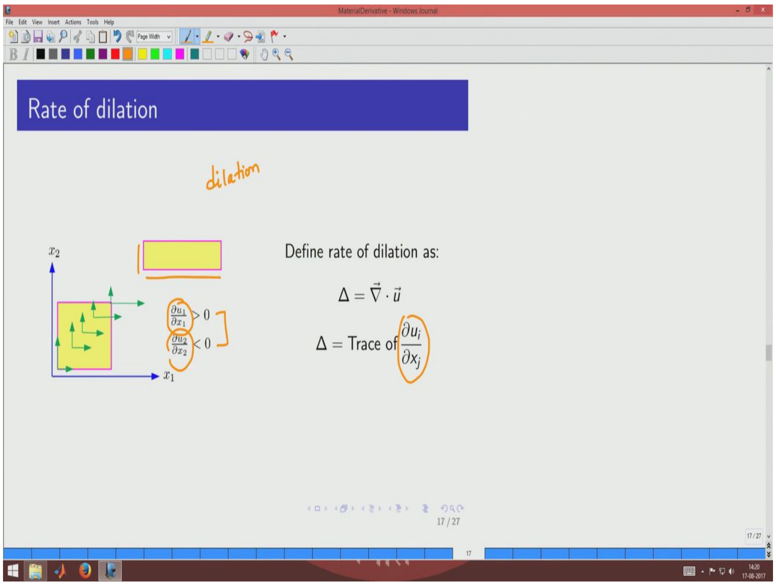

So, here we have seen that if you have a situation like this, where the velocity gradient in the x_1 direction is positive and y_1 x_2 direction is negative, then the control volume actually will expand in the x direction this is expansion and this is contraction. So, this expansion and contraction can just be called as just dilation. So, positive or negative does not matter, the size of the control volume is changing. Now you could see that in the x direction it is changing because of $\frac{\partial u_1}{\partial x_1}$ in the y direction it is changing because of $\frac{\partial u_2}{\partial x_2}$. So, now, you may suspect that in the z direction the same thing would happen, which means that if you were to look at this quantity $\frac{\partial u_i}{\partial x_j}$ as a quantity with 2 indices, then whenever these 2 indices are matching you are getting these terms and these terms are causing the shape of the control volume to change ok.

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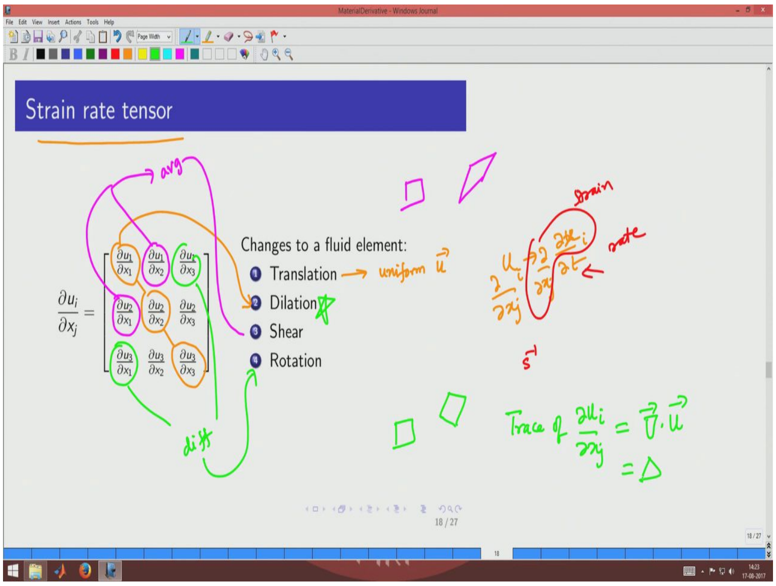

And we could then see that this is nothing, but called as a strain rate tensor. The reason why its called strain rate tensor is because u is already. So, u_i is already this and we are now talking about this. So, which means that which you see that when you combine these this is like strain change in the length by length and this is giving you the rate. So, this is nothing, but strain rate and the units are just s^{-1}. So, if it is having 2 subscripts. So, you can guess that it could be a second order tensor which is true and one can also prove it, and you could see this elements and of these what are causing the dilation.

So, what you we can see that there are 4 actions that are possible because of velocities and their gradients. So, translation without any shape change is possible whenever you have got the all the terms 0, but the velocities are non zero; that means, that uniform velocities would give you translation, but if there are gradients in other 3 things can happen. And then the dilation will happen whenever these quantities are non zero. So, dilation would be caused by these terms now what about the others. So, we would expand about the others later on, but at this moment I will just mention that these are actually causing something else. So, whenever you take the off diagonal terms, and if you were to look at what is common to them, the average then that would actually cause the shear and which means that a control volume which looks like that would go something like that. And if you look at the off diagonal terms and their difference then the difference is going to cause rotations which means that something like this would then become like that ok.

So, you could see that the velocity gradient tensor all the components have specific actions and therefore, you could see that the terms should be related to how we write different terms in the transport phenomena equations, and at this moment we are only using this term the remaining terms we will use later on in other equations. So, the dilation term is of interest because these 3 terms are going to appear and what would be the trace of this? The trace of $\frac{\partial u_i}{\partial x_j}$ is nothing, but this right.

So, what we are doing is that, this trace because of this we actually want to give a symbol and we want to call it as a rate of dilation and that is what we are going to write here.

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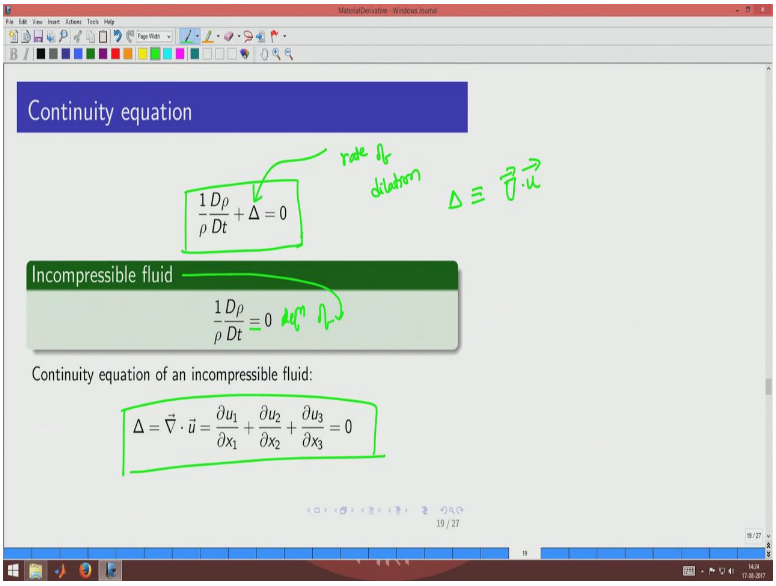

So, this equation continuity equation, wherever we have got the divergence of velocities we want to put a symbol there. So, that we define that symbol here and the symbol is basically rate of dilation that is defined like this. And you now see that once the continuity equation is written like this for general form, then you could also then start seeing how you can define the fluid flow itself in a different manner ok.

So, we define now an incompressible fluid by saying that the material derivative of density does not change as the material flows, then it must be an incompressible fluid. So, we are defining like that; if we wish you could say that this is a definition of incompressible fluid, which means that for incompressible fluid the continuity equation will turn out to be just that the rate of dilation is 0. So, if you wish you could also define an incompressible fluid as that fluid for which the divergence of velocity fluid is zero or the material derivative of density is 0, one of the 2 because they are anyway correct at by the continuity equation which is a mass balance statement, which is applicable whether the fluid is incompressible or not.

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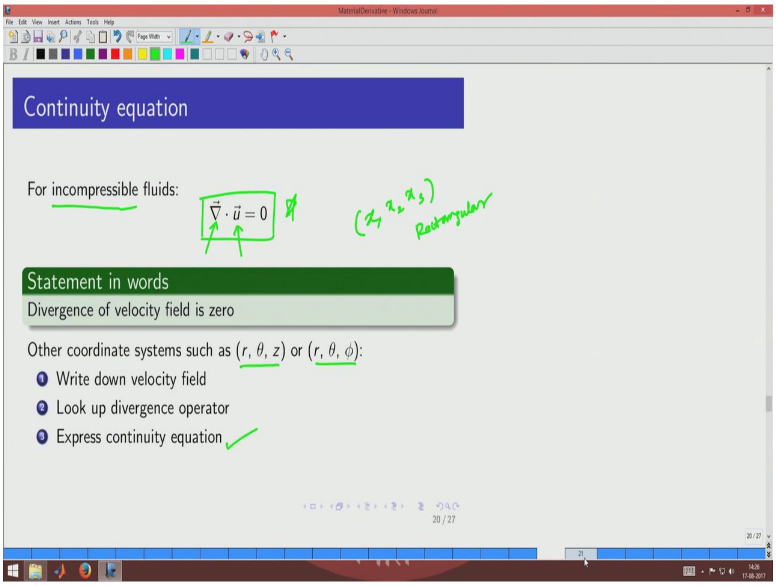

So, we now will assume that by and large in particularly in materials subject we have incompressible fluids, and usually in situations where the fluid flow is approaching the velocity of sound etcetera, then we have compressible phenomena coming in, but in materials process generally the velocities are quite small and therefore, by default we are only referring to incompressible fluids, and for incompressible fluids the continuity equation is very simple the $\nabla.\vec{u} = 0$. Now the reason why we are saying like is this because, we are able to derive this using the x y z; x_1, x_2, x_3 coordinate system the rectangular coordinate system, we derive using the rectangular coordinate system because it is easier and once we derive and we are able to write it with vectorial notations then the sense of what we are writing is already established. So, the sense that the divergence of velocity field is 0, once it is there then what we can do is that in any other coordinate system such as the cylindrical here or spherical, what we can do is that we can first look up what would be this operator and how the velocity are to be written in those coordinate systems and then express the continuity equation in those respective coordinate systems, which means that we do not have to derive this equation in all the coordinate systems again and again, because the sense of this equation is already established in this vectorial notation that we have written here.

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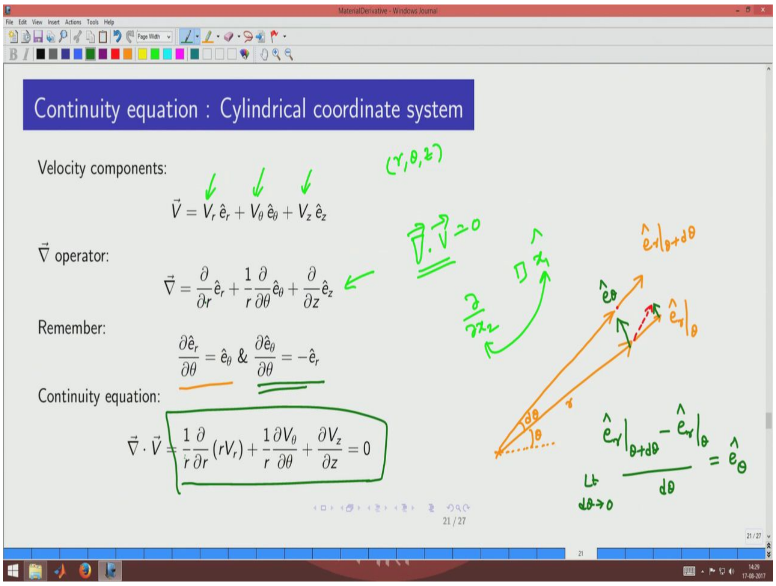

So we can now do this we write the velocity components of any velocity field in cylindrical coordinate system like this, the cylindrical coordinate system is defined by r θ and z. So, which means that we will have velocity components with v_r, v_θ and v_z and then the e_r, $\widehat{e_r}$, $\widehat{e_\theta}$ $\widehat{e_z}$ are basically the unit vectors in those 3 directions and once you have the velocity vector written down, then you can also look up what would be the ∇ operator for cylindrical coordinate system we look it up and then we then use this same idea $\nabla.V = 0$. Now when we write like this we are actually going to have each of this operators act on each of those elements, now when you do that in rectangular coordinate system whenever you encounter like this something like this and there is some quantity in this manner then you know that this and this do not act on each other that will be a 0, but in this case we do not have that luxury because the cylindrical coordinate system is a little different.

So, let me just illustrate what we mean by that. So, let us look at this. So, cylindrical coordinate system you have for example, the r and we take this at some angle θ this is r and this direction is then basically $\widehat{e_r}|_\theta$. Now at $\theta + \Delta\theta$, this would be dθ now this would be the vector corresponding to $\widehat{e_r}|_{\theta+d\theta}$. So, which we now if you translate this to meet the origin here, then you would see that that vector will be at an angle to the first one which means that if you were to look at the difference between the 2 which means that $\lim_{d\theta\to 0}\frac{\widehat{e_r}|_{\theta+d\theta}-\widehat{e_r}|_{\theta}}{d\theta}$ and of course, if we were to take this limit, this difference is nothing, but it in the direction of you could see what is this direction? This direction is nothing, but $\widehat{e_\theta}$. So, which means that when we act the operator on those terms, that is watch out that there are some more thing that are happening you sees that $\frac{\partial \widehat{e_r}}{\partial\theta}$ is $\widehat{e_\theta}$ such a thing in rectangular coordinate system all of those cross combinations will be 0, but not so in the case of cylindrical and spherical coordinate system.

Similarly, you could also prove the same thing is in same construction here. So, remembering that these kind of terms will exist, then when we dot this way this then we get the equation final form here, and which means that the divergence of velocity in cylindrical coordinate system equal to 0 as the continuity equation in cylindrical coordinate system, you see that it is nothing, but $\frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}=0$. So, we now have the continuity equation in the cylindrical coordinate system also.

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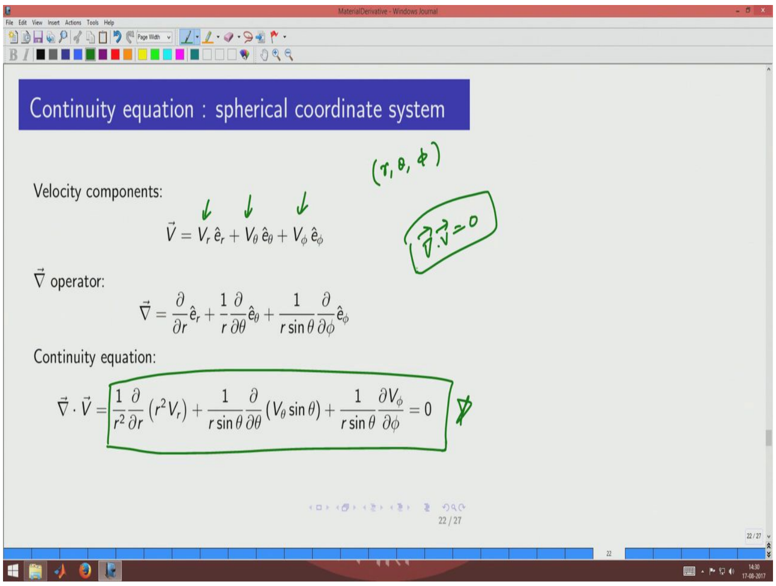

So, similarly you could also do it first spherical coordinate system, remembering that spherical coordinate system is defined by r θ and φ. So, the azimuthal angle the angle θ and r and accordingly the components are given here and the unit vectors are given and you have the operator defined when you look it up from any of the Maths hand books then you cat upon them and remember that there are some cross terms that will be possible. So, finally, you will also arrive at this kind of an expression, which you can derive and show that the continuity equation in spherical coordinate system would look like that; which means that we do not have to derive this we just look up what are the operators and then act upon them, because the derivation as in this is already done and this adequate for us to know. So, rest of it is only a matter of details.

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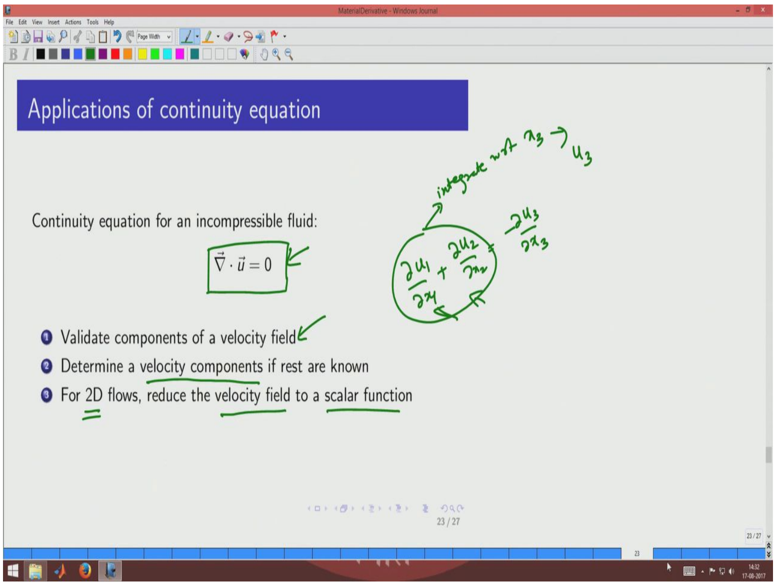

Now,. So, when we have the once we have the continuity equation, what are the uses of that equation in what way it is going to help us in understand the transport phenomena in materials processing. So, first thing is that the velocities are basically vectors and we already know that there is one check that we have to verify whether its a vector or not and that is the definition of vector that is tensor of order one. So, once we that is passed then we now need to see whether that velocity which is a vector is a valid vector for describing the fluid flow subject to the mass balance. And which means that you could actually plug in any velocity field you have and check whether the divergence is 0 and if that is satisfied which means that it is a valid velocity field for an incompressible fluid. And if you do not get 0 then we may suspect that may be it is not for incompressible fluid and very often that is how we can find out mistakes in the velocity field when you write down ok.

And the continuity equation is also useful in determining the velocity components. So, let us say we have got 3 components here and 2 of them are known, and what is the third component? So, you could actually substitute and verify and you can then determine. So, what happens is that you have got $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = -\frac{\partial u_3}{\partial x_3}$. And if these 2 are known then whatever it is coming up here then you integrate it with respect to x_3 then that will give you u_3.

So this is how you can use the continuity equation to determine unknown components of any velocity field, if the rest of the components are known. And in case you are actually limiting your domain to be 2 D, then you could also use this equation to reduce the velocity field from 2 unknown quantities to just one quantity, which is a scalar function you can do that so that, the number of equations you need to solve were reduced. So, we will come to that in detail in the next session, but at this moment you know that you have got 2 components of velocity in 2 D flow and you have got a one equation that is already being satisfied here this equation. So, you could use it to reduce the number of unknowns. So, this is how the continuity equation will be useful in determining the components in a general situations ok.

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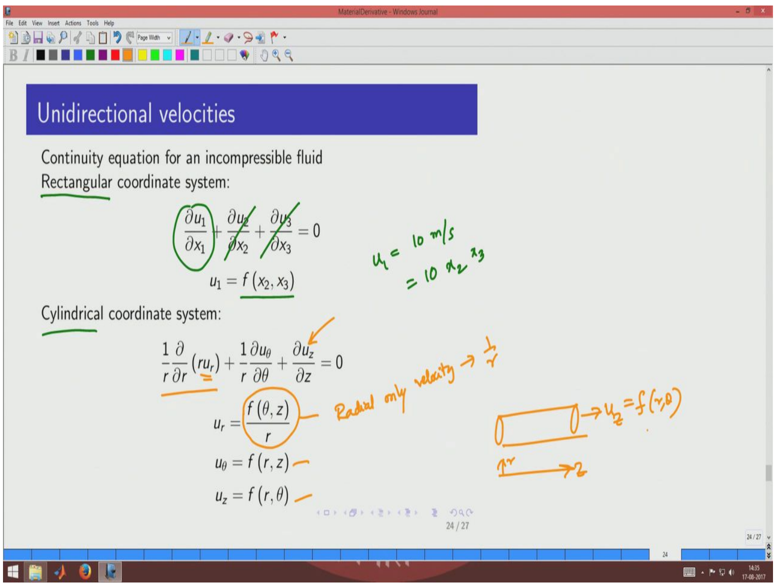

So, it is also helpful to see what kind of unidirectional velocities are possible, because most of the time in materials processing we may assume for simplicity the velocities to be uniform. And in one dimension and if it is one dimension then what kind of functional form are allowed that is something that you can see.

So, let us take for example, rectangular coordinate system. So, if it is rectangular coordinate system and we want to look at the unidirectional velocity and let us say we want to keep only u_1 and we do not have these two.

So, which implies that any function which makes the derivative with respect to x_1 0 is the valid velocity field in this particular situation. So, which means that any function of x_2 and x_3 will satisfy continuity equation in 1D, which means that; obviously, u_1 = 10 m/s of course, that is; obviously, valid because you have got only number there and when you differentiate with respect to x_1 you got 0, but you can also write like this. If you like you could do that also because when you differentiate with respect to x_1 you get zeros and then nothing will be remaining. So, these are all how we can go ahead and just see what will happen to the uni dimensional velocities. In the case of cylindrical coordinate system you have got little more variety that is possible ok.

And if you look at the first term and you want only the radial velocity to be there, then it must necessarily be having a form of $\frac{1}{r}$ .You could see that there is an r sitting there. So, when you have $\frac{1}{r}$ form these 2 r's will cancel, and then we have only a function of θ and z which when you differentiate with respect to r you get a 0. So, which means that a radial if you want radial only velocity, then it must be all the form $\frac{1}{r}$. So, which means that at as the r increase just the magnitude of the velocity should decrease you cannot have the same magnitude in a radially outward velocity. So, that is what the continuity equation is telling us.

Similarly, the second and third terms will also allow you the velocities to be of any functions. So, if you took for example, this is z velocity for example, if you have a cylinder situation, and this is a z direction this is r direction then this kind of a velocity can be any function of r and θ, it will satisfy the continuity equation ok.

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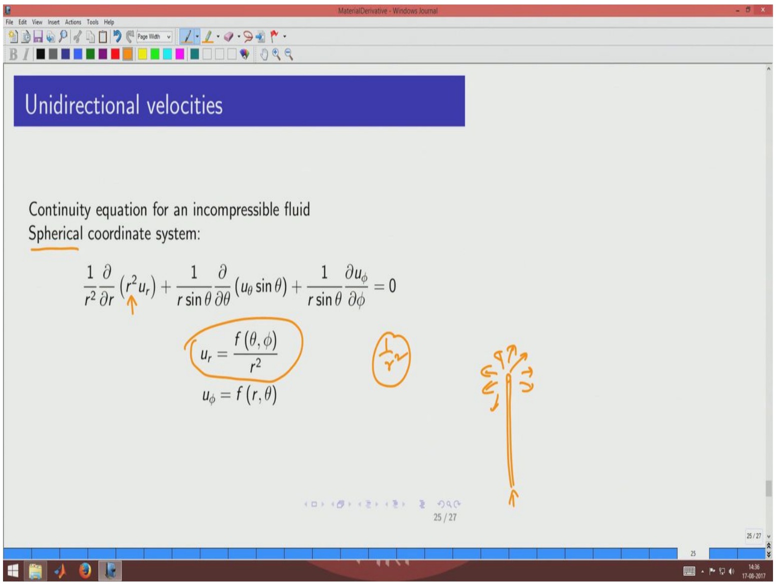

Similarly, you can also see what would happen to the spherical coordinate system when you have unidirectional velocities, you can actually tell what kind of velocities are possible. So, usually there is not much use of looking at unidirectional velocity in θ and ϕ directions, but radially it is definitely useful and you can now see that because of r^2 here the $\frac{1}{r^2}$ form is going to satisfy the radially outward velocity in the spherical coordinate system.

What this implies is that for example if you have a very thin pipe and pipe that is getting the water and then it is spraying in all directions, then radially outward direction in all 3 directions. It means that as you increase the distance the magnitude of the velocity goes as $\frac{1}{r^2}$. So, it dies down the magnitude has to die down faster than in the cylindrical coordinate system. So, this is how you can actually start to modal if you are interested in unidimensional velocities using the continuity equation. So, that the velocity field that you write is automatically satisfying the equation and therefore, it will also satisfy the mass balance ok.

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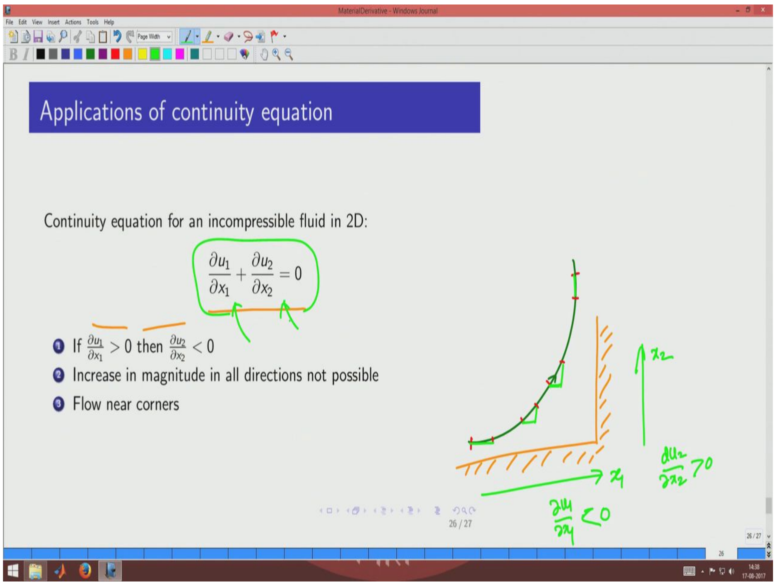

And there is one more last application before we close to see how the continuity equation can be of use. So, consider the continuity equation in 2 dimension, you have got this. what this implies is that if one of them is positive the other one has to be negative because the sum has to be zero. So, it implies that you do not have a luxury of having the velocities having a positive gradient in all directions you cannot have magnitude increase in all directions if it is increasing in one direction in another direction it must decrease this is the outcome of the continuity equation and this can be used to tell how the flow will be near a corner. So, let us look at a wall like this and you have a corner there and near the corner we already know from our commonsense that if the liquid have to flow then it would go like that. So, what we do is that we take segments that are of uniform length and we then see that whenever you take this segments then you could see that the x velocity is decreasing you can see that here the x component of the velocity is much smaller is in between and this is full and the y component is actually zero here it increase this and here it becomes fully y.

So, which means that in this direction if you were to take this as x_1 and in this direction you take as x_2 then the $\frac{\partial u_1}{\partial x_1}$ is negative because x_1 velocity is decreasing to zero the real white has to decrease to zero is because it cannot penetrate at the wall and it means that $\frac{\partial u_2}{\partial x_2}$ is positive which means that the in this corner the y velocities picking up in magnitude it is zero at this location and full velocity at by the time you go away from the corner. So, which means that recirculations can be explained by continuity equation by looking at only an equation of this simple nature saying that if one velocity has to keep decreasing in magnitude because of a wall then the other component will pick up which means that the velocity will take a turn this is how you can actually apply continuity equation.

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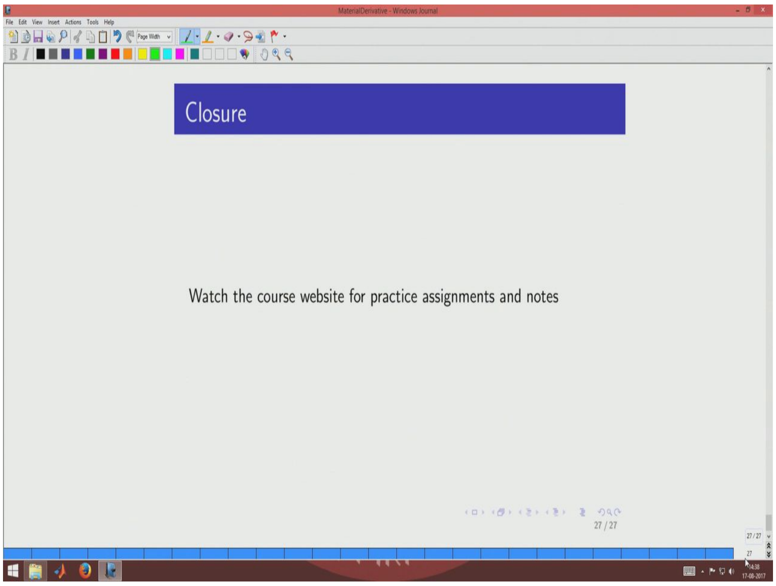

So, in the course website we will put up a list of some sample velocities you can plug them into the continuity equation to check whether they are valid or not and you will also have some notes to help with some of the derivations we have done