Transport Phenomena in Materials

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Lecture - 04
Introduction to Tensors

So, welcome to the session on Introduction to Cartesian Tensors. So, the concepts of tensors will be introduced to you, and we are going to take only a subset of the concept, namely the Cartesian tensors. So, it will be sufficient for us to handle the concepts that are required for this course.

So, the Cartesian tensors are introducing the following manner. So, the idea is as follows: why do we need the tensors at all. So, in the subject transport phenomena, we are going to basically describe physical processes; such as heating, cooling, or diffusion, or velocity that is actually evolving as a function of pressure gradients and gravity direction and so on. So, essentially whenever we are describing physical processes, it is important that we use quantities, which do not be affected by the coordinate rotations and transformations. So, tensors are basically mathematical entities that are used to represent physical processes, and we are going to use that in context of coordinate rotations and we are going to define in fact, the tensors in the same way.

So, normally whenever we encounter the quantity tensor so we are often coming up with this particular word order or rank.

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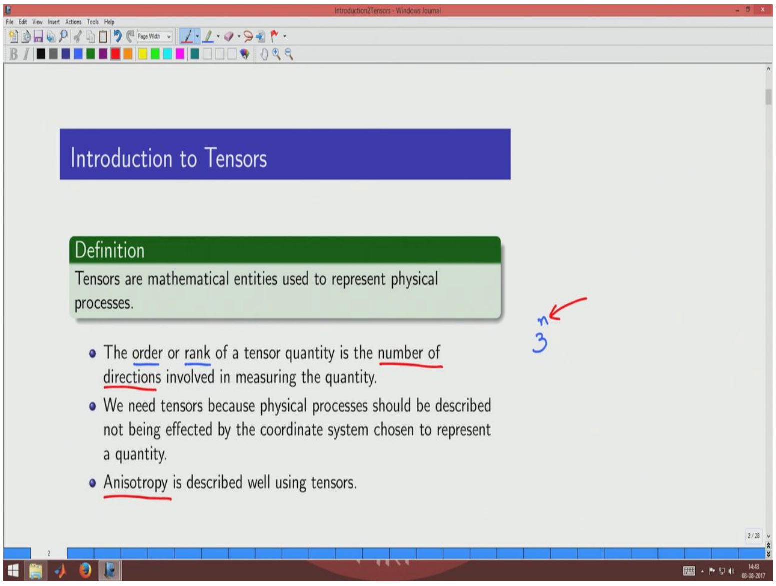

So, people say a tensor of order 1, tensor of order 2 or tensor of rank 1 or rank 2. So, what we mean by both order and rank is 1 and the same, it is basically telling how many elements will be there for a tensor quantity. So, if you want to mention a tensor of an order n for example, you can immediately say that there will be 3^n number of elements that will be involved. So, which means that a tensor of order 0 means that there will be just 1 number we are talking about for that quantity, which is basically a scalar. And then when we are talking about tensor of order 1, which means a bunch of 3 numbers we are talking about and tensor of order 2 is a bunch of 9 numbers and so on.

And it is also important to know that this order is also having one more important meaning, that is basically tells you how many different directions are involved while measuring that particular quantity that is being represented by the tensor. So, number of directions is important ok. So, as you can see for the tensor of order 1 we are having 3 elements. So, which means that we are talking about only one direction, and to represent one direction we just need vector and which has again 3 elements. So, that way you can actually be familiar with the tensor of order 1 is identical to vector and that is what is familiar to us, but we can then make a very general definition of what is a tensor for any arbitrary order, and then use these expressions to process the derivations that we need for this course ok.

And one of the reasons why we also choose tensors is because; tensors can actually handle anisotropy very well. What mean by anisotropy is; that there are different values of a particular quantity in different directions. So, there is a direction dependence of the quantity because directions are already coming in here. So, the direction dependence can be handled very naturally. So, these are also the reasons why we are using tensors in this particular subject. It is possible to do the entire subject without involving tensors, but if we can introduce tensors write at the beginning, then most of the mathematics that we need, will be quite simple and also it actually gives us a broader perspective of this subject, ok.

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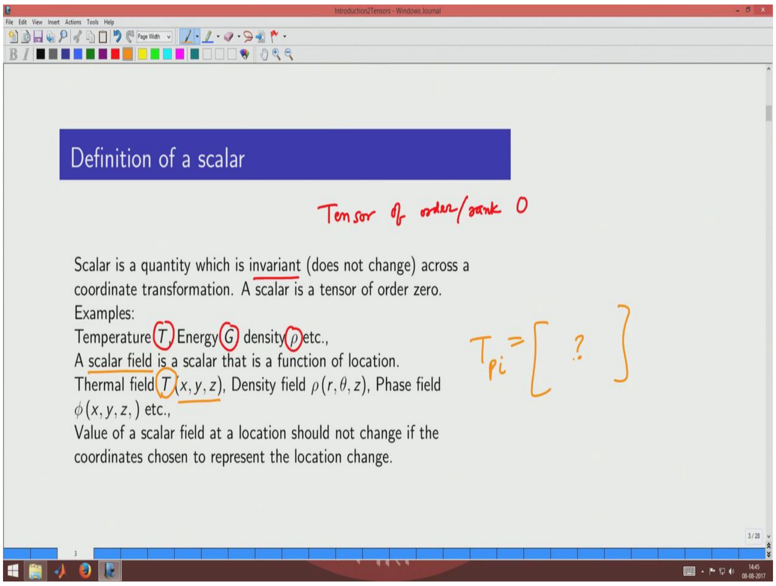

So, we start off with tensors of different orders. So, we talk about the zeroth order first. So, scalar is basically the tensor of order or rank 0. So, it is not common to use the word tensor when we are talking about the scalars, though it is technically correct to say it is a tensor of order 0, but people do not use the word tensor mainly because a scalar conveys the same thing anyway.

So, scalar is basically a quantity which is invariant across a coordinate transformation, which means that do not need any directions to represent that particular quantity. There are 0 directions involved in measuring that particular quantity and therefore, it is a tensor of order 0 and examples are basically temperature, energy, density, and so on. So, these are very familiar to us and we are again refreshing our memory about a technical term that we have used earlier scalar field. So, which means that it is a variable which has its value depended upon the location. So, scalar field is nothing but a scalar variable, which is a location dependent 1.

So, there is still no direction at because there is the location itself is specified and that is adequate, and the value of this scalar should not change when the coordinate transformation happening. So, which means that whenever we are having any arbitrary T, whatever is the values that we have given whatever it is, we must not have any change in the value of the scalar variable that we chose. So, this is actually to prove that something is a scalar or not ok.

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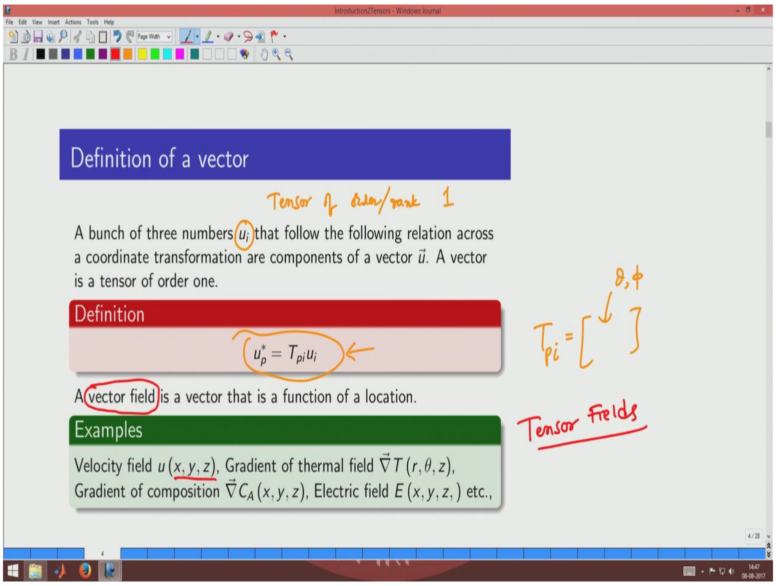

So, then we go on to describe or define what is the tensor of order 1. So, it is a vector which is again familiar to us in engineering, but it is also basically tensor of order or rank 1, is not again common to call vectors as tensors of order 1, but technically that is what it is.

So, again we go through the definition, a bunch of 3 numbers can be called as a vector whenever they transform using this particular relationship, whenever the coordinate system rotates given by the elements of the transformation matrix. So, the transformation matrix is defined by the rotation of any angle whatever if it rotated by multiple rotations, then you can also put those things in. So, the elements of transformation matrix can be obtained, and the elements of the $\vec{u}$ if they transform in the particular manner, then it can be called as a vector.

If it does not transform like that then its not a vector it is just any arbitrary collection of 3 terms, which vary randomly when coordinate system is changing, and similar to the scalar field we can also defined what is a vector field. So, vector field is 1 which has 3 components which are actually function of location. So, the location is coming in there. So, we will be encounters a vector fields in this subject and therefore, we must be comfortable with it already.

So, in other words we are already going to actually use what are called tensor fields. So, tensor fields may sound the quite technical, but we are basically talking about scalar fields and vector fields, and because scalars and vectors are nothing but tensors of order 0 and 1. Therefore, we are also going to use tensor fields as a part of this course ok.

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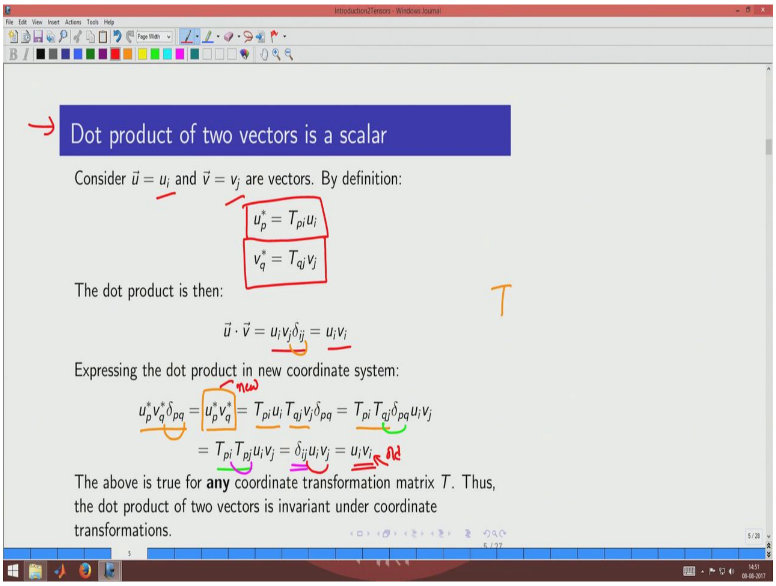

So, we are now going to go through the definition of what is a scalar and what is a vector to see whether we can use a definition to define those as scalars or vectors. So, what is going to be proven is actually indicated here. So, we want to show that dot product of 2 vectors is a scalar. So, what we mean by that is, when we take 2 vectors, and then look at dot product then in any other coordinate system components of these 2 vectors could change, but the dot product will not change is will be the same value, numerically it will be the same value. So, we want to prove that. So, how do we prove? The way we are prooving here the same steps will be followed for all the derivations. So, if you pay attention to the sequence of steps, then it will help you derive a many of remaining derivations also by yourself. The way it is like this, first let us analyze what is provided to us what is given to us is that u and v are vectors ok.

So, the moment you say that something is a vector, it means that it follows the definition of vector. So, this is a definition of u being the vector and this is the definition of v being a vector, and what is the dot product. In the summation convention using the subscript notation a dot product is then written in this manner. So, the subscript is the same. So, what are we using here? We are using that the subscript which is repeated can be replaced by the other subscripts. So, $u_i\ v_i$ will become the expression in subscript notation to indicate the dot product of the $\vec{u}$ and $\vec{v}$.

So, once we write the dot product here, then what we do is that we verify whether the elements of this particular expression are they different when we change the coordinate system. So, the way we change the coordinate system is to say that the transformation matrix T is having some values, and what we do is basically we see the dot product in the new coordinate system. The dot product in the new coordinate system is given by $u_p^*\ v_q^* \delta_{pq}$ and of course, I notice that the subscripts are repeated.

So, I just use this. So, this is the dot product in the new coordinate system. Now because u and v are given as vectors, then we use a definition of vector for both u and v and when we then look at this expression. And then see that there are two transformation matrices coming side by side, then we pull them together and when we realize that we have some simplification that is possible here. We see that we have got a subscript that is repeated here. So, q is repeated here.

So, I could get the index of p there and that is what I did here. So, what are we using here we are using the property of δ to change the subscripts. So, we are using that and then writing this expression. So, the moment we write here again we see one more simplification is possible, that is the subscript p same and i and j are here coming up. So, we again have the property of transformation matrix T, where we saw that whenever there are two T's coming together with one subscript that is matching then the other 2 subscripts can be taken for the delta and that is what we have written here.

The reason why this comes about is because the matrices transformation matrices have some properties, because they are actually transformation matrices of orthogonal coordinate system. So, therefore, whenever we see two T's coming up the orthogonality relationship is giving us the δ. So, once we have δ_{ij} and immediately we can see that we can add 1 of the j's on the other one then we can get the i. So, which means that element by element when we multiply the components of u and v whether we did it in the new coordinate system or in the old coordinate system we get the same numerical value.

So, we do not have the values of T sticking in at all that is the idea. Basically, the derivation is such that T is totally out of the expression before and after which means that the way u and v transform its elements such that when you dot them up. Then the number will be the same this is basically the proof of dot product of 2 vectors being a scalar ok.

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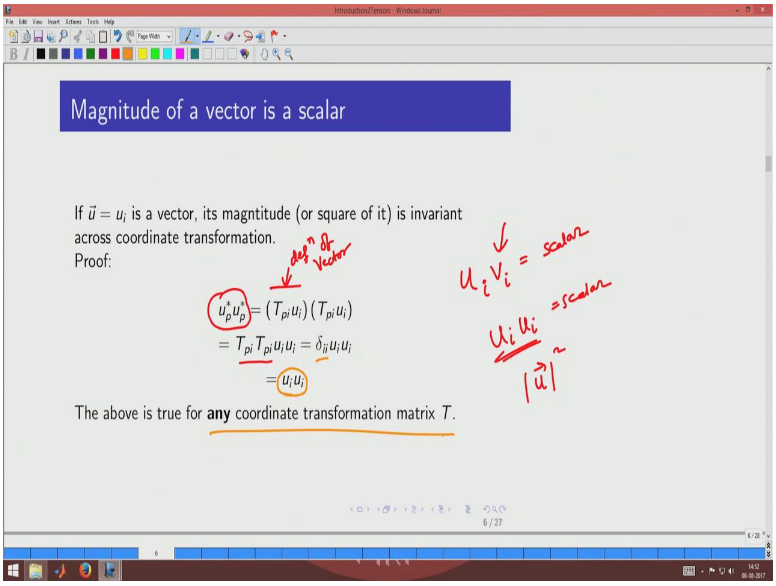

So, then we also can use this to prove few other things, you can see that if $u_i v_i$ is a scalar, then if v that will be the same as u then it is obvious to see that $u_i\ u_i$ is also a scalar it is a corollary and $u_i u_i$ is nothing but $|\vec{u}|^2$. So, it is quite easy to see that when dot product is a scalar, then the magnitude of a vector is also a scalar. So, that is what we are trying to prove here and the way we do is this is the magnitude of the $\vec{u}$ in the new coordinate system, and because U is given as a vector.

So, we write the definition, this is the definition of a vector and once you have the definition then we have got the two transformation matrices coming side by side. And then we use the δ coming up to simplify that and then we can see that finally, we get the same combination of the use. There is no trace of any transformation matrix elements are sticking around, which means that irrespective of whatever transformation we are talking about.

For any arbitrary transformation of the coordinate system, the dot product of the vector with itself namely the magnitude of the vector is going to be invariant across the coordinate transformation, which means that it is a scalar. So, this is for any coordinate transformation matrix T. So, this actually shows you that whenever you have a functional form for a vector. Then you can immediately see that if we can take some scalar quantities out by using the dot products, then we know that those expressions will not change when the coordinate system changes. So, this has been used in some theorems which we will come out with later on in this course ok.

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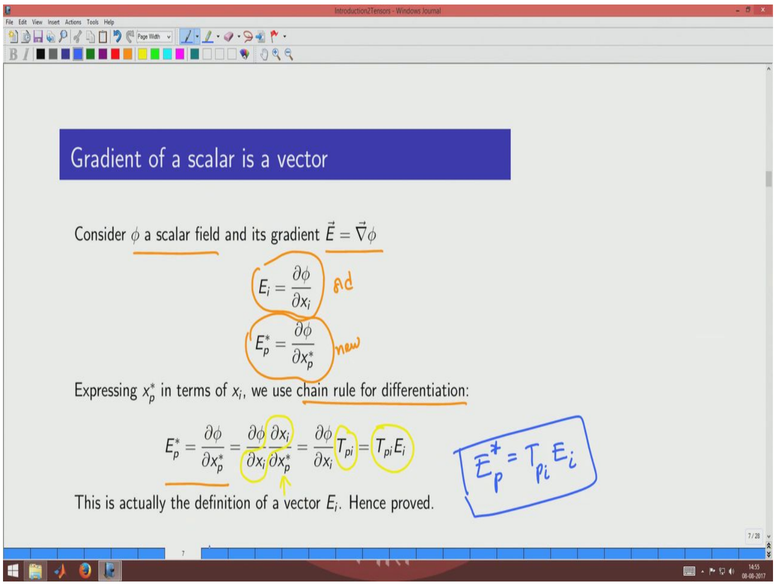

Now, there are also other proofs that we can do, and these appear to be quite silly because we thought always that they are vectors. So, for example, if ϕ were a scalar field then the $\vec{\nabla}\phi$ must be a vector. So, we always knew that gradients are vectors. So, what is that mean by proving that they are vectors? The idea is as follows; do the elements of the gradient change the way, we have defined the elements of the vectors should change. So, if we can show that then we show that the gradient of a scalar function is a vector. So, to do that what we do is, given that φ the scalar field φ does not change when the coordinate system is changed.

So, we do not to have to put a star on top of it etcetera. So, we define what will be the gradient in the old coordinate system and then the gradient in the new coordinate system. Now what we see is that we want to express the new in terms of old. So, this is the new in the new, coordinate system what is a gradient. So, here we use the chain rule so that we can differentiate the unit vectors with respect to the old ones using the chain rule. So, this is something that we already are familiar in mathematics, whenever this a variable that is a function of three other variables and you can use the chain function.

So, chain rule for differentiation, which means that we are pretending that x_p^* star is actually a variable which depend upon 3 variables namely x_i ; so x_1 x_2 x_3. So, if that was true then you can use the chain rule. And if we use the chain rule then we immediately recognize that here we have got in the second term, the elements on the transformation matrix T, because we defined the transformation matrix elements in terms of the differentiation also earlier. So, we recollect that and bring that in here. So, then we immediately see that, we get the new coordinate system expressed in terms of the old with that T coming up. But then we now realize that what we have written is basically we have written.

So, we have basically written $E_p^* = T_{pi}E_i$. Now this is nothing but definition of E being a vector and if E was basically gradient of a scalar function. That means, gradient of a scalar function is a vector. So, this is how we are proving essentially ok.

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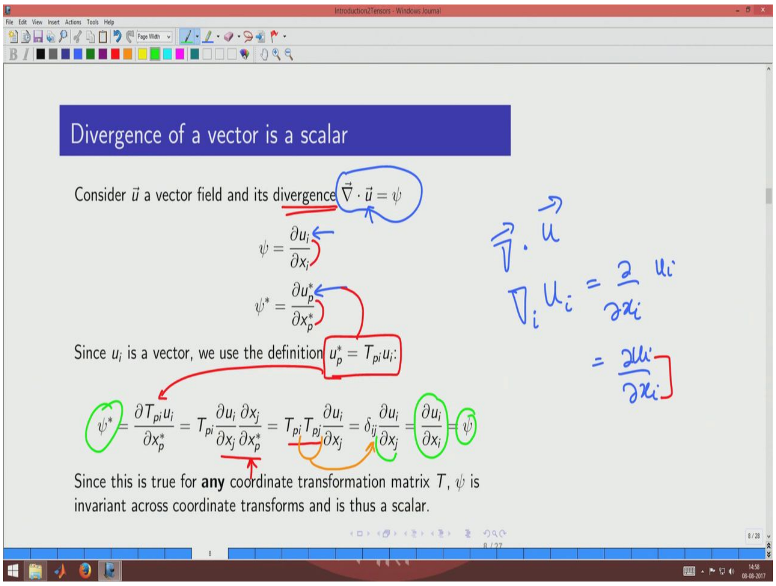

Now, furthermore we have a couple of more proofs to show. Divergence of a vector is scalar. So, this is analogous to the dot product being a scalar. So, divergence of a vector is scalar. So, we are saying that if u is given as a vector, then its divergence which is here given a ψ is then can be taken as a scalar. So, the way we do it as follows.

So, we express the u in terms of the old and new set of terms and divergence is expressed in subscript notation as follows. So, we have got this dot. So, we want to indicate sum subscript for this. So, you want to say u_i and ∇_i. So, this is nothing but $\frac{\partial}{\partial x_i} u_i$, that is nothing but $\frac{\partial u_i}{\partial x_i}$. So, which means that we see that the subscript notation is allowing us to write this very easily, that is what we are using here. So, we are using the same subscript here, same subscript here to indicate that we are talking about a divergence ok.

So, now what is remaining to be proven is that, if u were to be a vector then this is something that you can take for granted because there is a definition of a vector. So, when you substitute that into the expression then what we get is follows. So, we see here that when we substitute we have got the p that is ticking inside the differentiation, but T is nothing but a transformation matrix which is basically a number, which has owned a number for a given θ it just be a number. So, you can bring it out it is not location dependent. It is basically only a relationship between 2 sets of axes it is not location dependent and therefore, you can bring it out. So, when you bring it out then you can see this expression.

In which the second one is again familiar to yours, it is basically nothing but the elements of the transformation matrix. So, when we have got 2 transformation matrices coming side by side, using the subscript notation we see that it is basically a summation over the first index. And we again realize that we are seeing that there is a simplification we can do. The second indices are not the same first index is same. So, we can put the second indices into that δ because of the orthogonality of the old and new coordinate systems and we have put that in here.

So we then see that; finally we can see that when we use the function of δ to simplify the subscripts, you can see that i choose j as a common subscript. So, the other subscript i is then put in there. So, we then choose expression to be having the dummy index i, we could also choose the dummy index to be j that is not a problem. So, when we put that in here then we get the divergence, which is defend in the old coordinate system.

So, we see that whether you choose a new or the old, we see that the divergence is defined in such a way that there is no element of the transformation matrix coming in, which means that it is not dependence upon the transformation matrix, which means that it is invariant across the coordinate transformation, which means it is a scalar so that is how we prove that divergence of a vector is scalar ok.

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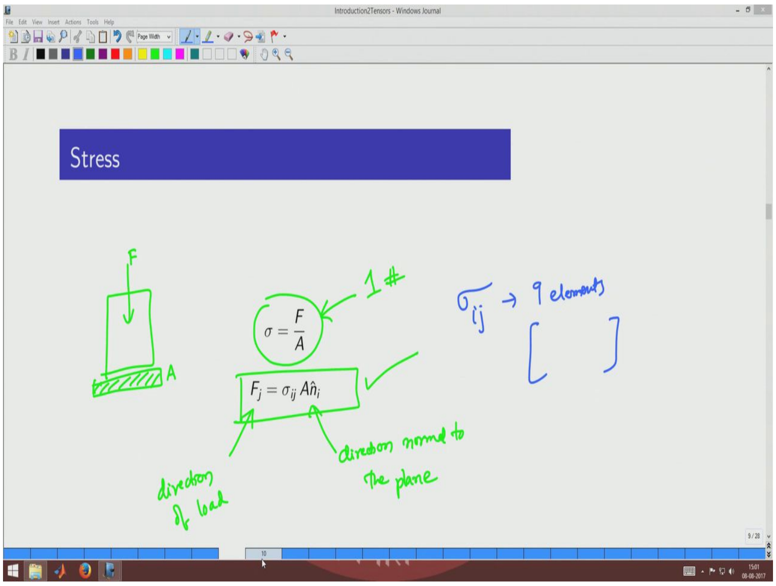

Now, we want to move on from scalars to vectors to tensors which are basically of order higher than 1; so order 2 for example. So, in this situation we have to remember that in high school mathematics or may be even in the first year engineering mathematics, we may not have used tensors. So, it may be a new concept.

So, here is where we have the opportunity to get them introduced, because it makes the rest of the mathematics quite simple. So, let us get that clear. So, most of us have already encountered stress, the stress we always defined as force for unit area. So, if you want to define stress in this manner force per unit area, what we mean is basically you have got when you apply force on one area element of area A. Then the stress that is expressed by that sample with the hatched area is now basically $\frac{F}{A}$.

But if you define like this then you would see that you cannot actually comment on how many elements are required to determine the stress. Because it is very clear that we have got only one direction for force and that is about it and we are just taking two numbers and dividing. So, very often people think that the way they remember this expression, they remember there is only one quantity that is required. So, may be stress is a scalar, but that is wrong because technically the correct way of defining the stress is like this.

The reason is like this, whenever you want to define stress we need to remember that there are 2 directions that are involved. One is the direction of load and another is the direction which is normal to the plane. So, we have 3 combinations of both these directions that are possible, so that totally 9 elements that you can have in a stress. So, technically we say that stress should have 9 elements, and it is a matrix of 9 elements and it is a tensor of order 2, because there are two directions we are talking about. So, this is the first quantity engineering which is of a tonsorial order higher than one, which we will be using very frequently. So, we must not forget that stress is a tensor of order 2. And so we can use that to then define further quantities as we go along ok.

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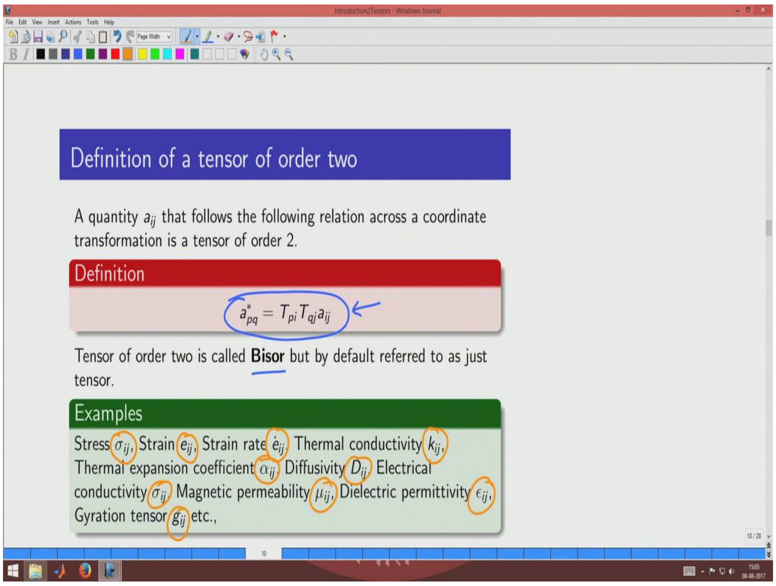

So, we let us now define the tensor of order 2, and then we will also use the expression that we have used for stress to check whether it is a tensor of order 2 or not we will come to that shortly. So, technically the definition of a tensor of order 2 is given here. Any bunch of 9 numbers a_{ij} is bunch of 9 numbers any bunch of 9 numbers which transform using this role, whenever the coordinate system has transformed using a transformation matrix T then that bunch of 9 numbers can be called as elements of a tensor of order 2.

So, you could see that transformation matrix is coming twice; the reason is that there are 2 directions involved in defining the quantity a which is a tensor of order 2. So, therefore, we can then say that is a tensor of order 2 ok.

And technically the tensor of order 2 should be called as a Bisor, but very often people do not use that they just use the word default is tensor. So, when people do not mention the order or rank of a tensor, what they mean is basically tensor of order 2. And what are the quantities that we are familiar with in our engineering education where these are actually known to us as tensors of order 2.

Just now we mentioned that stress is a tensor of order 2, and we have also strain is a tensor of order 2, strain rate is a tensor of order 2 and thermal conductivity which is again a concept that comes from the high school physics itself, thermal conductivity also is a tensor of order 2; thermal expansion coefficient, diffusivity, electrical conductivity, magnetic permeability, dielectric permittivity, gyration tensor for optical properties and so on. There are whole bunch of quantities that are actually of tensor of order 2. And some of these actually we are not familiar to be using 9 different numbers to represent them, some of them we have reformed only 1 number to represent.

So, under what circumstances a tensor of order 2 can have just one number and not 9 numbers, it will be clear in the next session. For now remember that the most generic way of defining these quantities that are listed here, is to call them as tensors of order 2.

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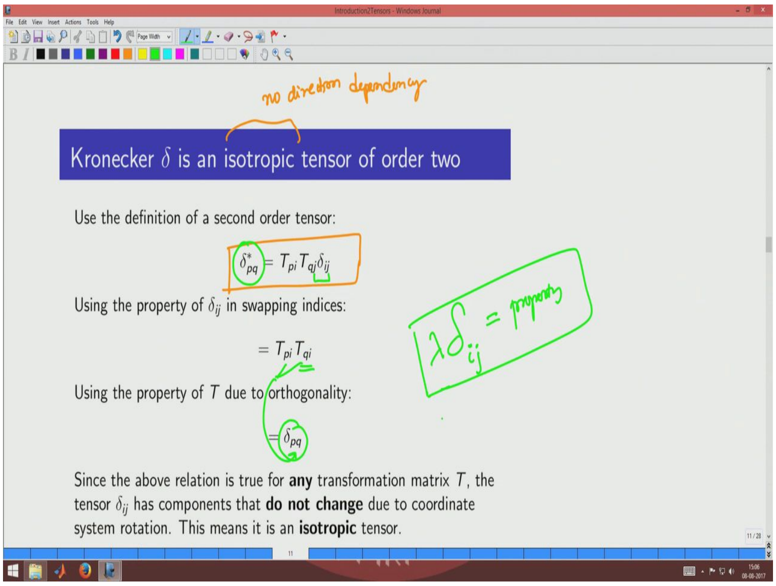

Now, here we want to show that Kronecker delta is a very special tensor, its a tensor of order 2, but it is a special one namely an isotropic tensor of order 2. So, what we mean by that isotropic tensor is as follows. So, what we mean by isotropic tensor is that, there is no direction dependency.

So, please not that I am not calling it as a scalar, it is a having no direction dependency. So, for any transformation matrix it will transform according to the form such that, the numerically the elements of a δ do not change there will still be 1 0 0, 0 1 0 and 0 0 1. So, first of all if we want to prove this what we need to do is, that we have to take the definition of a second order tensor which is known to us and then use the properties of the transformation matrix and the δ to see what happens.

So, first what we do is that we will look at this expression, and see that you can see the j index is matching here. So, I can get the i index and put it there. So, that is what we did here. And here again there are 2 piece 2 transformation matrices coming up. So, that i index is common. So, I take that index that is not common and put them here ok. Now, we see that on the left hand side we have got the δ and the new coordinate system on the right hand side we got the δ, and the old coordinate system the indices are same and whether we put a star or not the numbers are just simply again 1 or 0.

Therefore, we can say that δ it will transform in such a way, that it does not have the elements of transformation matrix sticking in, and it does follow the rules of a second order tensor. So, it is a second order tensor but an isotropic 1. And we use the δ conveniently to represent any property there is a isotropic property. So, what we do is that, we pick a quantity and then use δ before it and say that this could be my property as I want. So, I use this kind of an expression whenever we want to represent a property which is a second order tensor, but it is an isotropic one.

So, this is how we make use of the isotropic tensor, given by δ ok.

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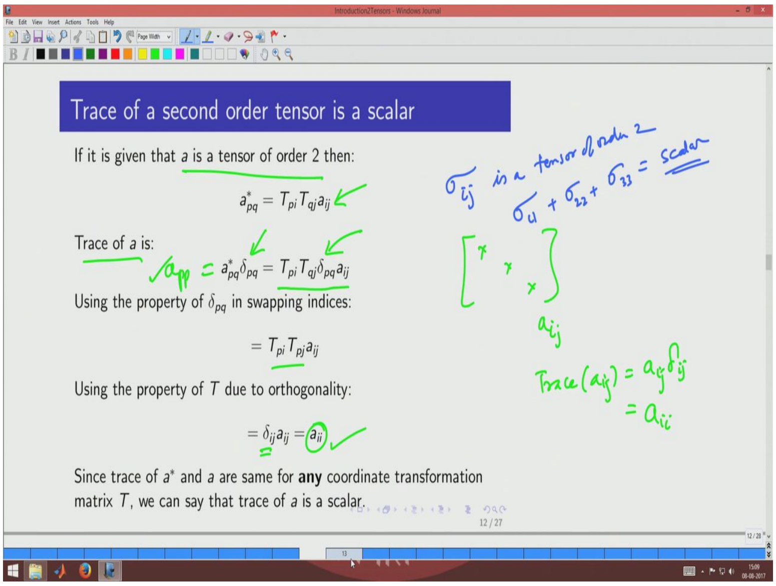

Now, there are some properties that we can now derive, based upon the definitions. So, one property is trace of a second order tensor is scalar. So, this is of actually tremendous importance. So, this is used in some of the constructions that are done in mechanical metallurgy using stresses, and we can actually realize that the meaning of those constructions comes from the very basic property of a tensor namely the trace of a tensor is actually invariant across coordinate transformations.

So, the way it works as follows; is a is a tensor of order 2, then first of all if it is a tensor of order 2 we then write this definition, and then we see what is a trace. So, trace is nothing but the sum of the elements in the diagram. So, these 3 sum. So, which means that the if a_{ij} is a matrix then the trace of a_{ij} is nothing but $a_{ij}\delta_{ij}$, which is nothing but a_{ii}. So, that is what we are doing here. So, trace of a is then given.

So, we now take this quantity and introduce δ because we want to take a trace. So, once you take the trace now you can see that the same properties of δ can be used to swap the indices and then we get the 2 t's and then we use a orthogonality of the coordinate systems to see how to reduce the 2 transformation matrices into the δ, and then we see the trace comes out here. So, you could see that on the left hand side. So, on the left hand side on the right hand side we see that is the same quantity a, and all the diagonal elements are being summed up.

Please see that this expression does not have any transformation matrix elements sitting in it. So, which means that this is free from any effects due to the coordinate system rotations, which means it is a scalar ok. Now, this has some connotation why it implies is that we already know that stress is a tensor of order 2, what it implies is that the $\sigma_{11} + \sigma_{22} + \sigma_{33}$ is the trace, and this is a scalar; which means that whenever you rotate the coordinate system the different values of the stresses will change, but the sum of the diagonal elements does not change.

So, this is a property that we have brought out. Now this is not a property because it is a stress, it is a property because stress is a tensor of order 2 and for a tensor of order 2 we have this property coming up ok.

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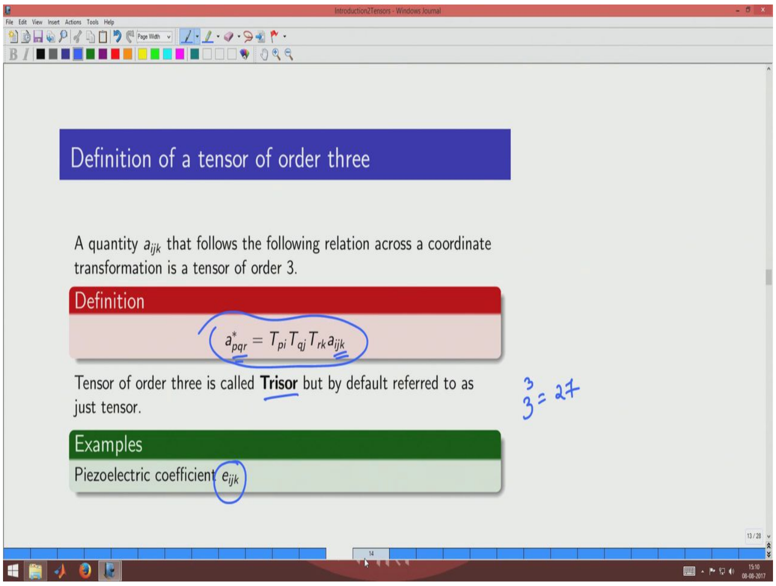

Now, we can actually now go on to define tensors of higher order. So, we will not go into depth for each of these, but we want to define them so that the generic definition of a tensor of any order is then evident. Tensor of order 3 is defined here you can see that the transformation matrix is coming thrice, which means that there are 3 directions that are involved in measuring a, and look at the indices the indices of the new coordinate system are the first positions of the transformation matrix T, and the indices of the old coordinate system come as a second index.

So, it is a same thing as the vectors, its only that we have got more instances of transformation matrix that are sitting in the definition and technically you should call as a trisor, but people do not use it and very often if we just say tensor 1 means tensor of order 2. So, when you are referring to tensor of order 3 then you also mention the order or you can also call it Trisor. So, there are some quantities that are known to be tensors of order 3 and piezoelectric coefficient is one such quantity that we are familiar with. Very often, again it might be used as just one number, but that is a separate concept, it is known that if it is a tensor of order 3 then you must have 3^3.

So, you must have 27 different elements that are required to determinant completely and the most general way ok.

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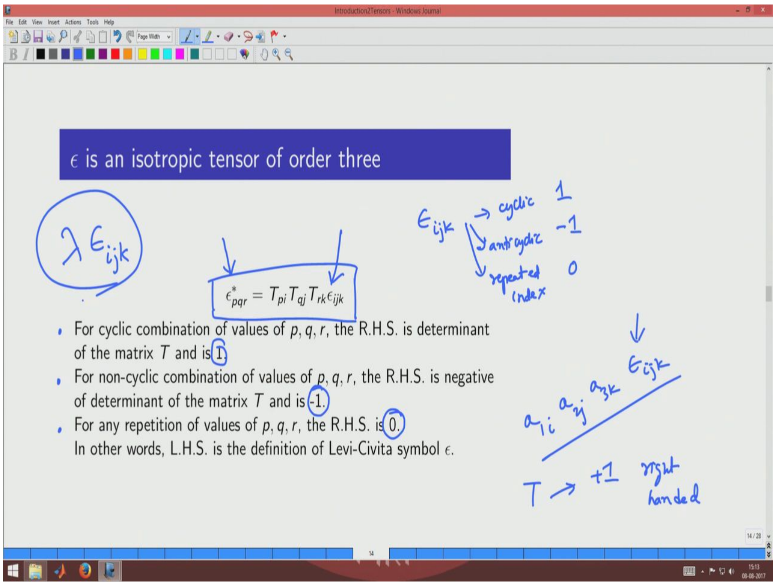

Now, we can extend this definition to see whether we know some quantity that is also isotropic in that sense. So, we see that the quantity which is having 3 indices is Levi Civita tensor or permutation matrix ϵ. So, ϵ just to recap ϵ is defined such that if i j k are cyclic, then you get the value of +1 if they are anti cyclic then you get -1 and if they have any repeated indices then you get 0. So, it is a bunch of 27 numbers with 3 +1's and 3 -1's and the remaining 0's.

Now this permutation matrix was used by us to define a cross product, and we also used to define the determinant of a matrix. So, we are going to use that to now check what happens whether if we want to call this ϵ as a tensor then what does it imply. So, we see that the definition of ϵ being a tensor now given here. Now from this definition can we talk about anything? Now you see that the way the argument is done is as follows; whenever the indices of ϵ_{ijk} are cyclic then what we are talking about is that the expression that we are written here are is nothing but it is a determinant of the matrix T which is +1. And to remember you also know that $a_{1i}\ a_{2j}\ a_{3k}\ \epsilon_{ijk}$.

So, this was a definition of the Levi Civita tensor being used to define the determinants of the matrix a. So, we are using this definition to check; now we already know that the determinant of the transformation matrix T is +1, because we are using the right handed system right handed coordinate system. So that means, that whenever i j k are cyclic then we get the value

+1 here. And by the same thing when they are not cyclic then they are actually going into the left hand system So, we get the magnitude of the determinant of T being -1, and then when we have any indices matching; that means, that there is a coplanarity that is being used.

And therefore, we get a 0 there, which means that gain the values that we are getting 1, -1 and 0 for cyclic, non cyclic and repeated indices is nothing but definition of ϵ itself, so which means that on the left hand side what we have is ϵ itself. So, we have started with the old coordinate system having ϵ, and we have landed up with a new coordinate system also having the ϵ with a same values for the same type of indices, which means that ϵ must be an isotropic tensor of order 3 ok.

So, this shows that whenever you have an isotropic property that has to be determined, then you could actually use an expression like this you could use $\lambda\epsilon_{ijk}$. So, you can use this expression to represent an isotropic tensor of order 3, which has just 1 number that is multiplied in front of it. So, tensor of order 4 is given here.

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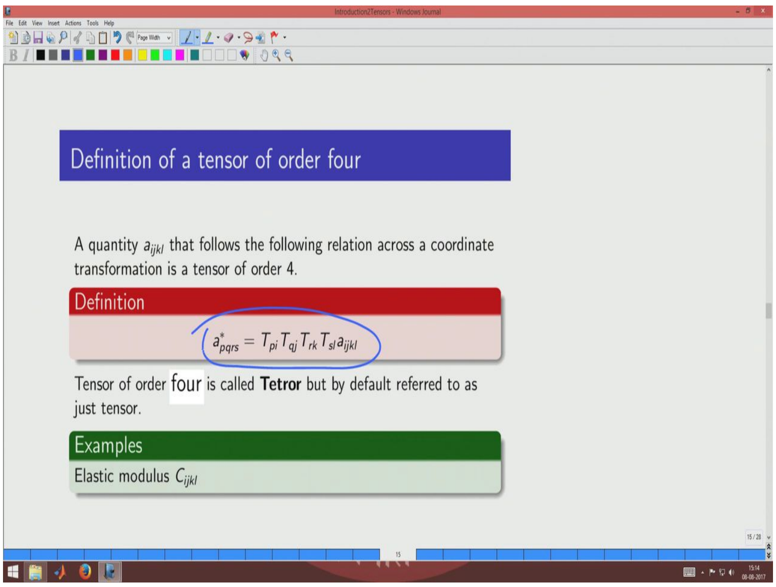

And you can see that the 4 instances of transformation matrix T that are coming in. You can see that it is coming with the indices pi, qj, rk and sl. And you can see that the indices are such the first one first letters belong to the indices of the tensor in the new coordinate system, p q r s and the second set i j k l are indices of the quantity in the old coordinate system.

So, the sequence of writing the transformation matrices same as for a vector definition, it is just that in the vector definition we have got only one term of transformation matrix coming in, whereas in the case of a tensor of order 2, we have got 4 of them coming in and you can extend it to higher order also. And you see the multiplication of the transformation matrix it is not a matrix multiplication.

Please note that the way the indices are it does not indicate a matrix multiplication. So, let us not take this as a matrix multiplication of 4 transformation matrices one after other.

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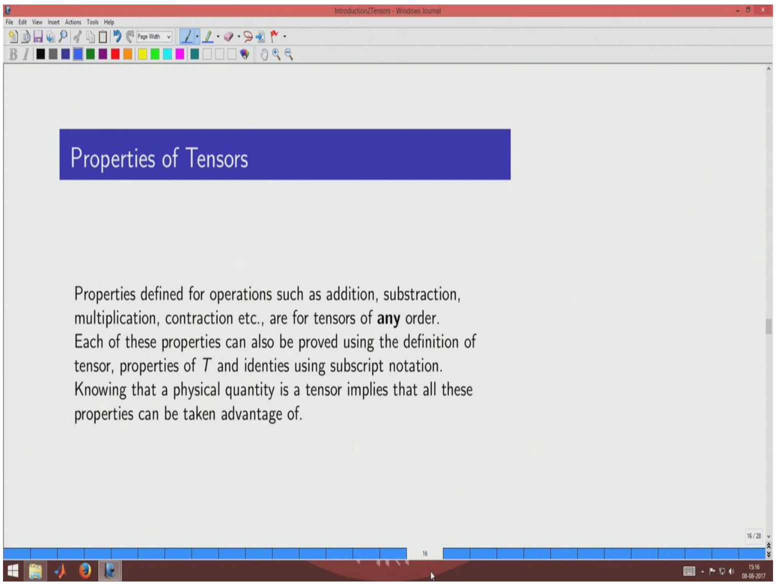

We have to be multiplied element by element the way the indices are given. So, just to practice on the right hand side what are all the dummy indices you can see that i j k l are repeated. So, they are dummy indices on the right hand side the free indices are p q r s which means that the 4 free indices, because tensor is order 4. So, we have got 4 free indices the technical name of a tensor of order 4 is Tetror, but very often people do not use that people say tensor of order 4 ok.

And there are quantities that we already are familiar. So, elastic modules or complains that is a tensor of order 4. So, tensor of order 4 is supposed to have 3^4 81 elements in it. So, there must be 81 different elements to represent completely a tensor of order 4. However, we see that when we want to represent elastic modules we do not have 81 elements, we have just 3 elements 3 different module. So, how did 81 become 3? So, we will come to that in a later class we will just know at this moment we will not discuss about that.

So, there are some properties of tensors that we can discuss now, and we discuss these to essentially make our mathematics little bit simplified. So, knowing that the quantity is a tensor enables us to make some operations. So, that is what we want to now go through. So, what we do is that? There are something that we already know which will work with numbers for example, addition, subtraction, etcetera some of them are commutative property some of them of distributive properties. So, which of these properties are applicable to tensors so, that we can make use of those in our derivations is our objective.

So, we will go through some of them now, ok.

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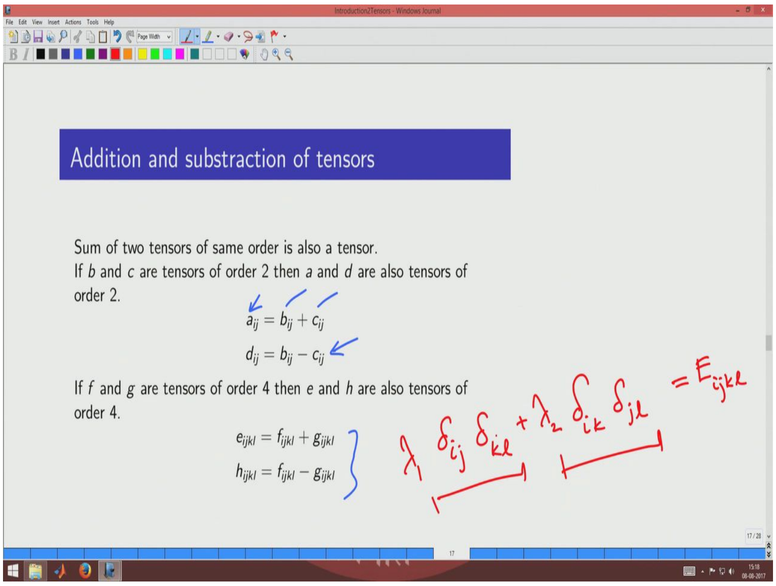

So, addition and subtraction are allowed for tensors. As long as the subscripts are same you can go ahead and add them then the tensorial nature of those quantities would not change. We should not do this when the indices are not matching; that is the free indices are not matching we should not do that in this case for example, if b and c are tensors of order 2, and we are using the same free indices i and j for both of them then a is also a tensor of order 2. And you could also do the subtraction here you have a + or a - it does not matter.

So, if b and c are tensors of order 2 then a and b are also tensors of order 2. So, that is guaranteed and you could also extend this to any order. So, you have seen here that we are using 4 indices. So, f and g are tensors of order 4, which means that e and h are also tensors of order 4 which means that we can use this to combine quantities to see how to generate tensors of higher order. So, let us take for example: something like this, let us take you know an expression like this. So, this is having 4 indices. So, it is a tensor of order 4.

And let us take another expression. So, this also is a tensor of order 4. So, what would be? So, if I make like this. So, what happens is that, this term and this term are both tensors of order 4 with the free indices i j k l. So, I can add them and then I get a tensor of order 4. So, I want to give some new. So, I can give it as; so we can extend it further.

So, the summation allowing us to preserve the tensorial nature is useful to generate quantities such as this and we will find a use for this kind of a quantity later on. Now there are certain types of tensor that we talk about, particularly for second order tensor they are very important.

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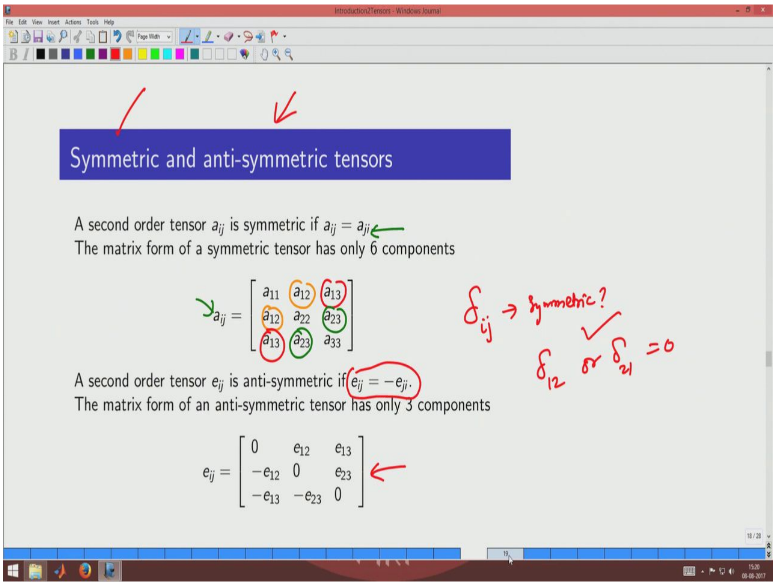

So, we characterize the tensors by some names. So, whenever we use the word like symmetric. So, when we say a tensor is symmetric tensor, what we mean is that the matrix representation of the tensor is a symmetric matrix that is what we mean. So, here you can see that i am using; so a_{12} in both the positions, and 1 3 in both the positions and then 2 3 in both the positions.

So, you can see that this matrix is actually a symmetric matrix. So, this tensor a is a symmetric tensor. So, it also means that symmetric as in the position of the indices. So, when I swap the positions of the indices then the quantity does not change, that is the essence of calling a tensor as a symmetric tensor. So, now, is it obvious that this δ is symmetric of course, because δ_{11} whether you talk about 1 1 2 2 3 3 they are all ones, but δ_{12} or δ_{21} they are both zeros. So, therefore, it is a symmetric tensor of course, it is also a diagonal tensor because you have only the diagonal elements that are present, but it is one very simple quantities that we know.

So, just like we have defined the symmetric tensor we can also define; what is an anti symmetric or skew symmetric tensor. So, what we mean by anti symmetric or skew symmetric is, that when we swap the 2 indices you get a minus of that quantity. So, that is what is essentially given here. So, when you swap the indices i and j then you get a negative of that quantity that is given here, which also means that the diagonal elements must be necessarily zeros, the reason is that when you swap the indices you must get negative.

So, 1 1 must be same as minus of 1 1. So, $e_{11} = - e_{11}$ it only possible when e_{11} is 0. So, that is what we are having here indicator, ok.

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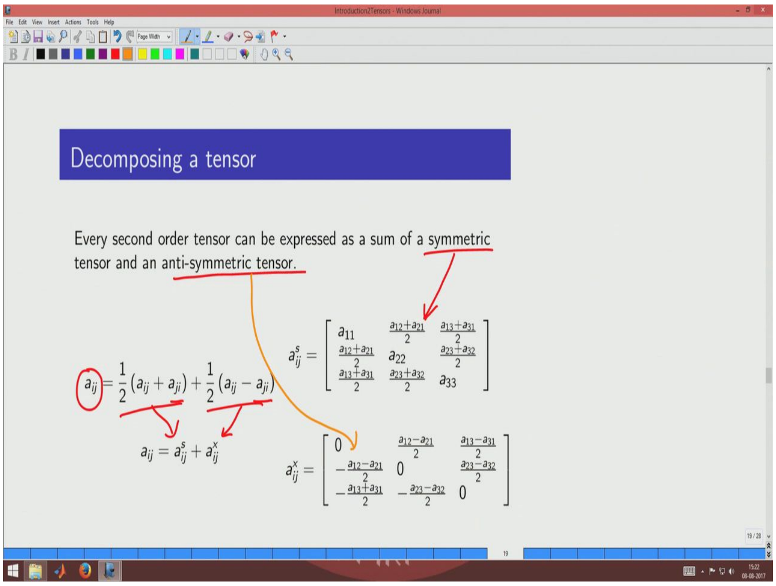

So, these are the type of you know characterizations we give for a tensor, and then there is a theorem which tells you that any tensor can be expressed as a sum of symmetric and anti symmetric tensor. So, you can take any second order tensor and then you can actually call it as 2 parts and one part is symmetric and one another part is anti symmetric. So, we can do that here.

So, let us say a_{ij} ; a_{ij} is any arbitrary tensor of order 2. So, what we do is that we add $\frac{1}{2}\, a_{ji}$ and we subtract $\frac{1}{2}\, a_{ji}$. So, we are not done anything to the tensor a_{ij}, but you see that the first term here is the symmetric part and the second term here is a anti symmetric part. So, we can see that any tensor a_{ij} can be represented using this kind of a summation that it can be a summation of symmetric and anti symmetric parts. So, this is the symmetric part, and this is the anti symmetric part.

So, this also helps us later on when we look at this stress tensor for example, we want to separate it into the symmetric part and the anti symmetric part because we do not want the anti symmetric part to come and make any presence in our equations for some reason that we will discuss later on. So, ability to spilt a tensor in 2 parts is something that we will need later on, ok.

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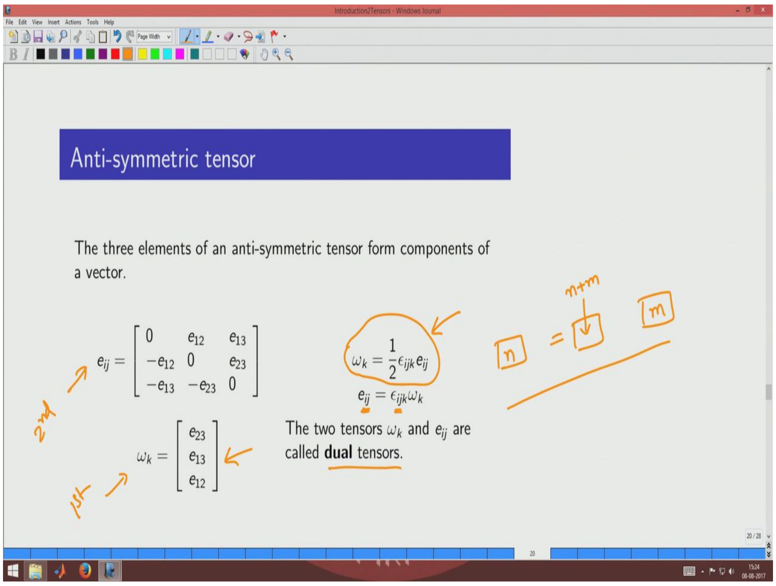

Now, the anti symmetric tensor that is the skew symmetric tensor which has only 3 elements in it 3 zeros- so 3 elements and then the other three are just negative of these. So, this can be also represented as a vector because there are only 3 numbers actually you have you can see that you have got only e_{12} e_{13} and e_{23}. So, they have only 3 elements.

So, we could actually generate a vector out of that. So, the way this vector and this tensor related is by the Levi Civita tensor or ϵ and that is given here. So, you can see that the quantity here this is second order and this is a first order. So, you could see that the there is a relationship between a first order and second order tensors if the tensor happens to be anti symmetric. And therefore, such quantities are also called as dual tensors, and this particularly as of use for us later on because we can actually use this expression to simplify some of the terms later on in some of the derivations.

So, the idea that when you swap i j you get a minus sign there is also preserved in the right hand side you know that if i, j, k were to be cyclic and i j are swapped, then you go anti cyclic and you get the minus value. So, the sense is preserved only thing is that we are actually relating a second order tensor, with a first order tensor using a third order tensor perhaps this is a first time that we are encountering, how tensors of different order can come together. So, we see that whenever you have got these quantities, you see that you can write equations where one is a tensor of order n.

And another is a tensor of order m and you can in many situations write the quantity to be a tensor of order n+m. And this kind of a form will come again and again in many of our expression later on.

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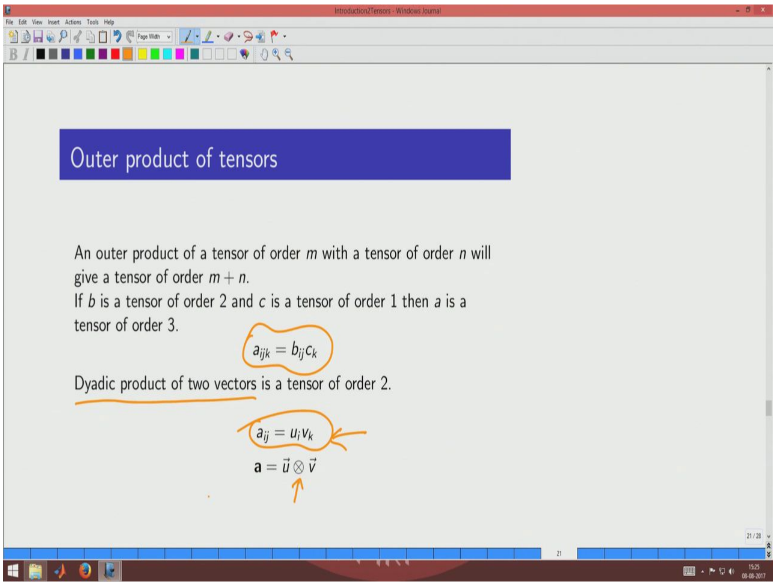

Now we can now go ahead and start proving some quantities to be tensors of order 2. Till now we have proven some quantities to be vectors or scalars. So, we can prove some of them to be tensors of order 2, and what do we need to prove them as tensors of order 2 we basically need only the definitions of tensors. So, one thing what we are writing here is that an outer product of tensors is a tensor.

So, this is a theorem you can prove that shortly. So, this is a theorem, and we already know that the dyadic product of 2 vectors is nothing but outer product of 2 vectors. So, which means that if we knew u and v to be vectors then a is a tensor of order 2. So, we can construct tensors of higher order from tensors of lower order here quite easily, and this is actually using the symbol □ and these 2 are basically saying the same thing, ok.

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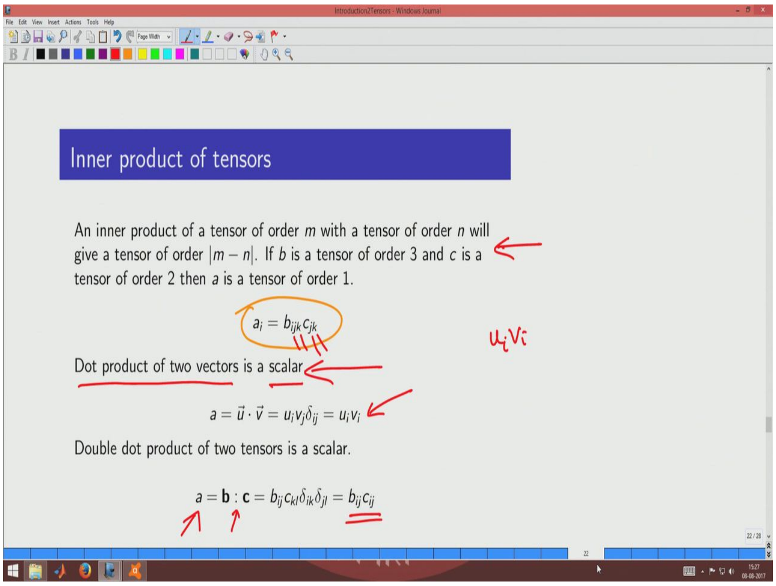

So, the inner product also is a tensor is another theorem, this also can be proven in various situations. So, what we mean by inner product is that some of the indices are dummy. So, you can see here that you can see j and k are repeated. So, on the right hand side j and k are the dummy indices and i is the free index on the left hand side i is a free index. So, you can see that the way b and c are multiplied is a inner product, you cannot call it as a dot product, because dot product we are familiar like this you know $u_i v_i$.

So, we are not familiar to using the word dot product whenever we are taking higher order tensor, but inner product is a very generic way to say that does not matter how many indices are there for the 2 quantities b and c, if some of the indices are repeated then it is a inner product. Now what is a non repeated index will come on to the left hand side as a index of a. Now dot product of 2 vectors is nothing but inner product of 2 vectors and the theorem says that inner product of 2 tensors is a tensor.

So, inner product of 2 vectors must be a tensor, and because is a inner product it must have reduce reduction of the order of the tensor. So, we go from order 1 to 0. So, tensor of order 0 is scalar and that is way we write it as a scalar. So, what we are stating here is nothing but the same statement as here. So, what we are stating is inner product of 2 tensors of order 1 is a tensor of order 0, and here we are saying inner product of a tensor of order m with the tensor of order n is a tensor of order m - n.

So, here it is 1 - 1 = 0 and that is why we get the scalar. So, this is a theorem and it can be proven. So, we can see that u_i v_i must be a tensor of order 0, we went add and proven it earlier, but we now know that there is a theorem to say very generically how this can be handled. Now you can also see that we can define double dot product of 2 tensors just like we have define the dot product of 2 vectors like this with a double dot, and that is nothing but the inner product of 2 tensors of order 2 where both the indices are repeating. So, there is no free index so; that means, this must be a scalar and as per the theorem this must be a scalar also.

So, one can actually use this to actually deduce lot of things about the tensors that come together ok.

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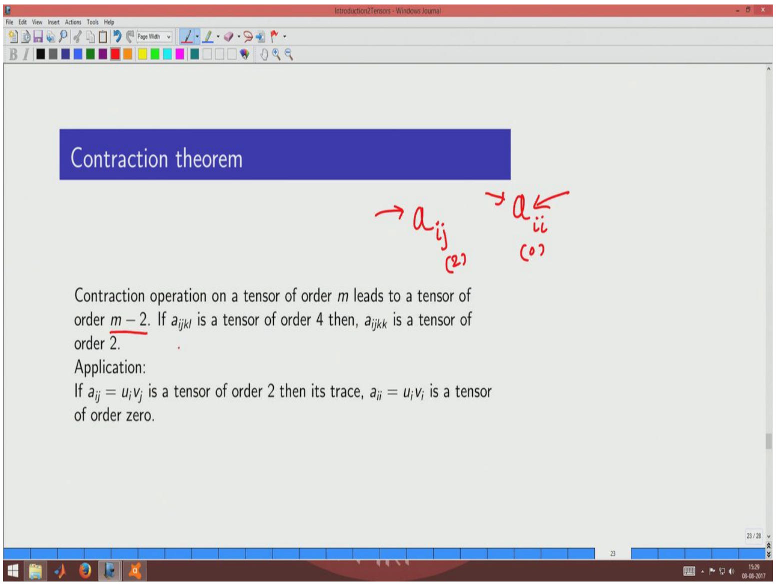

Now, contraction theorem is one theorem that is also useful to actually go ahead and assume some of proves has actually not required, what contraction theorem says is as follows. Whenever there are certain numbers of indices for a tensor, then if you repeat any of the indices then you are doing a contraction operation. So, let us say you have taken a tensor of order 2 a_{ij}. If i repeat the index; that means, if i say a_{ii} ; that means, what i have done is a contraction operation over the indices i and j, and the number of free indices here is 2 and this is 0 here. So, whenever we do a contraction operation we actually reduce the order of the tensor from m to m-2. The theorem says that whenever you do a contraction operation of a tensor, the resultant quantity is also tensor, but the order is of m-2.

So, what it implies is that when you do a contraction operation of a tensor of order 2, then the resultant is a tensor of order 2 minus to 0 tensor of order 0 is scalar. So, what we are also saying is that the trace of a is a scalar. So, this is something that we have proven separately earlier, but from this theorem also it is evident that is nothing but contraction theorem applicable to second order tensor is what is giving you the fact that the trace of a tensor is a scalar now you could also apply it to any higher order terms, ok.

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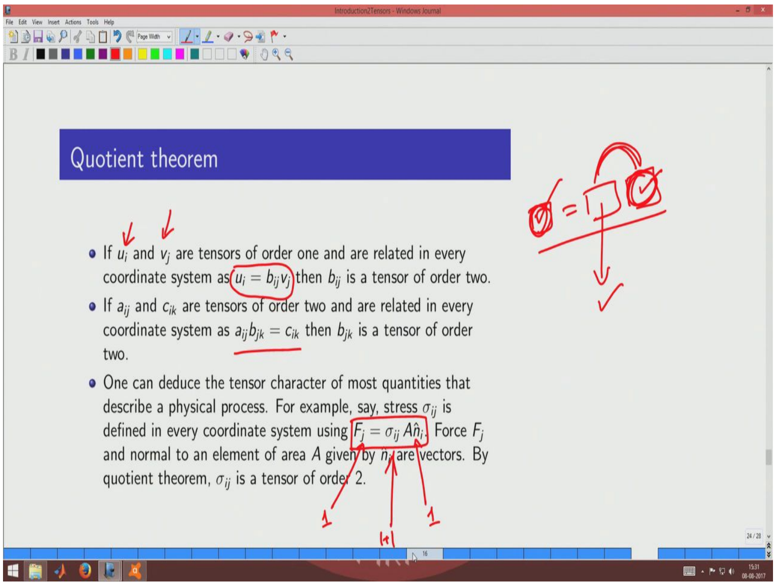

Now, this is the one of the powerful theorems that are used to derive quantities that we do not know as tensors, but we realize that tensors and this theorem is called the quotient theorem. So, the quotient theorem is actually a very settle concept, what it says it as follows. We will define it in the respective vectors and then we can extend it to higher order tensors also, quotient theorem is applicable for tensors of any order what it says is as follows. If you knew that in every coordinate system u_i and v_j which are basically 2 vectors are always related in this form, $u_i = b_{ij} v_j$ in every coordinate system they are they are represented in this manner then b_{ij} must be a tensor of order 2.

So, it is like this you know when you knew this way that this is a tensor of order something this is a tensor of order something and the way this is multiplied in every coordinate system then it implies that this must also be a tensor of order appropriately taken. So, this is idea. Now the outer product being a tensor is when you knew these two are tensors, but here it is not these 2, it is these 2 which are tensors and we are talking about the tensor will order of what is here ok.

So, it is not actually a direct extension of outer product or inner product being tensors. So, that is why we need a separate theorem to talk about this now when we apply it for higher orders we can also see that in every coordinate system if a_{ij} and c_{ik} are related like this, then b_{jk} also must be a tensor of order 2 and we can also go ahead. Now we can use this quotient theorem to deduce that a stress must be a tensor of order 2, because in every coordinate system stress is actually define in this manner and we know that this is a vector of order 1, tensor of order 1 vector and this is a tensor of order 1 vector.

So, here is 1 and here is 1. So, this must be tensor of order 1 + 1 tensor of order 2. So, the stress being a tensor of order 2 is actually a result of quotient theorem that is about it that is enough for us to believe that stress is a tensor of order 2 and there is a quotient theorem to also prove the same thing, ok.

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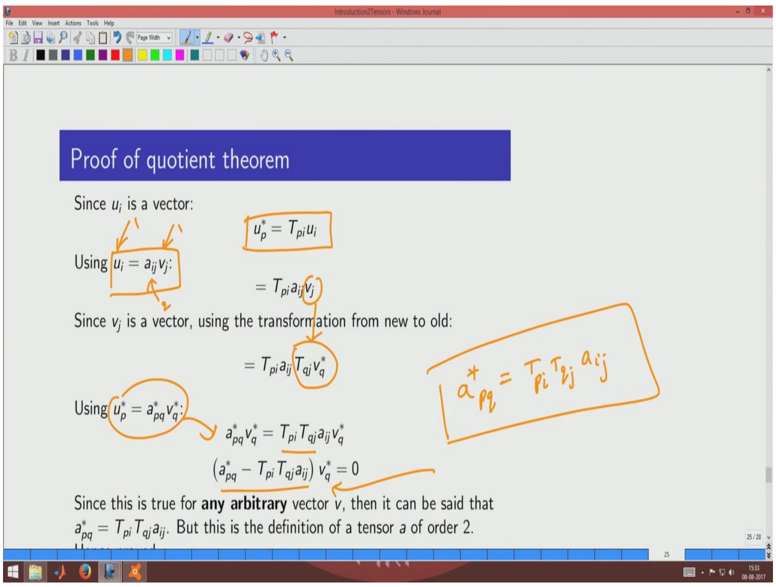

Now, the proof of the quotient theorem is given here, the proof is actually like this you go step by step first we say that because u is a vector we write this, this is nothing but the definition of u as a vector every vector has to follow this definition. And then we say that in every coordinate system u is written in this way. Therefore, I can write in the old coordinate system in this manner itself, and if v_j is a vector then I can expand v_j as this. Now what we realize that there are two transformation matrices that are coming together, then I bring them together here and then we realize that we can actually look at this subscripts, and see that we can actually collect them and we see that when we express the left hand side in the new coordinate system coming here.

Then when you subtract right hand side and left hand side, we see that in every coordinate system this expression must be 0. And this is for any arbitrary $\vec{v}$ and arbitrary $\vec{u}$. So, which means that this is only possible when a_{pq}^{*} is always equal to this, when this is always 0 and this is always 0 when for example, in every coordinate system this is true. Now this is actually a definition of a as a tensor of order 2 and that is exactly what the quotient theorem says. What it says is that in every coordinate system if you wrote in this manner, where U and V are tensors of order 1 then a is a tensor of order 2. So, that is what is been proven.

So, which means that we can now go ahead and use quotient theorem for the things that we do not know as tensors to deduce that they are tensors ok.

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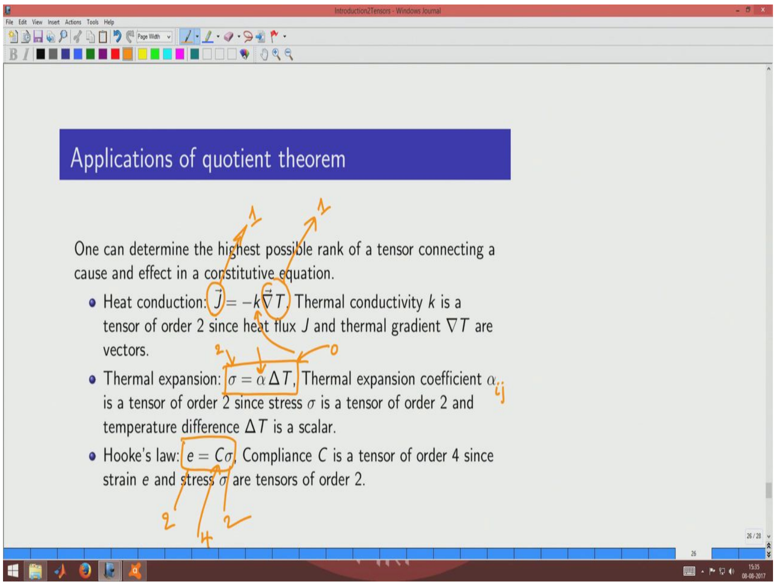

So, some examples are given here. So, let us look at the heat conduction for example, heat conduction, fourier heat conduction is given where heat flux is related to the temperature gradient in this manner where k is a thermal conductivity.

Now, we know that on the left hand side is a vector of order tensor of order 1, and on the right hand side this is a tensor of order 1 both are vectors the heat flux and the gradient or temperature or both vectors and therefore, the most general way to describe the k is that it is a tensor of order 2. Using the quotient theorem, we can deduce that it must be a tensor of order 2, that is the most generic way of saying it and then of course, we can go ahead and make that into a more specific form when we know more about the thermal conductivity.

Similarly thermal expansion coefficient, we know the thermal expansion coefficient is given by this expression we already know that on the left hand side σ is a tensor of order 2 on the right hand side we have got ΔT, Δ is a difference and T is a scalar. So, tensor of order 0 and this is order 2 so; that means, the thermal expansion coefficient should be a tensor of order 2 ok. So, that being tensor of order 2 is adequate coming from the quotient theorem.

And the later on we will see that when we have only one number to represent thermal expansion coefficient. There must be some other piece of information reduce in 9 elements of thermal expansion coefficient to just one, but the most generic way of representing α thermal expansion coefficient using a tensor of order 2. That is right way to do the last one last example is Hookes law. So, you could actually see the stress and strain being linked in a linear manner is in the Hookes law. So, the strain and the stress are tensors of order 2. So, the most generic way of representing a quantity that comes there using the quotient theorem must be using a tensor of order 4 and this is how for example, from expressions that we know already we can deduce the most general tensor order of any quantity.

Later on we can see whether we need such a higher order term or is it sufficient to have a lower order tensor quantity, it will be represent that what to the quantity, ok.

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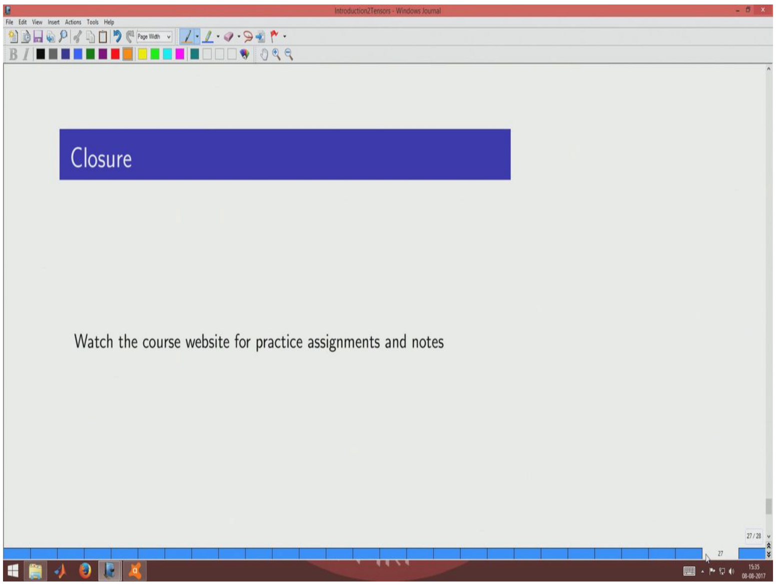

So, at this moment we stop, and we will have some assignments to let you practice with what we have thought. So, we will have situations to prove something is a tensor or not, and to use the various theorems about the tensors that we have derived till now to practice these definitions.