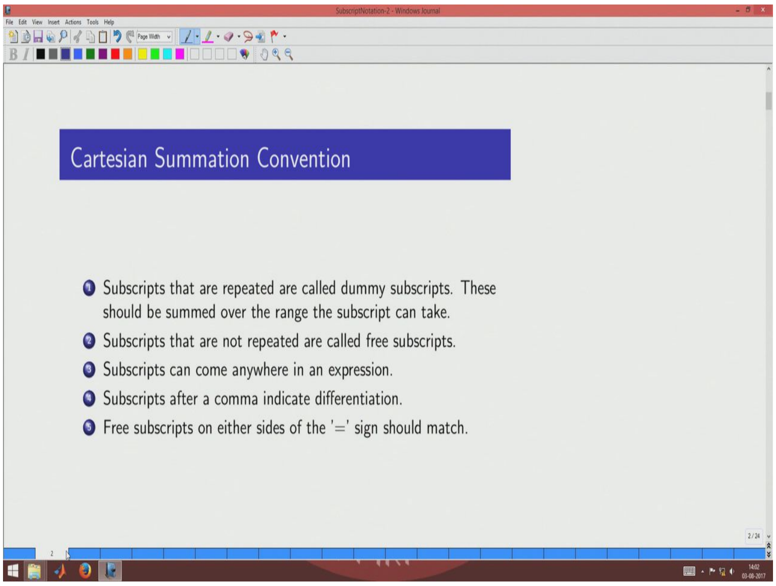
Transport Phenomena in Materials.
Prof. Gandham Phanikumar
Department of Metallurgical and Materials Engineering
Indian Institute of Technology, Madras

Lecture - 02
Subscript Notation Part 2

Welcome back to the session on subscript notation. So, in this session we will be looking at one more quantity ϵ, and then how we can combine the δ, and ϵ to make some of the derivations.

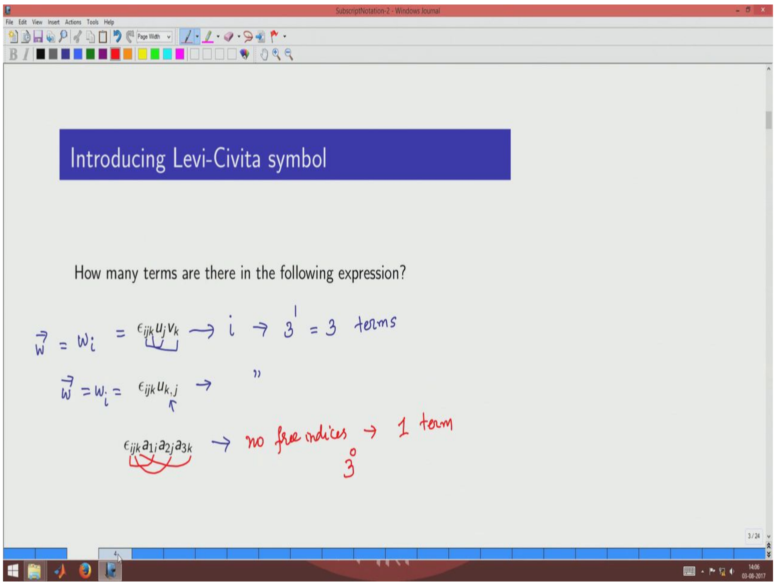
(Refer Slide Time: 00:36)



So, just a brief review of the subscript notation or the so called Cartesian summation convention, these are the 5 rules that we were using in the previous session. The first rule is that the subscripts that are repeated are called the dummy subscripts. And you can also call them as indices.

So, the dummy indices should be summed up over the range that the subscript can take which is basically from 1 to 3 and the subscripts that are not repeated are called the free subscripts or free indices. And subscripts can come in the numerator or in the denominator. And if there is a comma, then it means that the subscript that follows the comma is the distance variable with respect to which we are going to do the differentiation. And we have to verify that every term on either sides of the = sign should have the free indices that should match. The dummy indices in each term need not match because they are going to be summed up anyway.

(Refer Slide Time: 01:32)

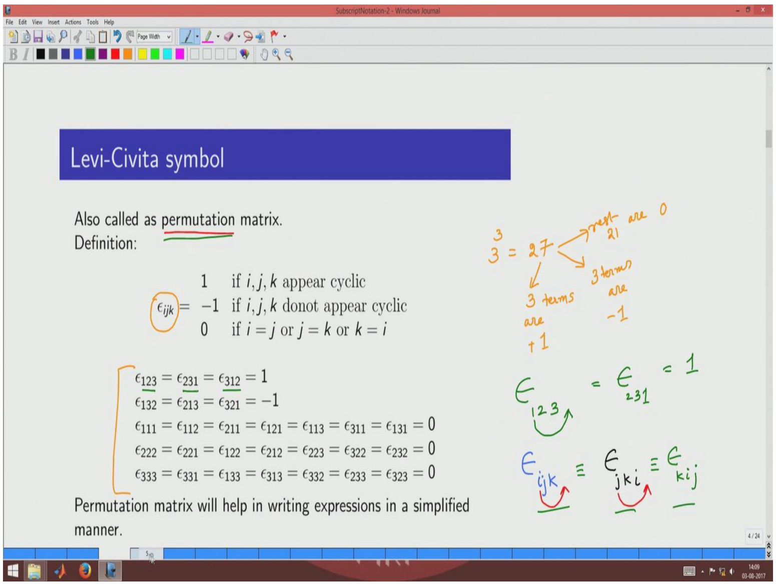


So, here we are getting ourselves familiarized with a new symbol; that is the ϵ symbol. This is called the Levi civita symbol or the permutation matrix. Now we will see the meaning of the epsilon shortly, but for now let us look at the subscript notation that has been used here in these 3 terms. So, how many terms are there in following expression? So, let us take the first expression, you see that in this expression the indices j are matching and also k are matching. So, the i is the only free index. And which means that there is one free index. So, 3^1, that is there are 3 terms.

In other words what we wrote was basically a vector. And if you want you may want to call this as w, and what index should I choose for w? So, it is clear that the free index on right hand side is i. So, you would call this as w_i which is also basically a $\vec{w}$. So, we have we have used ϵ as a symbol to change the indices of u and v into the index of w and there must be a relationship between the 3 vectors $\vec{u}, \vec{v}, \vec{w}$. What that relationship is evident a little while from now. Look at the second expression. In the second expression we are using a differentiation, because there is a comma out here I bring your attention here there is a comma. And k and j are repeated. So, again is a same thing as earlier i is the only free subscript and therefore, we have got this same expression, you could have some other $\vec{w}_i$ with a subscript i which is the free index on the left-hand side as well as on the right-hand side.

The third expression if you see the 3 indices are all matching. So, if you see the first index i is matching with this, and then j is matching here, and then k is matching here, which means that there are no free indices here. Which means that this will have only one term, because a number of free indices is 0. So, 3^0 is 1. So, there is a one term and what does this symbol mean will also be evident shortly.

(Refer Slide Time: 04:01)

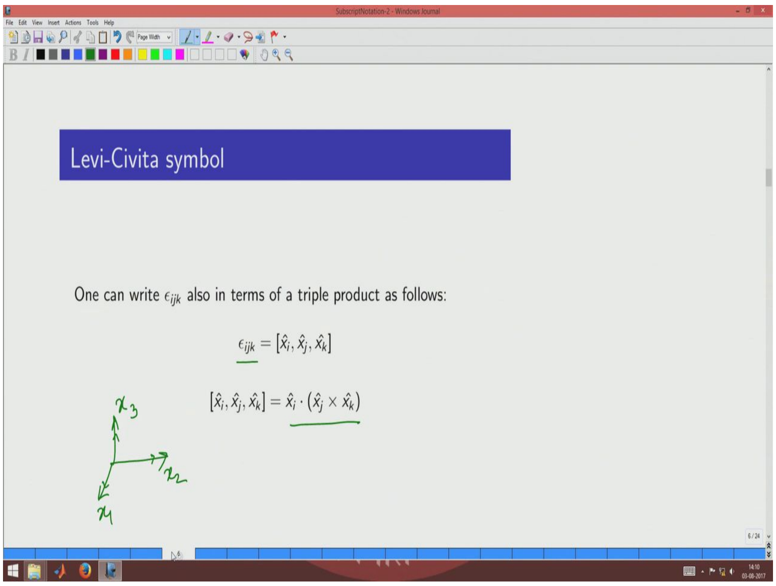


So, here we are using the ϵ symbol and I am giving you the meaning here. As you can see ϵ as a symbol, if you want to look at it has 3 free indices i j k. And how many numbers will be there in that. So, you now know that it should be then $3^3 = 27$. So, there are 27 numbers that will be represented by this one symbol ϵ. And out of this 27, we can see that 3 of them are 3 terms or 3 symbols are basically +1. And then 3 of them are -1, and the rest 21 are 0's. And these 3 values are when the subscripts of ε_{ijk}, if they are cyclic then it is 1, if they are not cyclic then it is -1, and if there is any subscript that is repeated then it have that has 0. So, individually all the elements have been listed in these relationships here, and we can see that there is a special property of ϵ which is why it is of course, called as the permutation matrix, the properties as follows.

Whenever we have the subscripts that are cycled, then the value does not change. Which means that when you look at a symbol like this ε_{123}. So, you take one and put it on the other end, and you will see it is ε_{231} and these 2 are basically the same value in our case it will be 1. So, even when we are using subscript notations where the indices are not expanded. So, the property of ϵ we are going to illustrate here.

So, let us say in any term we have got ε_{ijk}, then you could take the i on to the other side and write as follows. You could write it as ε_{jki}. And then again you could take this term to the other end and you could write it as ε_{kij}. So, all these 3 are cyclic variants of 3 indices i j k. And you could substitute one with the other and they are all the same as for as the subscript notation is concerned. So, this special property which is why we call it as a permutation matrix will be very useful when we are doing some derivations in the later part of the session.

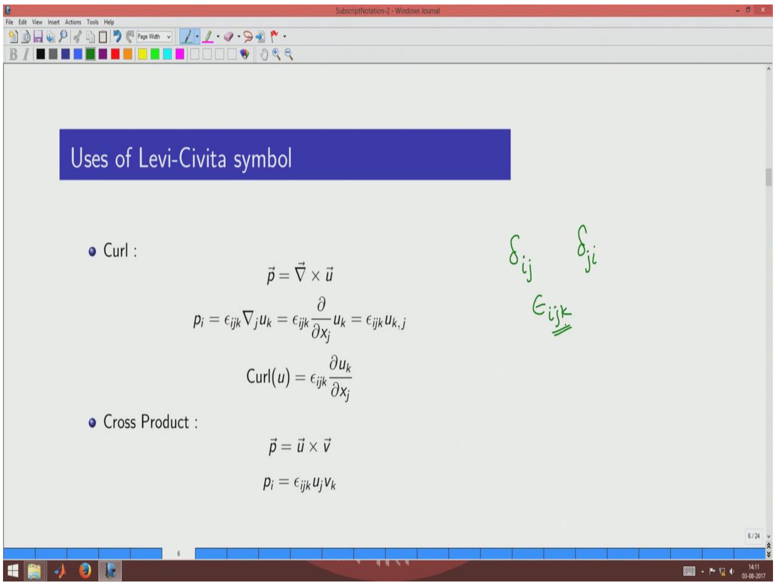
(Refer Slide Time: 06:41)



So, this symbol is then formally to be defined, and we could define it as a triple product of 3-unit vectors which will form basically the unit vectors of the axes that you have chosen. So, x_1 x_2 x_3. So, if there were these 3-unit vectors, then the volume of the cube formed by this 3-unit vectors is what is given by ε_{ijk} . And as we can see here, if there was a repetition of the index it means that basically the dot product will give you 0 and therefore, ϵ will take a value of 0. And if they are cyclic then we are using the right-handed system then you will get +1, and if they are not cyclic we are using a left-handed system which will give you a -1 for the value.

So, you could think that Levi civita symbol is nothing but the volume of the unit cell with 3 indices denoting the 3 vectors that are forming the cubic.

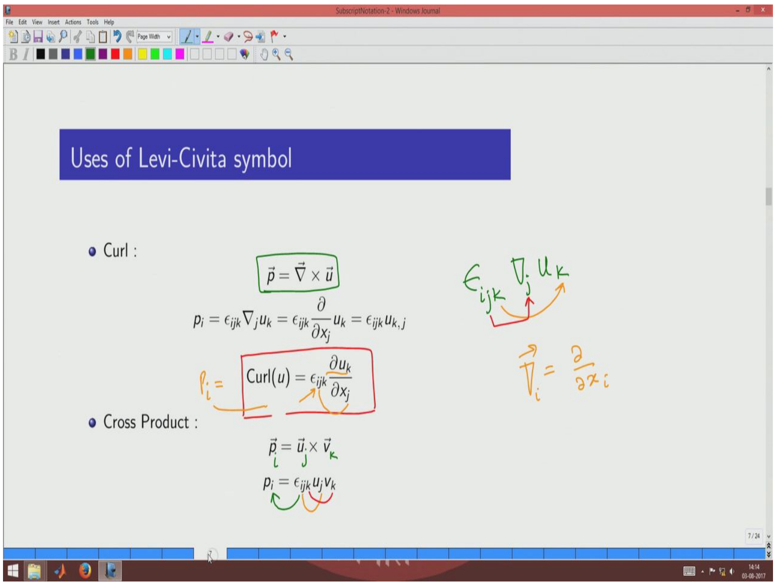
(Refer Slide Time: 07:45)



So now let us make use of this symbol to make some derivations. So, here the way we have to understand is to pay attention to the sequence of the subscripts. Unlike in δ, where if you swap the 2 indices it does not make any difference here it makes a lots of difference. For example, when you wrote a symbol like this δ_{ij} you could as well have written this as δ_{ji} because they are both the same value for a given set of numbers i and j they both will have a same value.

In other words, this is symmetric over the indices i and j whereas, when we write ϵ_{ijk} , the sequence of these 3 must be very important. So, we will illustrate how we are going to use this in our derivations as follows. Here let us take the ∇ X $\vec{u}$.

(Refer Slide Time: 08:43)



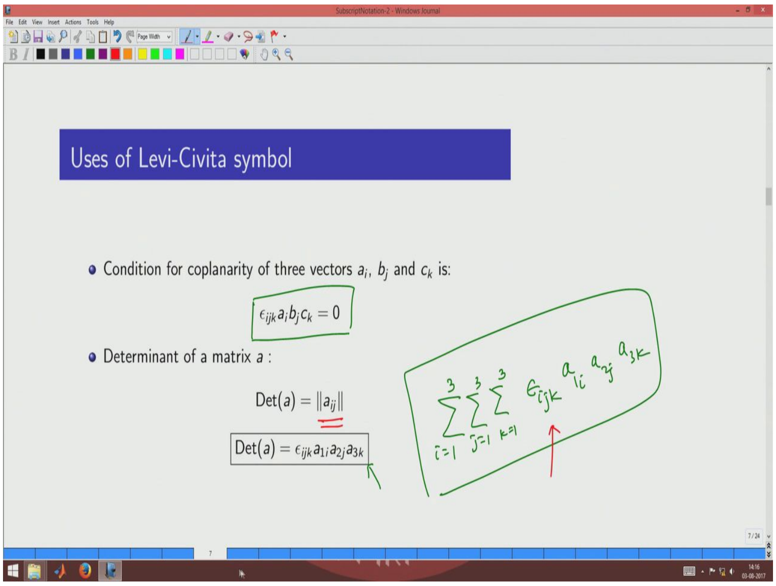
So, here we have got the $\vec{p} = \vec{\nabla}$ X $\vec{v}$. So, the way the curl operator is going to translate in the subscript notation is as follows. We know that operators can also be given a subscript. So, in this case the operator is given a subscript j. And a vector has to be given a subscript. So, that is given k.

So, we write the subscript for ϵ in such a way that, the second one corresponds to that of the nabla operator or the ∇ operator, the third index corresponds to the vector. So, here I would illustrate it here like this. ε_{ijk} and then you have got ∇_j and then u_k. So, it is very important to get this right. So, this is how it goes. This way we will not make a mistake, as we know that when the rotation axis is different then you get the sign changed in these kind of vectorial operations. So, keeping the sequence in correct is important. So, when we expand the ∇ operator we know that this is nothing but this kind of a thing. So, then you expand it here. And then you could then use the comma symbol and write it the same way here, which means that the curl operator is then expanded using the as follows.

So, whenever there is a curl operator, you could then immediately use ϵ and then you can make a subscript notation of the entire symbol here. And the output is basically a vector. So, which index should be use for that vector. It is already known to us. Because when we look at this expression you saw that the j is repeating, and k is also repeating. So, the one which is free index is i, and that should be the index for what is supposed to be here on the left-hand side. So, p_i is equal to. So, this index has to be shown such that we do not have a violation of the subscript notation rule, the last will be says that the free subscript should be matching on both hand side.

So, cross product and the curl operator are to be treated in a very similar manner. So, the way you write a cross product is very similar, instead of the nabla operator here you are writing for the $\vec{u}$ and $\vec{v}$. So, here also the same rules applied. You could see that the second index is used for u, and the third index is used for v, and the free index which is there which is i is what is used for the outcome p. So, So, the indices for u and v if you decide them, then you could immediately see what should be the index of p. And ϵ having the 3 indices will immediately give you how the notation has to be written. Now this is going to be used again and again later on. So, getting the sequence of these indices is very important.

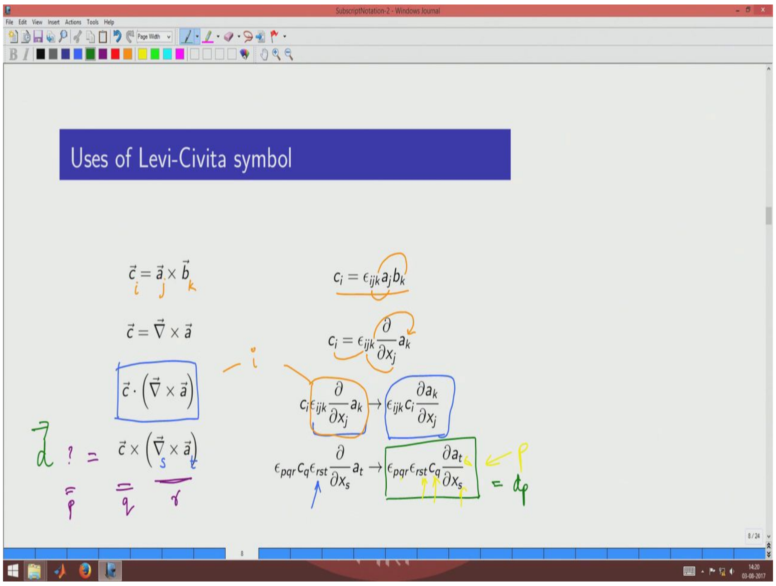
(Refer Slide Time: 12:03)



So now there are some direct usages of the permutation matrix, which are evident from the very definition. So, we can see that from the definition, we can use it to know whether 3 vectors are coplanar. And this condition shows how to evaluate that. It is evident that when i j k are corresponding to the indices along 3 perpendicular axes, then you get the volume. And if they are repeated which means that you get 0. So, it is obvious that when they are repeated it means that the vectors are actually being you know in the same plane and therefore, you will get a 0 there. So, this is a direct application of the permutation matrix from the definition. Another interesting quantity that we came across earlier is this quantity. So, when you expand it you basically have 9 terms.

Because this is nothing but so, $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}a_{1j}a_{2j}\ a_{3k}$. So, if you expand this, and knowing that ϵ has a values like +1 and -1 you could immediately see that this must be something to do with the determinant. And if a were to be a matrix then this gives you the determinant. And this actually allows you to write programmatically how to evaluate the determinant once the matrix ϵ is already available to you. So, these are the simple usages of the permutation matrix. We will see how to use it to actually prove some of the vector identities as we go along.

(Refer Slide Time: 13:46)



So, here are some identities that I am writing. So, this is a practice to check whether we got symbols correctly. So, $\vec{c}= \vec{a}$ X $\vec{b}$. So, I am just repeating the concepts once again. So, that we are practicing the subscript notation. So, here we can see that if I want to give the indices of a and b, then that of c_i and here, that is how it is and you can see that the second index is matching that of a the third index is matching that of b. So, the remaining index which is a free index that becomes index of c.

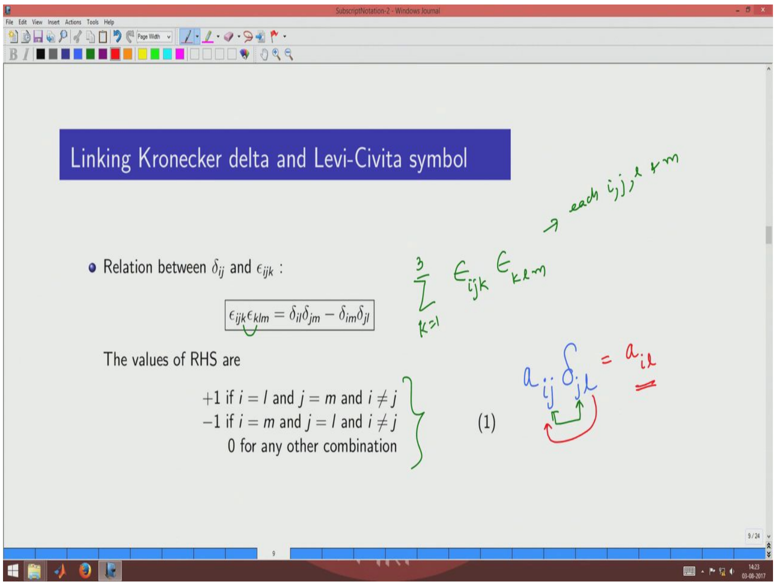
So, similarly we can write it for the curl product here also we have seen that j is matching here, and then the k is matching here, then the remaining index is here. Now whenever there are multiple operators like dot and cross, then we must then combine the rules that we have used for δ as well as the permutation matrix. So, what is the rule for the dot product the indices have to match? So, that is the idea of dot product, and what about the rules for the cross-product ϵ will come into the picture. So, we are going to use that concept. Let us look at the $\nabla X \vec{a}$. So, because it is a cross then I want to use the indices in such a way that I get a i index free. So, I will put a j here and I will put a k here. So, that when I have $\epsilon_{ijk} \frac{\partial}{\partial x_j} a_k$ then I got this term covered.

Now, the free index of this expression, the free index of this expression which is written here, the free index is basically i. So, I must use the same index for c. And that is how I write the dot product as $c_i \epsilon_{ijk} \frac{\partial}{\partial x_j} a_k$. Now once we write like this then we realize that ϵ being just a number can be just brought in and out of the expressions. So, that is why we bring it out and then we can write the expression as follows we can write it as $\epsilon_{ijk} c_i \frac{\partial a_k}{\partial x_j}$. So, you can see that an expression like this here. So, an expression like this can be written in this form.

So, whenever there are multiple operators like there are 2 cross operators, then we basically will require 2 ϵ's, which means that we will require 2 sets of 3 indices, and we have to then judge very carefully whether we will pick them in the correct order. So, let us look at that here. So, the very first symbol I am going to write here is here this one. So, I want to write it as s and t. And I use ϵ_{rst} as the first ϵ . And then you can see that the free index of this part of the expression the free index of this part will be then r.

Now if this was r, then I choose this index of the c to be q. So, that the free index that is coming out for whatever is the output is that p. So, that is why I write here $\epsilon_{pqr} c_q \epsilon_{rst} \frac{\partial}{\partial x_s} a_t$. So, we then use the idea that we could actually move ϵ in and out of the terms. So, we can write the expression in this manner. So, here i just repeat once more you can see that c_q and r c q these symbols if you see how many are repeated and how many are free you could see that q is repeated you could see q here. And here r is repeated, s is repeated, t is repeated. So, the only index that is free is p and which implies that this must be a quantity which a index of p which I may want to call it as $\vec{d}$. So, if I want I could write $\epsilon_{pqr} c_q \epsilon_{rst} \frac{\partial a_t}{\partial x_s} = d_p$. So, then you can see that you can write the expression in this manner now there are 27 values of ϵ . And there are 2 ϵ's sitting side by side. So, when I want to expand this then is really shortcut that is possible yes, there is a shortcut. So, whenever there are 2 epsilons then you can expand them as δ's. And that will be there in a slide form now here.

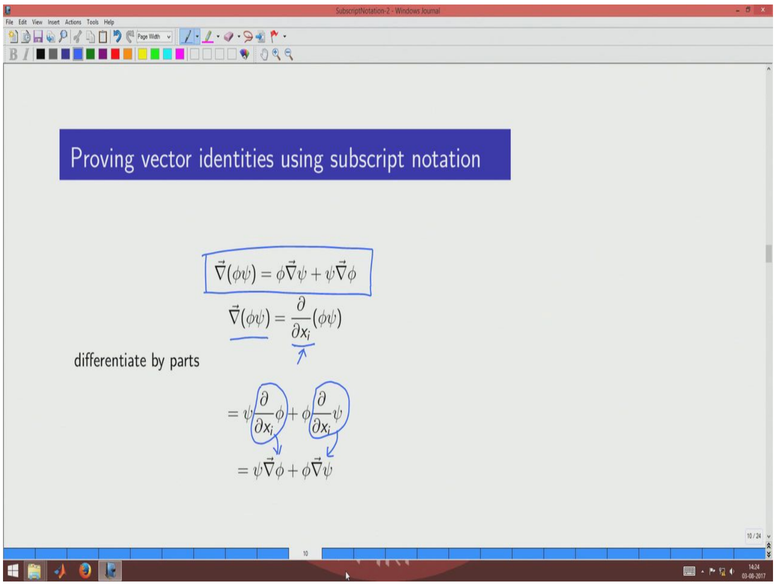
(Refer Slide Time: 18:19)



So, whenever there are 2 ϵ's this is what you do. So, when there are 2 ϵ's with the only one index matching. What we do is that we take the other 2 indices and make them into 2 pairs. And this expression is then an identity by identity. I mean, you it is applicable always and how to verify. So, for each combination of i j and l m you could verify and that is going to be quite tedious, because you have got 81 combinations that are going to come nevertheless you could just pick few of them and then verify whether it is true or not. So, this expression when you want to write you would see on the left-hand side we have got $\epsilon_{ijk}\epsilon_{klm}$. And this is nothing but basically $\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{klm}$. And this is basically for each i, j, l and m.

Then you have got each value of i, j, l m you are summing up 3 pairs of ϵ's and then getting the value. And these are the values that will come out. And this expression is very useful because of the property of δ. We already saw the property of δ. I am just reminding you that property here. Whenever you have got any expression a_{ij}, and then it comes in beside δ with indices that are matching let us see j and l, then we saw that the property of δ is such that whichever index is matching. Then take the other index and then put that in it is place. So, which means that this must be a_{il} do δ is going to be helpful in reducing the number of indices. And then whenever there are 2 ϵ's it will can be converted into δ, which means that while working with this expression even if you have many ϵ terms coming in. This is nothing worry because eventually they will be converted to δ's and which then reduced a number of indices and then we will retrieve the quantities that we want. So, these are going to be used in derivations that I am going to detail now.

(Refer Slide Time: 20:41)

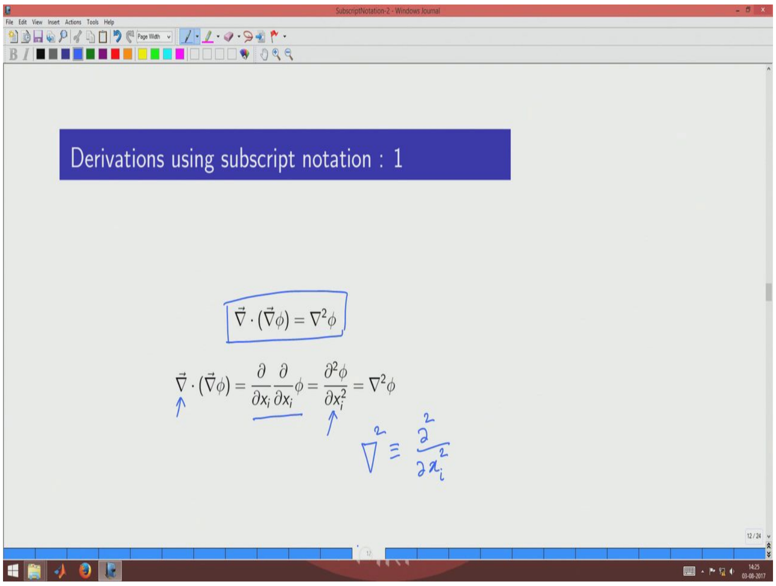


So, we can now practice some derivations of a vector identities to see whether we can use a subscript notation to reduce the amount of algebra. So, as we go along we will see the benefit and here I am starting with a very, very simple example. The first example is essentially to prove that this is true of course, you can visually inspect and say that this must be true, but the way we do it in subscript notation and illustrating here. So, this is essentially to guide you to do the derivations further down. First what we do is we take the left-hand side, and then write it in terms of the subscript notation. And so, the ∇

operator is going to be written in this form. And then once this is over, what we do is we differentiate by parts because this is a differentiation operator.

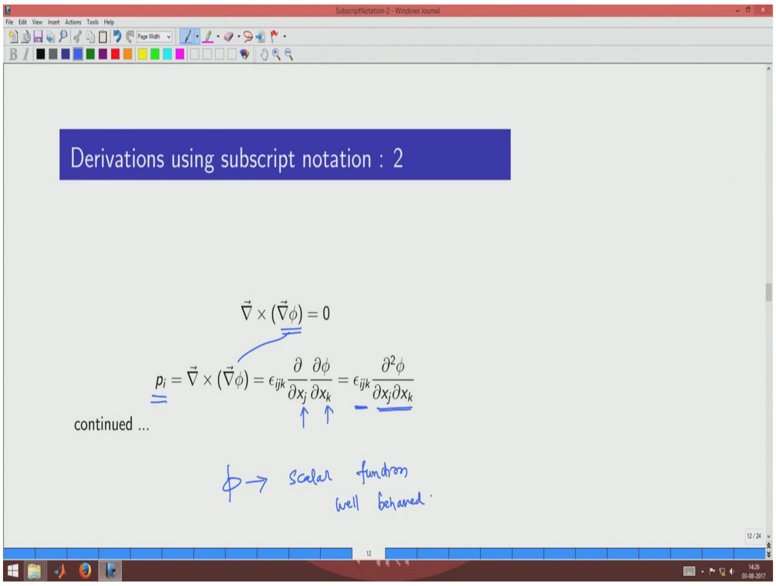
So, when we differentiate it by parts first we can take the ψ out differentiate ϕ, and then take the ϕ out differentiates ψ. And then we realize that we have got terms that we can again bring back into the vectorial notation, and that is what we have done here. So, we can see that in one step we can actually derive this. Of course, there is not much complication in this kind of a derivation, but you can see that the subscript notation can be used to reduce the algebra. If you want to prove this by using different components of the ∇ operator then the algebra would be a lot more.

(Refer Slide Time: 22:05)



So, let us talk with some more operators that will come in as we go along. So, here is a an identity that we must now prove there is nothing but to prove because we are actually going to define operator ∇^2 here. So, what we do is we write the left-hand side in terms of the expansion of the operators ∇, which is nothing but $\frac{\partial}{\partial x_i}$. And then they come twice then we can differentiate twice and that is what is this. And we are now defining this operator ∇^2, we are defining it as this kind of an operator. Once we define then the right-hand side as been arrived at and therefore, the proof is here.

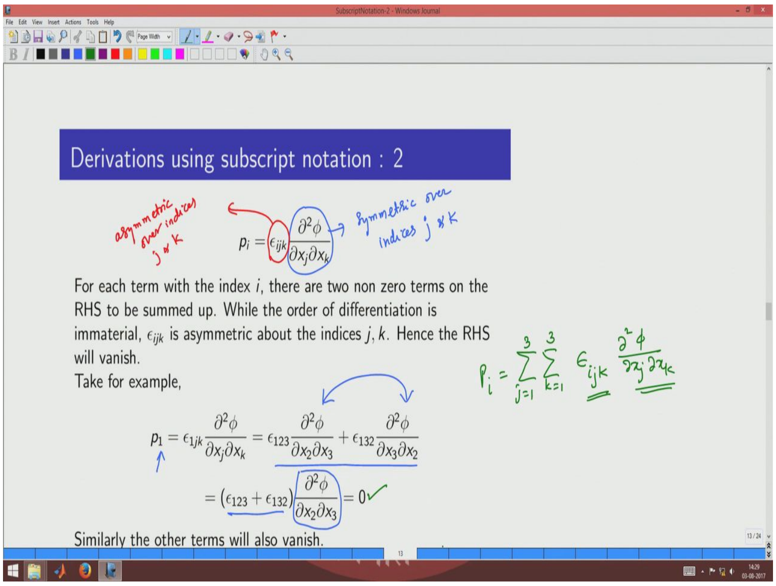
(Refer Slide Time: 22:55)



Now, we are going to use this definitions in some of the derivations later on. So, here is a derivation again to show you that the $\nabla X(\nabla\phi) = 0$. And this can be proved as follows. Whenever you have got something like that you know that it is a vector, because is any scalar function. And then the operator is basically giving you a variant. So, that must be a vector. So, that vector should have some index. So, we are going assign some index, and index that is assigned is k. And we have got one operator here you want use an index for that and we have chosen the index to be j. So, we choose the i j k as the cyclic sequence of indices. So, the remaining index is i. So, that must be what is for the quantity that we are actually expressing in this form p_i, but then what happens is that we saw that you have got a combinations of 2 terms. One is ϵ_{ijk} and other is $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$.

Now, there is a certain property of differentiation, whenever ϕ is well behaved. So, we assume that it is a scalar it is a scalar function that is well behaved.

(Refer Slide Time: 24:16)

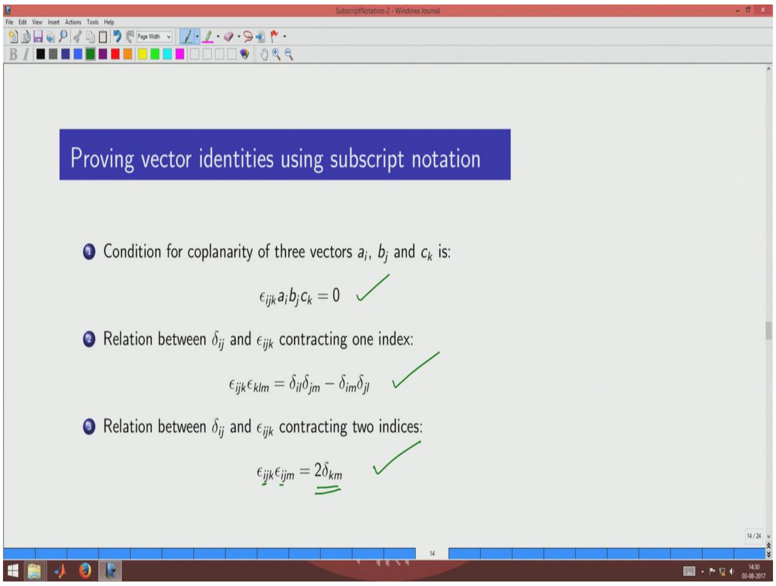


What we mean by well-behaved is, that whenever you change the order of differentiation then there will be no difference in the quantity that is coming out. So, what it means is that when you take for example, the p and look at one of the terms p_1 then we see that then we expand p_i with i = 1. On the right-hand side, you see that there are 2 terms that are coming here. And then you could sum it up and you get $(\epsilon_{123} + \epsilon_{132})$. And I am equating these 2, because I differentiate first with x_3 and then x_2 or first with x_2 and x_3 does not matter. Because we say that the order of differentiation should not matter.

So, we are going to use that to take that as common, and that is what is taken as common here and therefore, we can sum up these 2 ϵ's. And then when we saw these values one is +1 and other is -1. So, there will be get a 0. So, which means that we are actually now discovering something very interesting with respect to the symmetry, the idea is as follows this quantity $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$ is called as symmetric over the indices j and k. Because when we swap the indices j and k the quantity does not change. Similarly, this quantity ϵ_{ijk} is called asymmetric over indices j and k. The reason is that when the values of j and k are swapped, then you get a negative value. So, which means that you are now doing a summation first of all let us see p_i, p_i is actually summation we see that $p_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k}$. So, it actually summation over the indices j and k, and we see that when we are summing we are seen that there is a symmetric part and there is asymmetric part the asymmetric part is going to change the sign and the symmetric part does not change which means that when we sum like here, we will get 0.

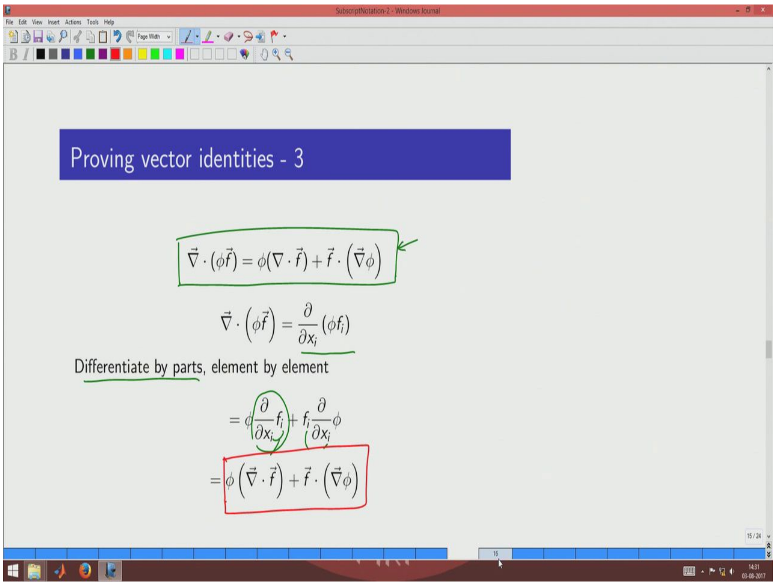
So, we are going to use this property later on in many of the derivations, whenever we see an asymmetric form of a quantity coming in contact with a symmetric form and then when you are going to sum up over the indices over which the symmetry is differing then the sum will be 0. So, we do not need to then expand all these terms to decide on what is the magnitude. Visually we can see whether the symmetry and asymmetry coming together, if they do come together then we will immediately assign that quantity as 0.

(Refer Slide Time: 27:03)



Now, there are some more relationships that I want to just summarize here, that when we are deriving the vector identities using subscript notation, then this will be use of you. So, we are already come across this. So, one is the coplanarity, and another is whenever you have got 2 ϵ's how to convert them into δ. And then we have got also whenever there are 2 contracting indices what will happen. So, you have seen that here i and j are common and k and m are the free indices. So, you could see that you get a quantity that is similar. So, these 2 identities are going to be used in the derivations as we go along.

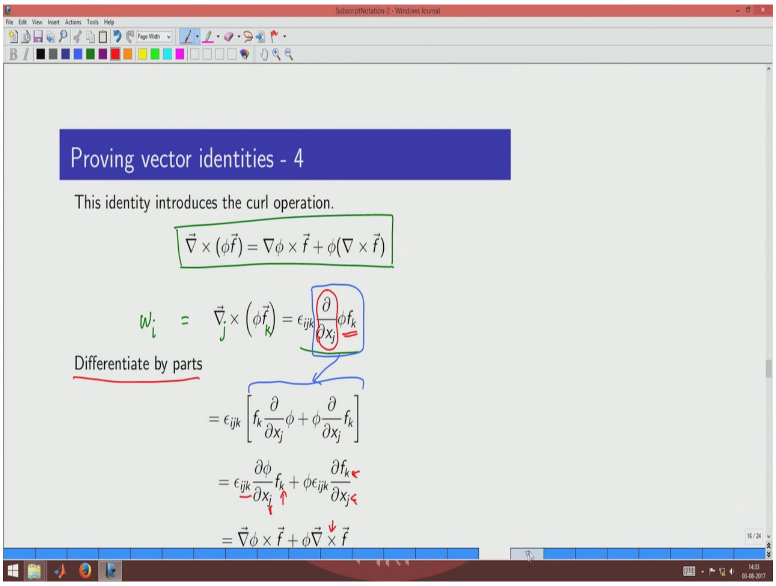
(Refer Slide Time: 27:47)



So, we are going to have about another 5 more derivations. So, we will go through them and I will only highlight the points that we are making use of instead of narrating how the derivation is done. So, this is the quantity that has to be derived using the subscript notation. So, we start with the left-hand side, and we expand it using the subscript notation, and then the differentiation by parts we expand and then write recognizing that when the subscripts are matching, then it must be the dot product and then when you have a scalar quantity with i there then it must be a grad and the subscripts are matching here.

So, that is the reason why we have got a dot product there. So, subscripts are not matching here. So, that must be a dot product here. So, this is how we write the expression given here in terms of the subscript notation, and then using the differentiation by parts we can then immediately see that the outcome is on the right-hand side. So, we could see that the r h s is coming out just by only one step, namely the differentiation by parts. So, there is only one step derivation in this kind of a equations.

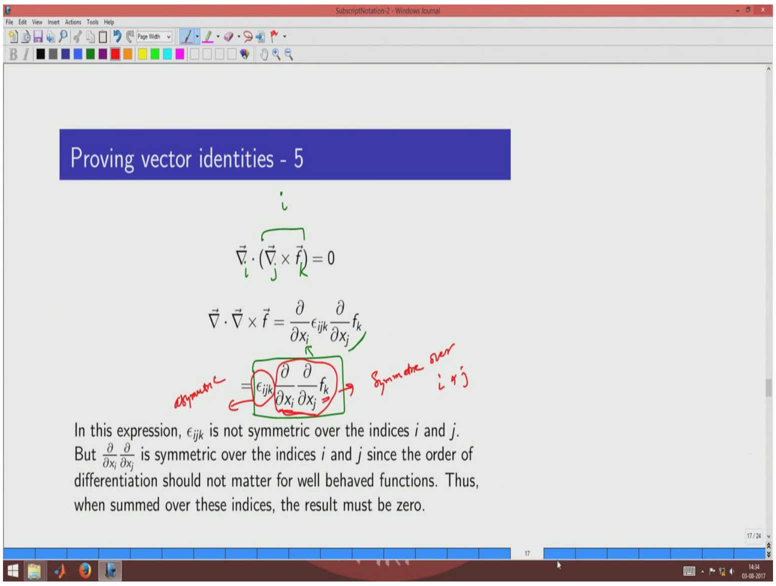
(Refer Slide Time: 29:03)



So, there will be multiple steps in quantities that are of more complexity. So, here we have got an expression that has to be derived using the subscript notations. So, we will start with the left-hand side and we have a curl here and there is a vector here. So, we choose the subscript for f to be k subscript for the first operator to be j. So, that the free subscript is i. So, this is some quantity i and then we can write the expression here. And once you write then we recognize that the differentiation indicated here has to be acted upon 2 terms and therefore, we do it by parts.

So, when we expand then we get this expression here. So, let me indicate this by parts is going to come here. Now we multiply epsilon with each of those and we could take epsilon in and out and that is what we are doing here and the second term. And then we realize that quite straightforward we could see that you have got a j here, you got a_k here, and then there is . So, there must be a cross product, and here also we have got a j here we have got a_k here, there must be a cross product. And we can write the final expression straight away. So, in about 2 steps we are able to derive. So, imagine trying to derive this vector identity using the components of f, components of the ∇ operator. So, that would be a lot of algebra which is now saved because the summation convention is making us save the space.

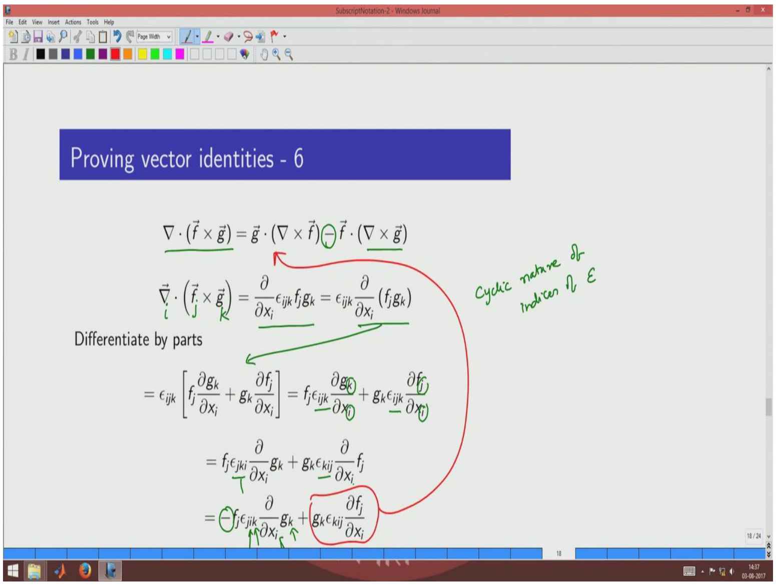
(Refer Slide Time: 30:36)



So, one more derivation, which means basically $\nabla \cdot (\nabla \times \vec{f}) = 0$. And that can be derived using the symmetric property that we discussed earlier as follows. So, we have now the task here you need to decide on the indices. So, I will choose j and k for the indices here. So, what would be the free index that will come out of this quantity that would be the i. And therefore, I should the same index here. So, that is what we do. So, we choose i here and then the j and k i choose here and then I get this term. Now immediately we see that in this term we have got the same symmetry argument i and j are basically acting on f. So, for each component of $\vec{f}$, if each component is well behaved function then the differentiation order does not matter.

So, this part $\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f_k$ is then symmetric over i and j you can see that this part ϵ_{ijk} of course, asymmetric. And then you can immediately see that this must go to 0. So, this is how you could prove that this identity will take you to 0.

(Refer Slide Time: 31:55)

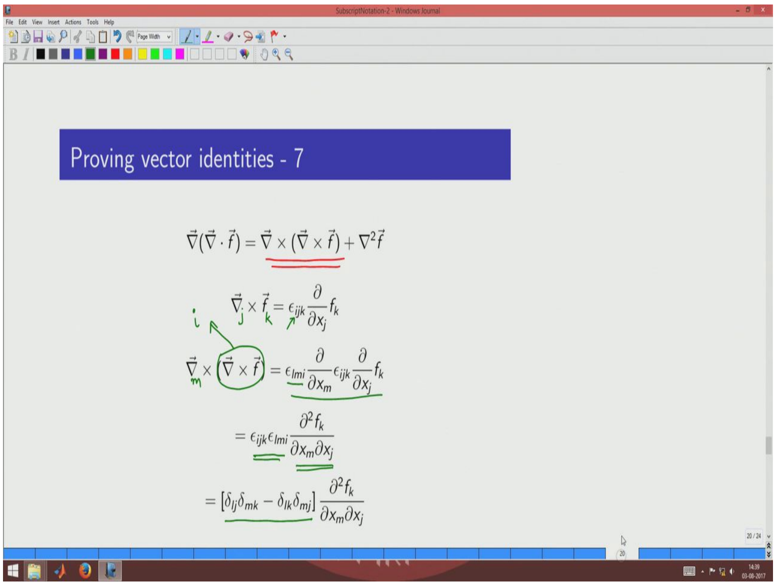


So, one more derivation here, which then is little bit more complicated because we have got more terms, and we can see that this can be also done in a very straightforward fashion, we start with the left-hand side we have see left hand side decide the indices. So, let us decide the subscript for f as j, and the subscript for g as k. And then there is a dot product. So, I must have the same index as what is remaining from ϵ. So, if has i j k then i is remaining. So, that is what in the index I will put here, and then this is what I get. And then I take the ϵ out and then I have then a differentiation by parts coming up and that is why we have got 2 terms here, and then we multiply it term by term. And then here there is some small manipulation I am doing which is basically the idea about the cyclic nature of indices of ϵ , which should not change the quantity.

So, what I am doing here? So, what I am doing here is I will highlight here. You can see that I change the order of indices. Why we are change the reason is that the indices are here k and I. So, I must have I as a second index and k as a third index. Similarly, here i and j. So, I must have i as second index and j as a third index. So, that is the reason why I am cycling the indices. So, have cycle that is k i j, and then it matches quite well you know the second index is i and it is matching here the third index is j it is matching here but when I try to do the cyclic rotation of the indices. So, for the first term then you can see that the sequence is not matching, which means that this quantity should be negative of whatever is cross product. And that is the reason why from here to here when we go we introduce a '-' sign. So, (j k i) (j i k) i am swapping the last 2 indices and therefore, I put a '-' sign. The moment I do swapping of indices, then I see that then I see that the second index is i, third index is k second index is i third index is k. So, then I can write this quantity as $\nabla \times \vec{g}$, and that is what exactly what is then here $\nabla \times \vec{g}$.

So, minus sign is coming because I did the swapping of last 2 index and that is the minus sign that is coming here. And the first term is shown here, and you can see that this term is matching with this term, because you could see that k i j $\frac{\partial f_j}{\partial x_i}$ is nothing but and then k is the free index and you are dotting with g there because the k is matching there and therefore, it is g dot that. So, you can see clearly that proving this vector identity has involved 2 concepts. One concept is the differentiation by parts, and another concept is the cyclic nature of the indices of ϵ. So, these 2 are basically conventions of the summation convention and therefore, by using the subscript notation we have made the differentiation very simple and in about 2 or 3 steps we have got this identity proven.

(Refer Slide Time: 35:09)

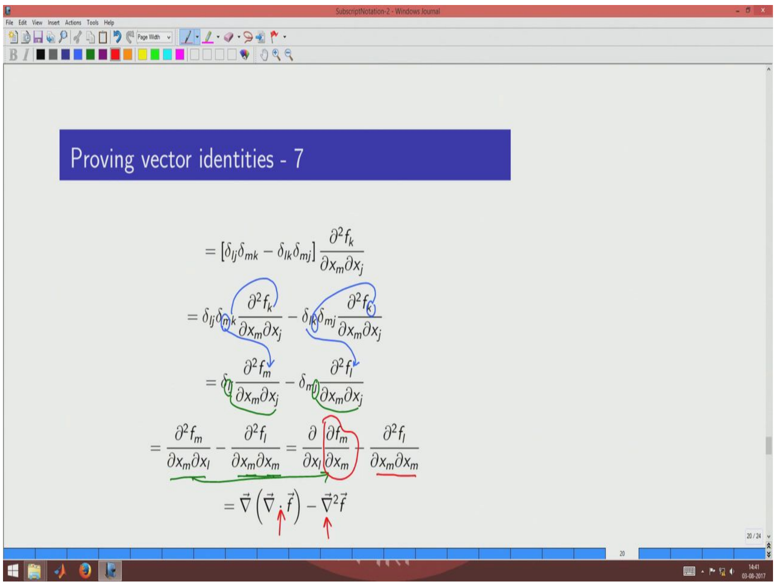


One more such identity this is a bit long one here, we are going to make use of the identity that involves 2 ϵ's. So, this is the identity that we are supposed to prove. So, what we do is we do not start with left hand side and go to the right-hand side, what we do is that we start with this term, and see if we can get the remaining 2 on the other side.

So, $\nabla x(\nabla x f)$ is going to be written. So, first we write $\nabla x f$, and here we write here with the indices as follows, we take the index for f as k the index for the ∇ operator as j. So, the remain index is i. So, i is coming up as a free index. So, for this quantity for this quantity the free index is i. So, then I must then have another 3 sets of indices in such a way that they are also cyclic.

So, I choose them as l m, and therefore, then I could then choose this as i. So, then here I would take m and therefore, I could put an ϵ_{lmi} here. So, the m corresponds to the operator here the i corresponds to the remaining index here. And then I have got this term written. So, once you have written here then we can take the ϵ out and therefore, we have got the ϵ out here, and the ∂ operators are coming together here. Now once we have got 2 ϵ's. We have already had an identity that will convert that into 2 δ's. So, that is what is expanded here. So, what is the common index i? i is the common index between the 2 ϵ's. So, j k l m are going to be used here to expand that, and the other term is left as it is.

(Refer Slide Time: 36:48)



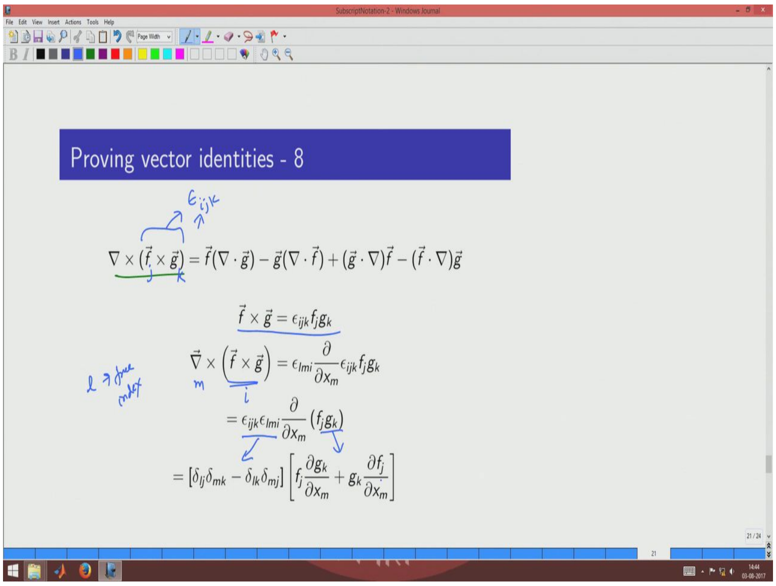
So, in the next step we can then see how it has been simplified. The way it is simplified as follows we then basically multiply each of these terms and then separate it out. So, here we are going to use the property of δ in reducing the number of indices. So, we could see that how we are doing it. So, you could see that the first we see that the index k is matching. So, the m should go there under f, that goes here similarly here. So, k is matching. So, this other third index has to come go here.

So, this is how we have reduced the number of δ's from 2 to 1, because we have used one of the δ's to change the index. Now once we have change then we can use the other δ again to change further. So, that we have done here. You could see that the j is matching. So, we bring the l into that and similarly here also you could see that j is matching you could bring the m there. Now that is you got 2 m's as subscripts here because one of the js has been substituted with m because that is the property of δ.

Similarly, here $\frac{\partial^2 fm}{\partial x_m x_l}$ because j has been removed and l has come there. Now we have got these 2 terms which can then be expanded further. And it very easy to see that from here to here, what we have done is we have swapped the 2 terms in the denominator ∂x_m and ∂x_l. We have swapped the reason why we swap is because of the property that the order of the differentiation should not matter. So, once we swap the indices, then we recognize that the term that is coming out is with the same index and that is nothing but the dot product. So, whenever we have got the same index coming in the denominator numerator that is nothing but the divergence operator that we have done earlier. So, that is what is going to be used here.

And similarly, the other one will nothing but denominator is coming twice. So, we have already defined this as the operator ∇^2 and that is why we are going to be used. So, this is how we have then found that the derivation here, we have got this term and this term coming out and this has been proven here within just 3, 4 steps. So, what are the concepts that we have used in this derivation we have used the concept of what happens when there are 2 ϵ's, how that it will be used to convert into the δ's, and how a δ can be used to replace subscripts, and how the order of differentiation can be used to swap the indices. And then how we can then sum it up with the operators that we have defined earlier.

(Refer Slide Time: 39:43)

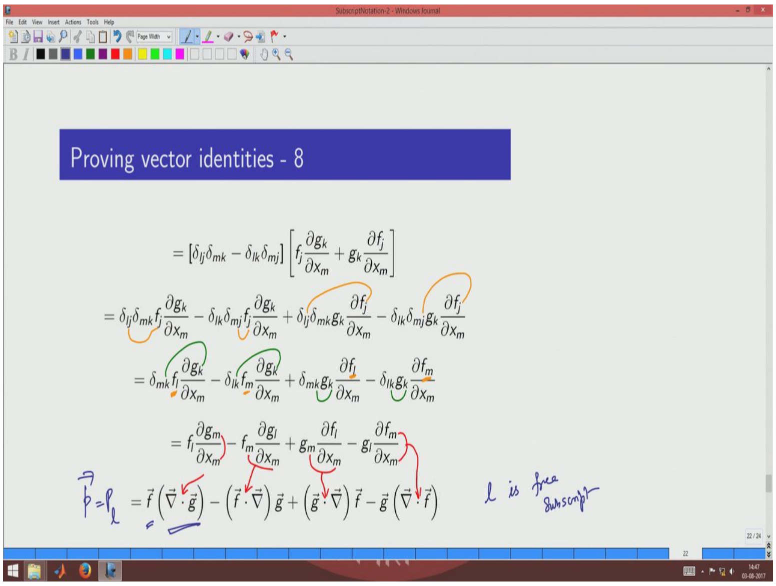


So, the vector identity derivation can be quite simple using the subscript notation, one of the last of this few derivations that I am illustrating to you here. So, this is the last derivation it appears quite formidable because if we attempt this kind of a long expression using the various components of f and g as well as that of the nabla operator, then it is going to be a (Refer Time: 40:03) task keeping track of all those terms there are a lot of terms there are going to come in. And we may make algebraic mistakes as we go along deriving, but when we choose the subscript notation we can see that the amount of algebra is quite limited and therefore, we can do quite correctly. So, I want to illustrate that here now.

So, let us take the left-hand side, and choose to have the indices as we want. So, we choose them as j and k. So, f g and g k there is an is that ϵ_{ijk} . So, this term I want to use an ϵ. So, that the i part is then coming out free. So, if the i part is coming out free. Then I can then choose that as one of the indices for the next ∇ operator. So, I would then choose it here. So, this is the first thing that we have written i is the free index. So, for this term i is a free index. So, then I want to use another one. So, I could then use this follow as m and then l will become the free index. So, l is the free index that is taking out. So, we have got l m i and i j k. So, i is the repeated index. So, we can then convert this term into the 2 terms that we have done just now, and we also have the $\frac{\partial}{\partial x_m}$ acting on

2 quantities. So, we have got the differentiation by parts that is coming here. So, we have got 2 terms on the left 2 terms on the right. So, when we multiply we must get totally 4 terms. So, they are the 4 terms that are written here.

(Refer Slide Time: 41:36)



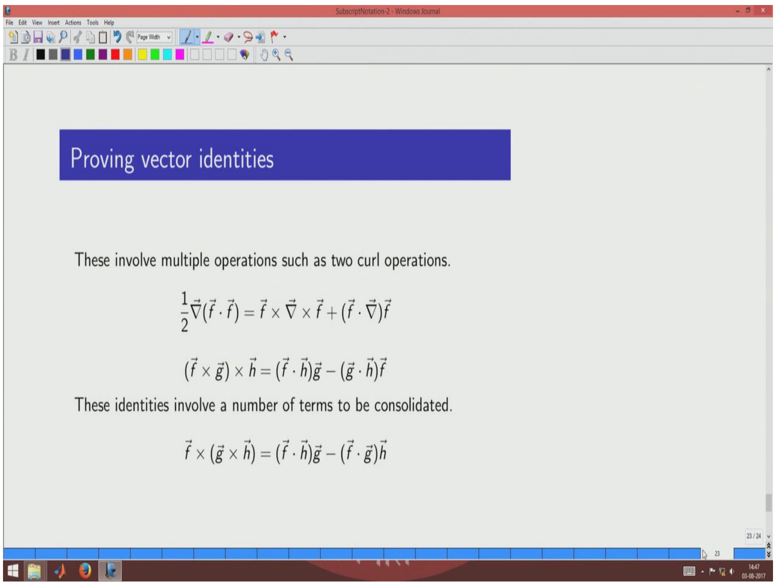
So, there are 4 terms here, and we are just blindly expanding the terms and then we will see how we can make use of the properties of δ to get rid of the indices. So, I am now illustrating that here. So, first let us see how the j is knocked off. So, I comes there, and then here j is knocked it off. And so, where j is there I want to just knock it off. So, the other index of the δ containing the j will be used to replace. So, the δ that is used here is with l. So, that is a index that will come for f and here it is m. So, that will be coming here and here it is the l. So, that is becoming here and here it is m.

So, that is coming here. So, I have reduced now in each term instead of 2 δ, now I have only one δ. And then I have reduced the indices and the indices have now modified. So, I do not have any more the j index in my expression. So now, we need to simplify further and we use the same principle of δ and simplify further. So, let us see how that is done. So now, you can see that we can now use the k to be knocked off. So, you could then see wherever k is matching. So, look at the k is matching and then knock that out. So, you could then see that this implied that the m is a remaining index that is coming here. And here it is l that is coming here, here it is m that is coming here and here that is l that is coming here. So, remaining things are untouched. So, you could see that we have successfully removed all the δ's and modified the indices accordingly.

Now, it is a matter of just recognizing the expression that we got in terms of the subscript notation. So, we could see that now let me erase this and then show you the terms. So, you see that you have got the m matching both ways. So, which means that it must be a dot product and you see that there is a m matching here it must be the dot product here, and then again here and here. So, whenever you have got the subscripts matching then you have got the dots that is coming in. And then whatever is a free subscript is basically the vector. And in our case what is a free subscript it is l. So, that would become the vector that is actually being mentioned here.

So, every of this quantities if you notice we have got a dot product here, and that is a scalar and then multiplied with a vector. So, it is a vector that is coming out. So, the entire expression can then be chosen if you if you like to call that as p then that vector p is basically p l, and then written in this kind of a form. So, you could see that such a long expression with just few steps. Differentiation by parts using the δ's and the properties of epsilons we are able to now derive.

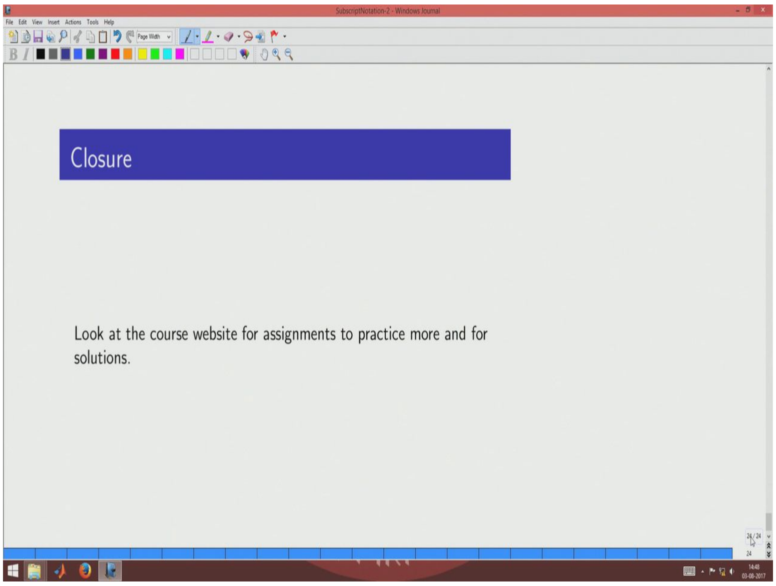
(Refer Slide Time: 44:59)



So, here and there some more such derivation that I would not do it in the slide projection, but I will leave it to you to do it for your own practice these do not involve any more concepts than what has been talked about till now. So, in other words you should be able to do it yourself. And in the course of sight we will put up the detail solutions. So, you can verify the way you have derived with the solution that is provided to check whether you have got it right.

You must be able to finish each of this within a one page of the subscript notation derivations. And for you to see the value of this notation, you may try a part of the derivation using the vector components. I am sure that very soon you will realize that subscript notation is a way better way to derive and then practice the notations so that we can make the algebra much smaller. So, with this we have had enough of practice of subscript notations. So, the idea is to use this in many of the expressions that we are going to derive later on in transport phenomena. So, we will wind up this subscript notation session now.

(Refer Slide Time: 46:02)



So, look at the website for the assignment for practicing more, and for the solutions of the 3 exercises that have been given here.