Transport Phenomena in Materials
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Lecture – 19
Heat conduction cases – Transient state

Welcome to the session on heat conduction. We take up cases of a transient state in 1D in this session. This is part of the NPTEL MOOC on transport phenomena in materials.

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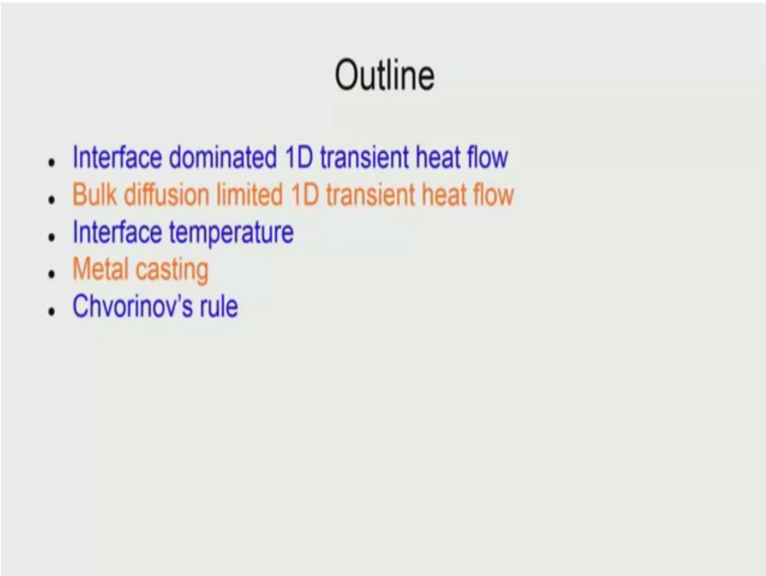

So, the outline for this session is as follows. We take up two different situations of 1D transient heat flow. We take the two extremes; one extreme being the interface dominated heat transfer, the other extreme being the bulk diffusion limited heat transfer and then, we will use these learnings to detect what would be the interface temperature when, two metals at different temperature are brought in contact and in transient state we will be able to arrive at some correlations, that will be useful for us in we what kind of combinations are good for metal casting and then, we will go ahead and use these expressions to derive what is called the Chvorinov's rule, which helps in the casting designs.

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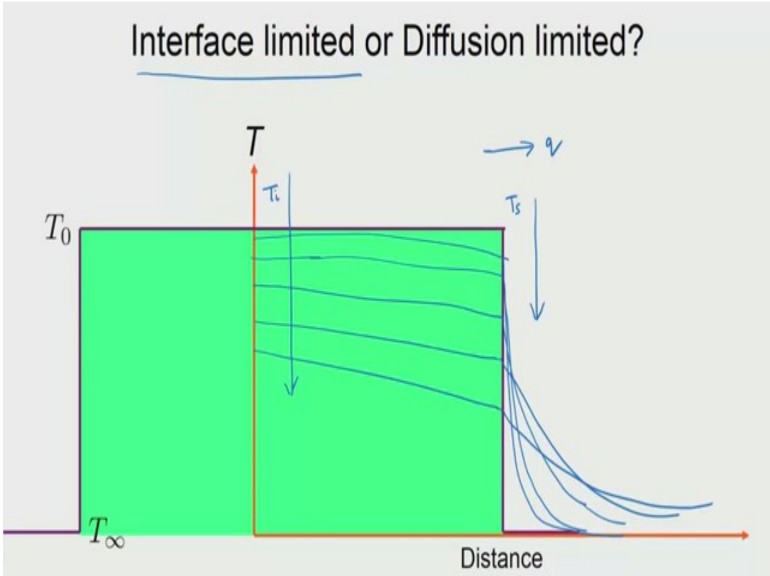

So, here is a situation where the body, which is shown in green, is the domain, also for us and the surrounding environment is at a lower temperature. So, the body is at high temperature T_0 and the environment is at lower temperature T_∞ and we have this interface, which is experiencing a change in temperature from high value to low value. Now, how would the temperature profile change from interior of this body to the air outside as a function of time?

So, we can have very different profiles, depending upon the heat transfer being dictated what is happening at the interface or what is happening in the bulk. So, let us take situations as follows. Let us say that, this body happens to be, let us say a metal and the body is also small. So, what would happen is that, the temperature profile would look like this. I am just drawing schematically. So, you could have situation like this and as time proceeds, we would see that a temperature drops and we start seeing that, we would have profiles going like that.

You could have temperature dropping both on the surface as well as the interior and if this is a situation, what it implies is that the surface temperature is dropping roughly, at the same way as the interior temperature is dropping and the gradient inside the bulk is very small and this is a situation where the material is made up of metal, which is a good conductor and you can now see that, the rate at which the heat is lost from the body is limited by how the temperature is dropping at the interface and how the heat is actually taken away at the interface.

So, this becomes a limiting factor. So which means that this kind of a heat transfer is called as the interface limited heat transfer. Now, we can take the other extreme and see what would happen. So, in this situation, you have basically a ceramic body for example, and the ambient air is at low temperature.

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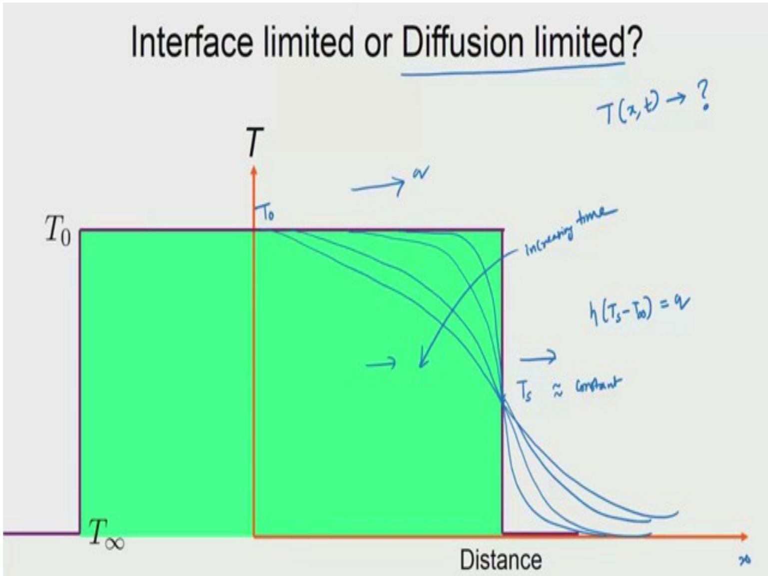

And the surface of the body very quickly cools to a temperature, but the interior is still at the T_o and what we expect the temperature profiles would be looking like this. So with time, we see that the temperature inside the bulk is still evolving. So, you could see that the T_s is almost constant in the duration of time that we are looking at the analysis and the heat is being lost mainly dictated by the bulk. So, the heat transfer is now diffusion limited, because as soon as heat arrives at the interface, it is able to go away into the ambient air by the heat transfer coefficient. Here, you have got T_s - T_∞, this is readily available; however, the temperature profile inside is not fully evolved and very interior at the center of the body; you see that the temperature is still T_o ; it has not come down at all.

So, this is a situation where, for example, you have a non conducting body like a ceramic and which is exposed to air and then, the temperature is supposed to drop. So, this is the other extreme and in reality, you would have situation that is in between these two; however, we can look at the extremes and see how the temperature profile would look like as a function of time. So, here for example, this is increasing time and this is the distance let us say x. So, we are interested in for example, T as a function of x , t. So, how that would look like. So, we take one dimensional case and see how the transient heat transfer would take place. So, we will take one after other these two cases.

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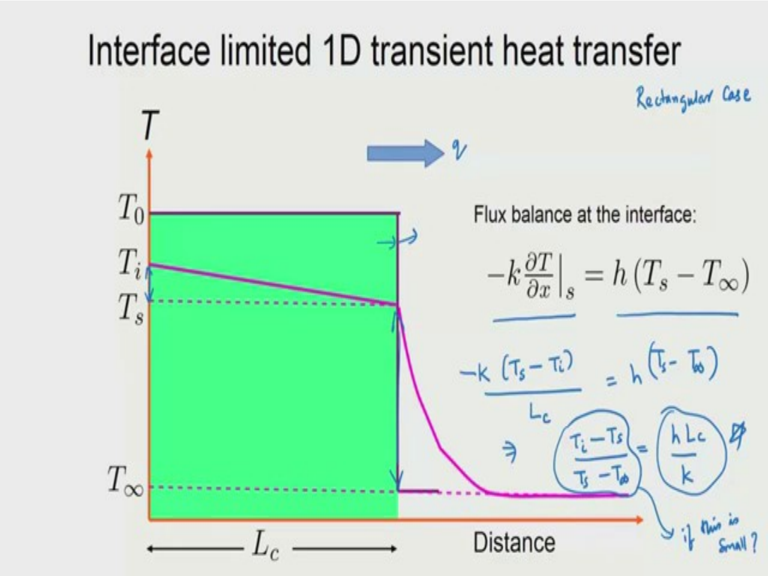

So, let us take the first interface limited 1D heat transfer. So, as we have discussed the temperature gradients are very small inside the bulk. So, that is why we see that T_i - T_s a small number and therefore, the inside the bulk you have got a small variant and the heat loss is in this direction and you could see that, whatever heat flux is arriving at the interface should leave. So, what comes here should leave. So, they both must be equal, but a flux balances because there is nothing special that is happening at the interface.

So, what heat flux comes up to the interface is given by the Fourier heat conduction equation and then, what is lost is given by the Newton's law and that is given here and the cross section area is canceled out on both sides, because we are taking the rectangular case. So, we also mentioned that, we are taking the rectangular case and we now then, approximate $\frac{dT}{dx}$ to be just the differences of the temperature, because these gradients are anyway small. So, you can make this approximation. So, - k × T temperature on the right hand side that is, T_s - T on the left hand side T_i divided by the distance between them L_c = h × T_s - T_∞.

So, we just take them, minus sign inside and then we write (T_i - T_s)/(T_s - T_∞) is then given by hL_c/k. Now, if you took, look the left hand side, this is basically telling you how the temperature differences inside the bulk are related to the temperature differences outside. So, outside temperature differences are basically here given by this and inter differences are given by this.

So, what happens if this is small? So, what does it imply? It implies that basically the temperature gradients inside the bulk can be neglected, which means that this quantity could then be of some importance and you could see from the left hand side that, there are no units which means that, on the right hand side also there are no units. So, hL_c/k will then take some importance and we will see how we can use that as a non - dimensional number.

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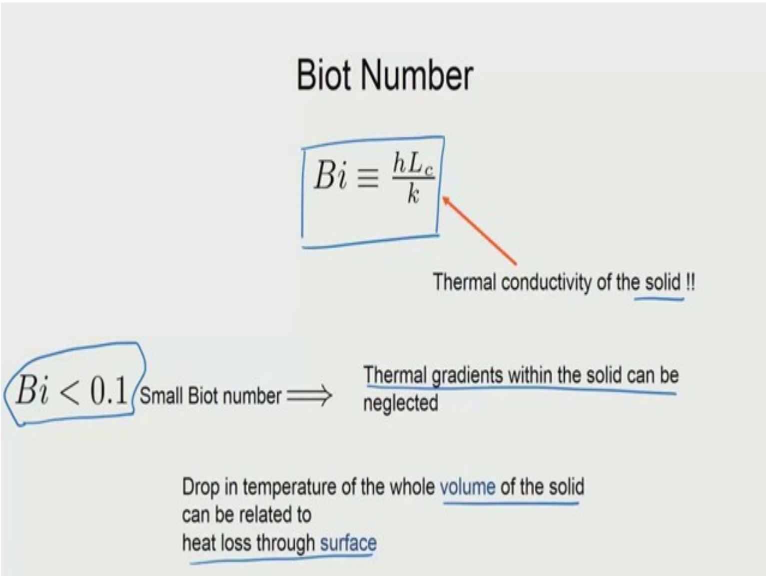

So, that number has a name and that is called as a Biot number or Biot number as you would like to say. So, Biot number is given as $(h \times L_c)/k$, which basically tells you whether the temperature profiles within the bulk are having flat gradients or not. So, in situations where Biot number is very much small less than 0.1, we can assume that inside the bulk there are no gradients and in situations their Biot number is very large, it means that the gradients are very steep.

And we have to alert here, the k that is here, you can already see from the derivation that the k is coming from here, which is basically the flux that is coming from inside, which means the k must be that of this solid. So, we must always remember that the k which is coming in this situation, is of the solid reason I am mentioning is because, there is another non - dimensional number called the Nusselt number, where the expression looks identical, just the k belongs to that of the liquid in this because the expressions are same; we should not get confused.

So, here is the situation we are saying that, in the case of Biot number being small, we neglect the temperature gradients. So, when we neglect the temperature gradients, we can make on approximation. The approximation is, that the way temperature of the entire volume of the body is dropping is related to the heat loss to the entire surface and this means that we do not want to look at the temperature variations within the body and we want to lump the entire heat capacity of the body into one entity.

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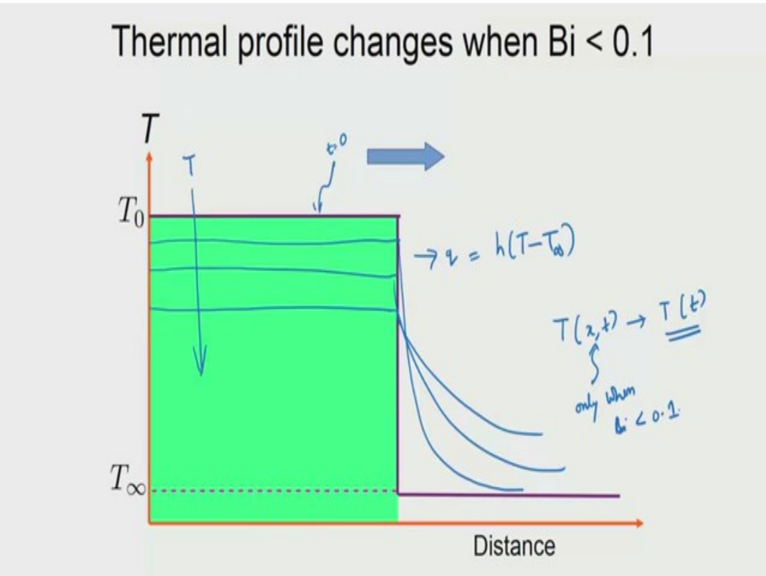

And that is a reason why we take up this approach and which means that, with time how are the temperature profiles looking like. So, there is no gradient inside the body, we have just now made this approximation which means as time proceeds. So, this first curve is basically at time is equal to 0. So, at the next instant for example, the temperature profile would look like that and then, you would have this way so on. So, which means that inside the bulk, the temperature is dropping and outside the dropping temperature is then matched with the heat, that is lost and that is of course, given by h × T which is the temperature inside, which is same as the surface T_s - T_∞ .T_∞ is given. So, this makes it little simple ok?

So, this kind of an approach is what we are going to see, to look at how the temperature will be available which means that, we no longer are looking at this. We actually look at this only because, we see that this is not important and this is only when the Biot number is very small. So, let us look at that extreme case and see what happens.

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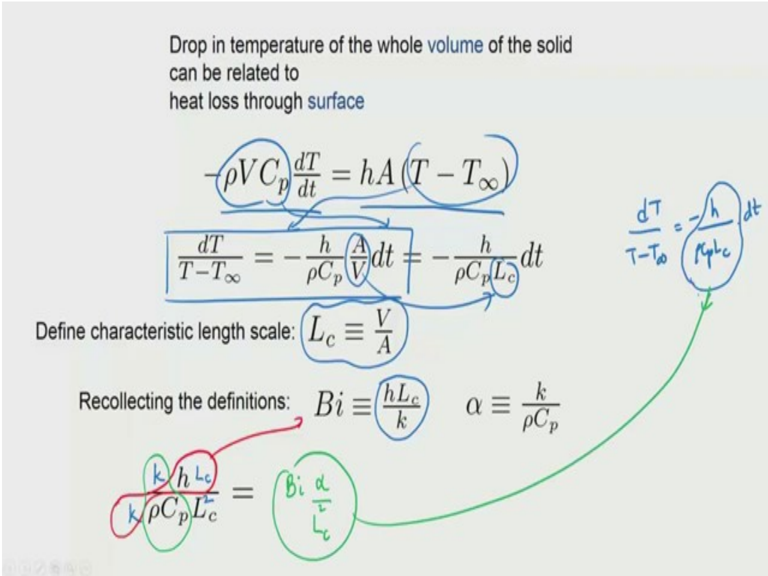

So, if that is the situation then what we then, say that drop in temperature of the entire volume of the solid which is being given here. So, ρ V is MCp × dT and then, we say that the rate at which the heat is being lost, that will be the time derivative below is equal to then whatever is lost in the surface. So, the total surface area is A into the heat flux from the surface is h × (T - T_∞). So, heat that is actually lost because of the drop in temperature is equal to what is lost through the interface. So, this approach will then give us an expression and what we do is that, we take this quantity and take it to the other side and this quantity, we bring it here and then that is when we got this expression ok?

Now, once we got this expression, we are then defining a characteristic length scale L_c. So, this length scale characteristic is very simple, just a length in the case of q, but when you look at cylindrical or spherical cases, then there will be some multiplicative factors that will be coming in. So, watch out else is always to be evaluated by looking at the volume and the area and once you introduce that, then this goes to the denominator here and so we have got the expression on the right hand side; giving you $\frac{dT}{T-T_\infty}=\frac{-h}{\rho Cp L_c}dt$. So, that we could integrate then.

So, what we now do is that, we look at only this expression and see whether we can make some manipulations and we do that remembering that, we have hL_c/k having a meaning namely, we want to use the term Biot number there. So, what we do is that we see we have $\frac{h}{\rho C_p L_c}$. So, we have to have hL_c. So, what we do is that, we multiply and divide with L_c. So, we got hL_c in the numerator. We need the k in the denominator. So, what we do is that, numerator and denominator we multiply with k. So, we have not done any manipulation, we just simply multiplied and divided with k as well as L_c.

So, once we do that then, we recognize that there are some terms that we could gather. So, we see that hL_c/k. So, this has come out and we also see that, one more expression is coming out $k/(\rho C_p)$. So, that has come out as α. So, you could see that Biot number is there, α is there. So, first is Biot number and then, you have got α there and then by L_c^2. So, which means that this expression can then be replaced with this expression ok?

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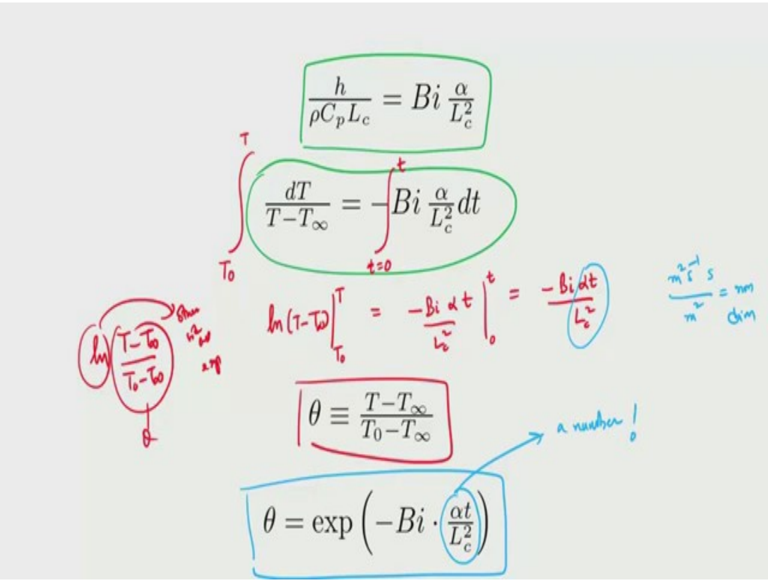

So, we do that. So, we have done that here ok? So, we have then $Bi \times \alpha/L_c^2$ being used for $h/\rho C_p \times L_c$ and then, we look at the governing equation here. This itself is written here. Now, what we do is that, we integrate this guy going from T_o at time t = 0 to temperature T at time t. So, which immediately sees that when left hand side will be log($T - T_\infty$), evaluated between T_0 and T and this will be $- Bi\ \alpha t / L_c$ evaluated between 0 and t. So, that of course, becomes nothing, but

$\frac{-Bi\,\alpha t}{L_c^2}$ and on the left hand side, you have a ratios and then the ratios, we can flip them to have the way we want.

So, we see that on the left hand side we have got log(T - T_∞) / (T_0 - T_∞). So, the logarithm can then be taken to the other side as exponential. So, that is what we have done here and we have defined this entire thing as θ. So, that is what is defined here as θ, which means our solution is now looking quite nice. It looks like this:

$$\theta = \exp\left(-Bi \cdot \frac{\alpha t}{L_c^2}\right)$$

Now, you could also see from here that α T has the units of meter square second, inverse T has units of seconds. You will see a square has units of meter square. So, this is non - dimensional. So, this means that this quantity could also be a number.

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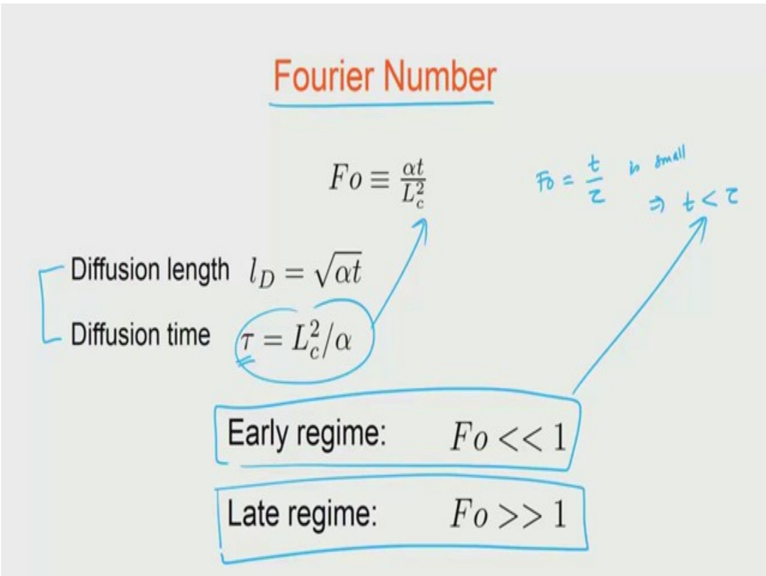

So, we need to find a name for that and that is what we call as a Fourier number. So, Fourier number is defined as α t/L_c^2. Now, the reason why we want to define this way is because the non - dimensional numbers allow us to expand the scope of these results, to multiple situations the concept of diffusion and then, the coupling with temp, time and length are given in this sense here.

So, there is something called diffusion length, which means that for the amount of time t how far the thermal diffusion would take place and if thermal diffusivity is α, the $\sqrt{\alpha t}$ would give you basically the length over which the thermal effects are taking place. That is the diffusion length. You could also think of diffusion time which is $L_c^2/¿\alpha$, which means if the domain is L_c, then how long it takes for the thermal diffusion to affect the entire domain. So, that also can be done.

Which means that if you now use this concept and substitute what happens is that Fourier number can be thought of as T/τ , which means that if this is small, it implies at the time that you gave was very small compared to τ , which means that it is in a very early stage of thermal diffusion. The time is not enough for the diffusion to take place in the entire domain with a size L_c which means A that must be basically the early regime. So, that is what we say.

Similarly, we can also argue what would be called as a late regime Fourier number greater than one, means the time available is much more than τ , which is the time required for diffusion of heat in the entire domain of size L_c, which means that the late regime can be talked when Fourier number is greater than one. So, which means that this quantity which we just now, cooked up as a non - dimensional number, Fourier number is actually having a meaning. If the number is actually small, it means that we are in the early stages of the thermal diffusion taking place and if the number is large, it means in the late stage of thermal diffusion taking place ok?

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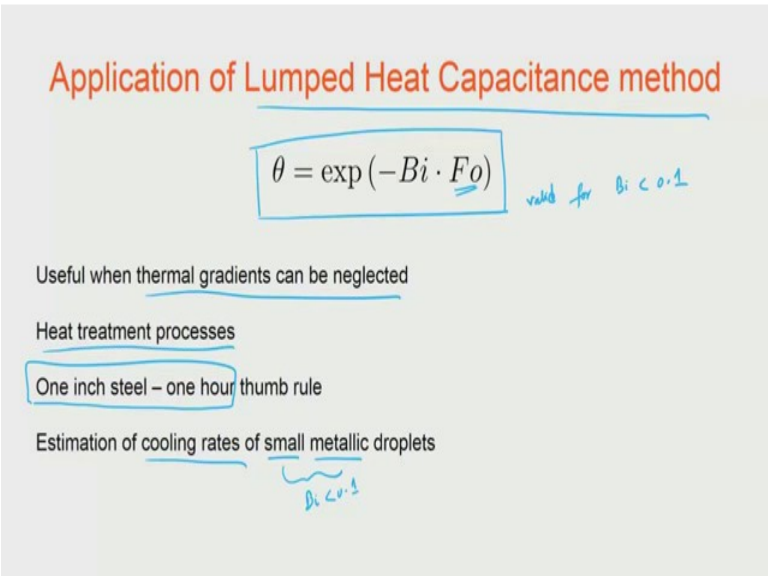

So, now our solution can then be written with the non - dimensional number used in the solution expression and; that means, that the expression is coming out quite elegant, it means that the scaled temperature $\theta = \exp(-Bi \times Fo)$ and this is referred to in the method as Lumped Heat Capacity Method, because we are lumping the entire heat capacity of the body as having just one temperature and seeing how the temperature is dropping, because of the heat transfer from the surface.

Now, this is valid. We already saw that it is for $Bi < 0.1$. Now, what are the uses of this heat capacities method lumped heat capacitance method. So, we can use this expression whenever the temperature gradients can be neglected or when you want to neglect or when you are unable to estimate. So, first order estimates can be obtained immediately by using this kind of an expression. So, if you take for example, a gear and you want to see how long it takes for it to cool down. So, straight away you can substitute this expression and get one number or the duration time duration, which is available in the expression for Fourier number and that will give you first order estimate of how long you must wait when you heat it a gear and then, drop it in water for it to cool down etcetera ok?

Now, heat treatment processes normally, we need to estimate how long it takes for us to heat up and cool down the materials because that is the time that we need to give which cannot be counted in the isothermal holding. So, depending on the size these durations can be significant. So, we can also estimate those durations using the Lumped Heat Capacitance Method and in situations where those durations are large, it means that the sample is too large or the thermal conductivity is too poor so we must have this transient effects can also be taken into account.

And when we do these kind of a calculations, substitute the properties of steel, then we would see that for a heat treatment temperature like about 75^0C and to cool to a temperature of easy handling like 50 ^{0}C , a one inch thick steel would take about an hour to cool down when the heat transfer coefficient with air is about 5 SI units. So, this actually leads to a thumb rule that look at the characteristic length scale of your sample being heat treated and in terms of inches, then you just simply multiply the number of inches and you will get the time in number of hours for you to wait for the sample to cool down before you can touch or handle that particular material. So, this is the thumb rule. So, directly you can estimate from this expression.

And sometimes we have situations like in spray forming, where droplets of metal are solidifying in their path through argon gas, for example, and they are cooling down to room temperature and in such situations also you can estimate the cooling rates because they are first of all small and metallic. So, these both together imply that most probably Biot number is small and in such situations, you can directly estimate what with the cooling rate. So, these are the kind of typical examples in metallurgy, how we would use lumped heat capacitance method.

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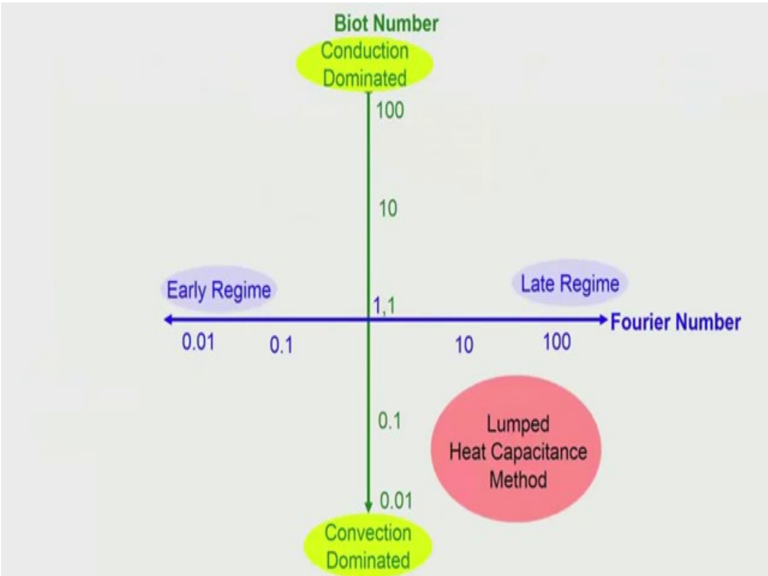

So, we see that regime, that we have to apply this method is when the Biot number is small. So, Biot number is small is on the bottom side of the axis and it is also in situations where the thermal diffusion is actually getting established. So, we are in this quadrant. So, we must estimate whenever we get the answers out, evaluate the Fourier number and Biot number and ensure that we are in this domain. So, if the Biot number happens to be large, or if the Fourier number happens to be very small, then this expression may not be valid. So, ensure the validity by looking which quadrant of the map between Biot number and Fourier number the following are in.

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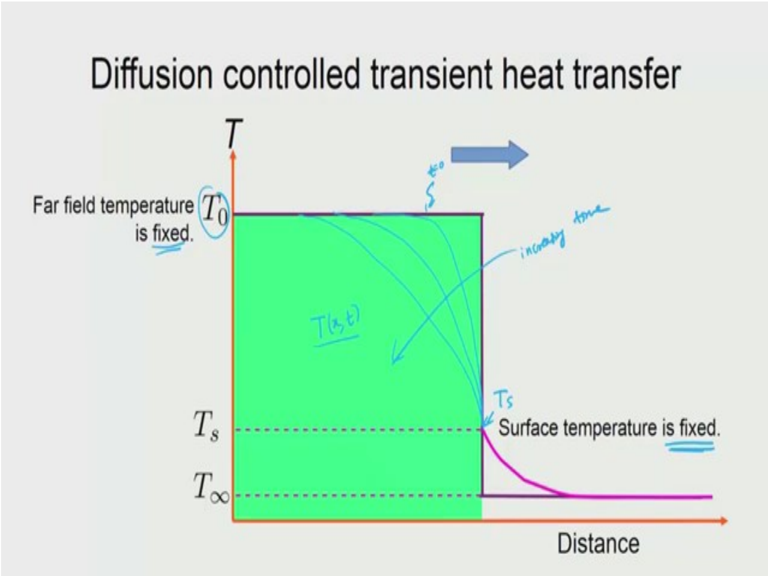

So, now let us move on to the second case, where the heat transfer from a hot body is dictated by a transient heat transfer limited by the bulk diffusion. So, which means that at T = 0, the profile is given here, but at higher times it has to be given by a plot, that I draw here. So, it is evolving very slowly. So, that interior temperature is still; not come down from T_0. So, this is the increasing time.

So, if this is the situation, then how would we go about evaluating and we then see that, the temperature is varying as a function of distance within the bulk. So, definitely we are interested in T as a function of x and it is also changing with the time. So, we are interested in T as a function of x and time t. So, in this situation we want to restrict it. So that we can get some solution out. We want to say that the surface temperature T_s here is fixed. So, it is important because, that allows us to make some approximations and look at the solutions.

So, we want to say that, we are looking at a regime, where the interior temperature has not dropped below T_0 yet. So, that is also fixed and the surface temperature has become stabilized at T_s value and the temperature variation is happening between these two values, that are pegged and then, these curves that we have drawn, are what we are interested in a function of x and t. So, that is possible.

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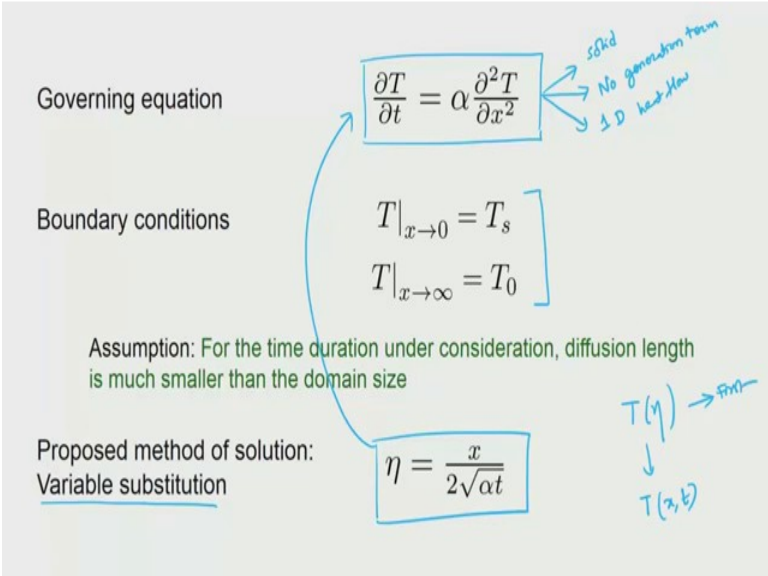

So, the equation that needs to be solved is written here. It is in 1D and you can already list what are the assumptions that, when behind from the generalized Fourier Heat Conduction Equation, it means that it is for solids, because on the left hand side there is no advection term, on the right hand side there is no generation, though it means that there is no volumetric generation of heat and it also means that, it is a 1D heat flow because the temperature variation is given as a function of only x and not with respect to y and z. So, these assumptions, we have written the equation and we want to look at the solution subject to the boundary conditions that are listed here, that at x = 0 temperatures is T_s at x = ∞, then temperature is T_0 which means that our distance is counted in this manner, x = 0. Here, x → ∞ here and this is how it is counted ok?

So, T_0 x tends to ∞ x is going this direction and T_s, x to 0 is given here. So, we look at these boundary conditions which will be useful in determining the integration constants. Now, when we see how to go about the solution of this problem, there are multiple ways that can be adopted. We could actually use what are called in a variable separable method. We can also use substitution of a new variable method and we are picking the second one. So, we are using the variable substitution method, the reason is of course, when you use variable substitution method, you will get a series and the beauty of this solution is not evident unless you plot and see. So, we are using this method because it gives us a very elegant expression.

So, we create a new variable called η and the way we have created the variable is such that, it is a expect $2\sqrt{\alpha t}$. A units are made use of here and the 2 here, is coming with hand side that is, if you do not put then later on the expressions will have some multiplicative factors, then we come back and then, use it here and then, we turn out that the expressions will be very neat later on. So, you do not have to worry how we cooked up this 2 or a root. Here, it comes out basically by iterating back and forth and once you have arrived at the variable functional form of η , then the solution comes out quite nice.

So, we substitute this variable into the equation and see. So, the strategies as follows. We evaluate the left hand side and the right hand side of the governing equation and then, we seek the solution as a function of η and then of course, because we know the η as a function of x and t, then it implies that we also have the solution as function of x and t. So, this is what we seek first ok?

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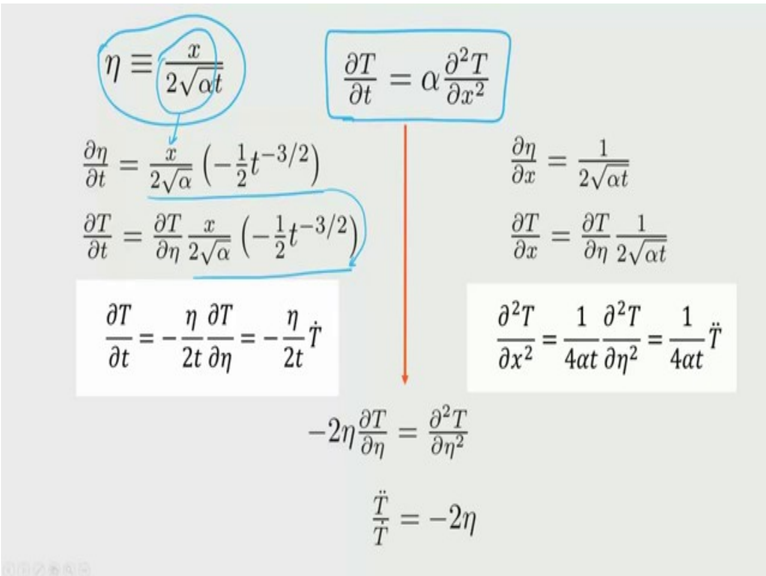

So, that is what we do here. So, this is the governing equation and we have got this substitution that is being used.

So, $\partial \eta / \partial T$, can then be straightaway seen that it will have this, straightaway coming here and then, $1/\sqrt{t}$ would differentiate to give $-1/2 \times t^{-3/2}$ and then, when you look at the first term on the left hand side $\partial T / \partial t$ and then, that will be given as a differentiation by parts. So, first you difference with η and then, η with respect to time and that is available here. So, this is straightaway coming from here ok?

Now, which means that you could actually see $\partial T/\partial t$ can be given as $\dot{T}$ or this one. So, this is then given as $\dot{T}$. Now, we do the same thing on the right hand side, also with respect to the x and you see that ∂T, $\frac{\partial \eta}{\partial x}=\frac{1}{2\sqrt{\alpha t}}$, because numerator is x that is gone and we do the second time, then you already see that it should be 1/(4αt). Now, this $\ddot{T}=\partial^2 T/\partial \eta^2$. So, we also note $\dot{T}$ is this.

So, we just then substitute these two into this expression and we see the governing equation has modified itself in this manner and therefore, we write now $\frac{\ddot{T}}{\dot{T}}=-2\eta$ok?

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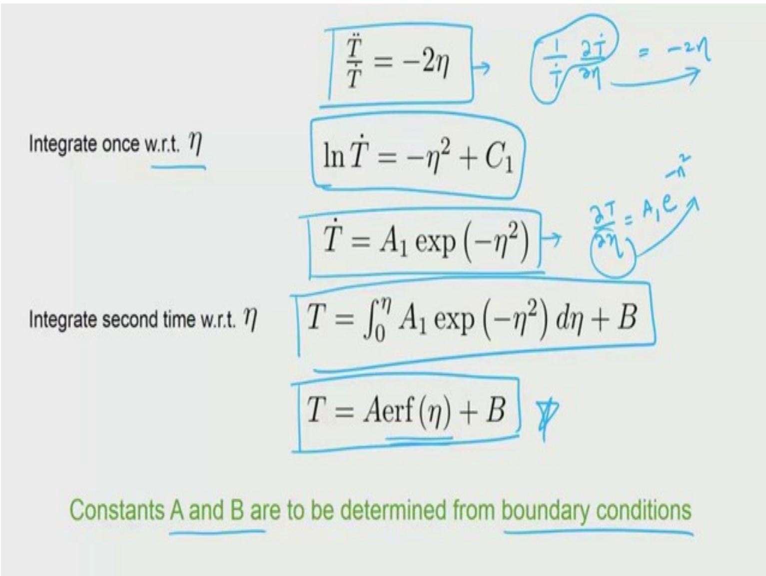

Now, what we do is that, we want to integrate this with respect to η. So, with respect to η, if you integrate now, you could look at this expression to be as follows: - it is basically is $\frac{1}{\dot{T}}\frac{\partial \dot{T}}{\partial \eta}=-2\eta$that is what it looks like.

So, which means that when you integrate, then this part would actually give you logarithm and this, when you take it to other side, it becomes η^2 with the minus sign. So, that is what we write as a solution here and of course, we exponent on the both sides and we can write the $\dot{T}$ as this expression and we have an integration constant C_1. So, when we exponent, it becomes $\exp(C_1)$. So, that we call as A_1. So just some integration constants on the way.

And left hand side, we now expand this expansion is given like this. Now, again we write it is nothing, but this is equal to $A_1 \times \exp(-\eta^2)$ and now, if you want to integrate, then you take it to the other side and when you integrate, you get this solution. So, we are not simplifying the integration of $\exp(-\eta^2)$ because we do not know how to do that. So, we leave it at that and we want to say that entire thing is a function which is called error function.

So, we leave it at that and then write the solution as $A \times \text{erf}(\eta) + B$. So, this is now the solution available. The reason why we leave it like that is, because error function is a standard function for which the values tabulated are available readily. So, most of the calculators already have this function. So, we can directly go ahead and use it. So, we do not have to simplify the integral of $\exp(-\eta^2)$ at term ok? Now, A and B are integration constants. So, they can be determined from the boundary conditions ok?

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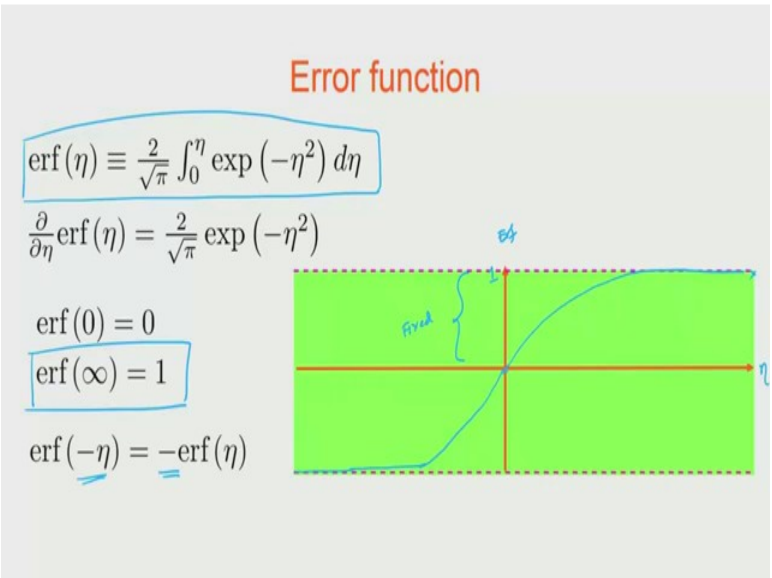

So, now what about this error function, how does it actually get defined? So, actual definition of errors function would have $2/\sqrt{\pi}$, because of the following requirement we would like to have a very nicely calibrated value for error function. So that when the argument goes to ∞, the value should be 1. So, to ensure that it is 1, you have a multiplicative factor of $2/\sqrt{\pi}$ other than that, there is no special meaning one can derive, that also separately. So, we can do that in tutorial if necessary, this function, it looks like this error function at the argument zero is zero. So, which means starting here and it is ∞, it is one. So, this is one. So, error function is in this direction, the H is in this direction.

So, it should look like that. So, asymptotically approaching one and on the other side it would look like that. So, this is how the error function would look like and you could see that the slope is given by at slope, at the zero is immediately given here and you could actually see that the slope at ∞ would be flat, because - η^2 have to. η goes to ∞ is zero and of $-\eta^2$ at $\eta = 0$, would be some number and that number is a slope, that you are seeing here and this function is an odd function, which means that when the argument actually changes, the sign, the value also changes the sign.

So, that is the reason why, when we go to the second half of this axis, η is negative. Then you get the negative values. So, this is how the error function is now. This range is the fixed range, which means that the temperature variations we are looking at should be calibrated to this height. So, that the functional form is given by these curves ok?

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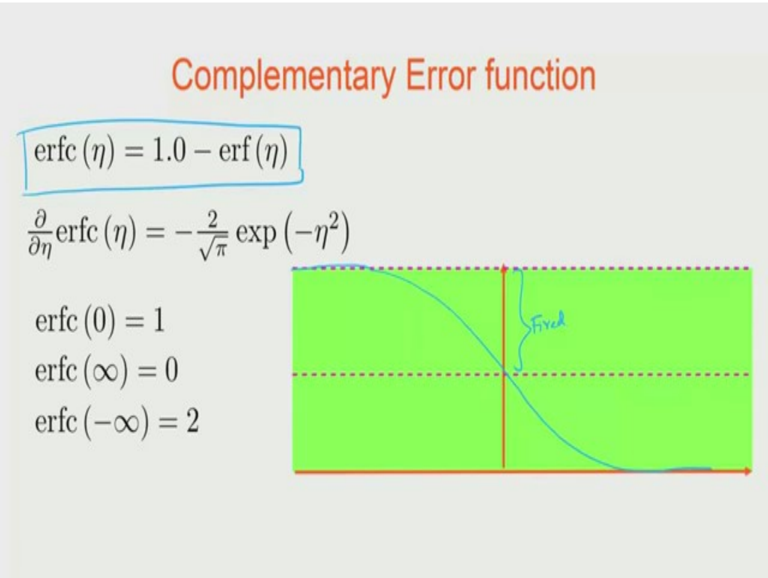

Now, there is something called complementary error function, also we just quickly go through that because it is nothing, but one minus of error function. It is a useful thing and rests of them are worked out. So, you can already see that error function, complementary error function at to zero has a value of one, because erf(0)=0. It has a value of one erfc of ∞ erf(∞)= 1.

So, erfc(∞) =0. So, you would see that, the functional form should be looking like that. So, the complementary error function goes that, you could see that it is basically flipped version of the error function. So, depending upon the way your slopes are, you could actually use these two functions to write your solution in any case. You could already see that, this difference should be fixed in the problem, only then the solution is valid ok?

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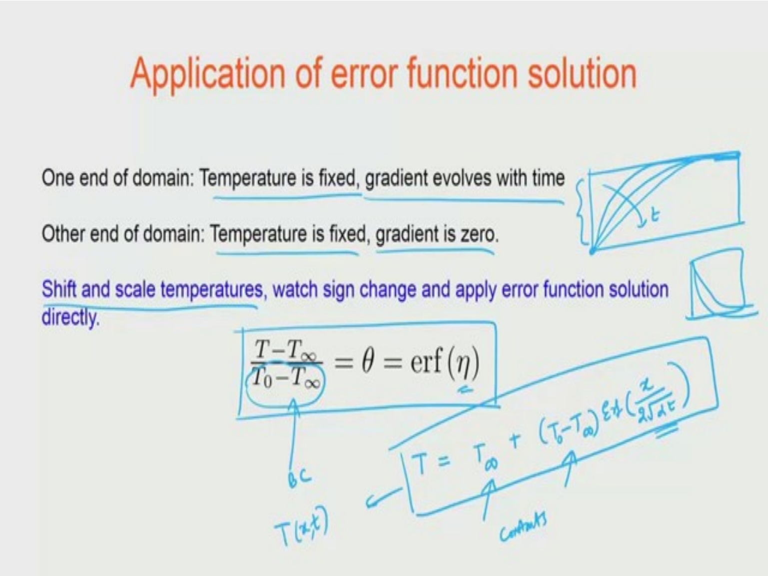

So, once you have it, then you can go ahead and write the solution. The solution is written in this form and here, you can see that we have already used the boundary conditions to write the values. So, you could then expand, for example, this you would expand it as $T=T_\infty+(T_0-T_\infty)\,erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$. So, this is what we have written here and you could see that, these are nothing, but the boundary conditions are the constants integration constants and you could see that the argument is now expanded and you now, have basically this is nothing, but what we were asking for namely T as a function of distance in time ok?

So, we have the solution available now and this is very important to note that, this is applicable only when on one end of the domain, you have got temperature is fixed and gradient is changing with time and the other end the temperature is fixed, but the gradient is zero which means that, this is valid when you are able to imagine the domain to be semi infinite, such that the profiles look like that, as a function of time. So, as a function of time, the gradient is changing on one end, but they are all zero at the other end and then, the values are fixed on both ends.

So, in such situations, you can use Error Function and then, the way to do it is, scale this difference to be the magnitude here and immediately write down the answer as Error

Function of η. So, this is as simple as that. So, shift and scale the temperatures and then look at the change of the sign, because this plot may be on the other side. So, if the plots look the other way. So, let us say if the plots in this manner, so then, look at the sign difference and adjust it. So that you can write the solution directly. So, very often Error Functional solutions can be written by visual inspection of the appearance of the temperature profile.

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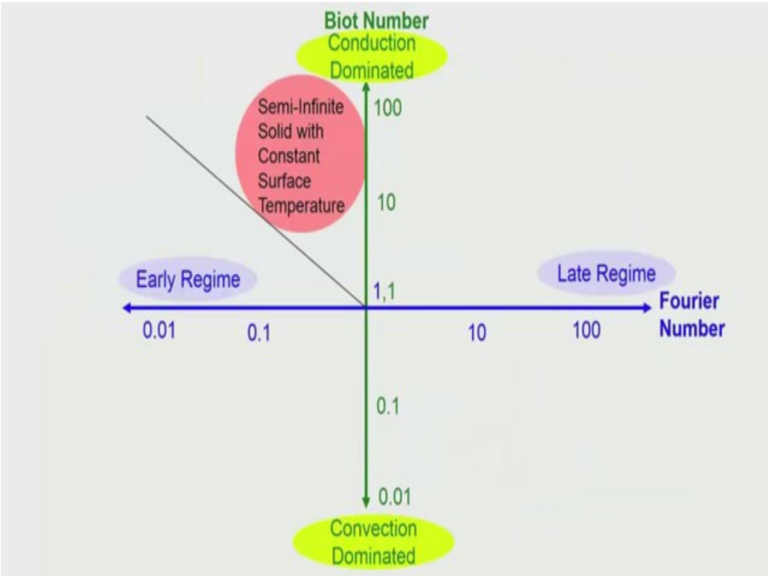

Now, in what part of the domain of the Biot number and Fourier number map, do we use this Error Function solution that is actually to be used in this part of the domain? So, which means that we should use them when Biot number is large and also for relatively small Fourier number. So, this is the domain and we could then expand and see what happens for other domains? What kinds of solutions are available? We are not going through them in this session, but we can already look at them.

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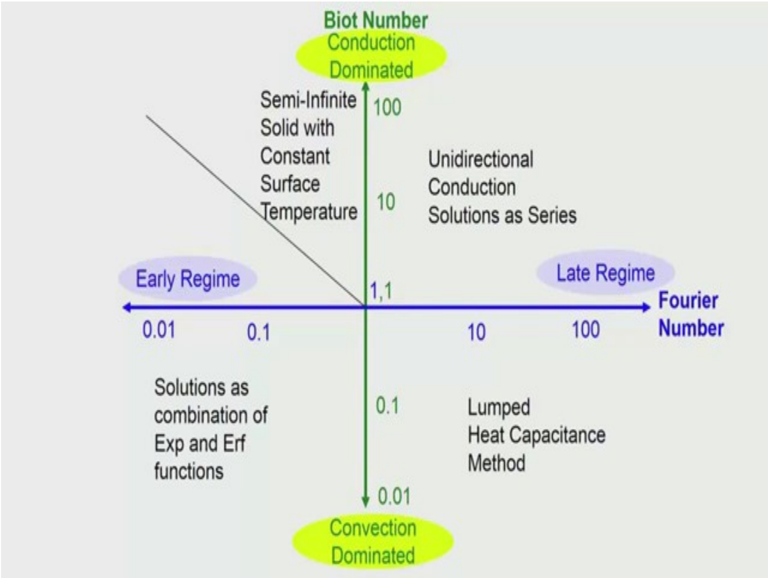

So, depending upon the choice of the problem, the Biot number and fluid number can be anything from a very small number to a large number and then, you have got various solutions. So, what we have done is this. What you have done is this, see that you can also use series solutions and we could also combine the exponential error functions and then, we can handle any part of the domain that we have. So, these two extreme situations are looked at, because they are very common in metallurgical literature.

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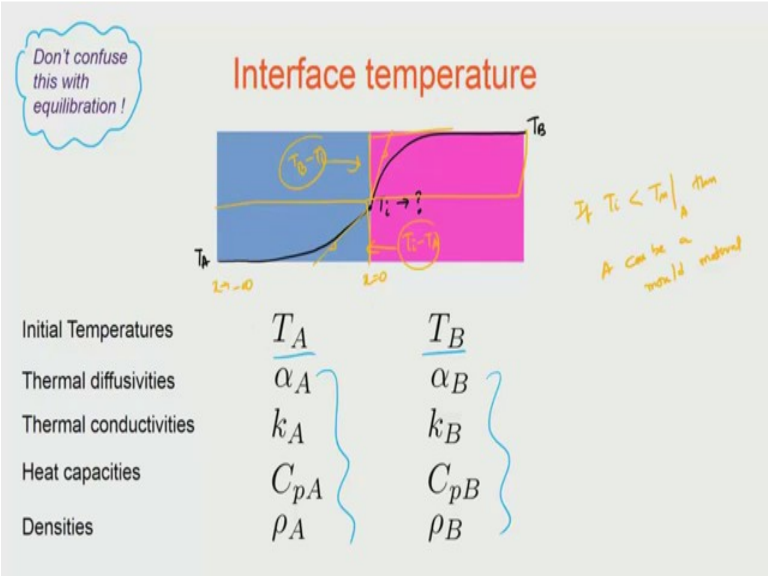

Now, we now migrate to an interesting application of this particular analysis, namely in metallurgy. We have liquid metals being cast. So, very often we are interested in what would be the temperature of the interface, when we bring hot metal in contact with a cold solid, which is basically acting as a mould. So, we should not confuse this problem with equilibration. So when, we have two bodies of temperatures A and B, with different properties then of course, under equilibrium, they both will act in a temperature T bar at after long time, but we are not interested long time, but instantaneously, as soon as you bring them into contact, then what would be the interface temperature.

So, for arguments sake, let us say, T is small. So, we would then think for example, that the temperature profile may look like this and at the interface, there is some temperature and on the other side is that there. So, you could say that this is a temperature B and this is the temperature A and the interface, there is a temperature T_i, which we are interested in finding out and we are not drawing the slopes on both sides, same the thermal conductivity on both sides is different. So, we basically do not want to make the slopes same.

Now, we are interested in T_i, because if T_i for example, if T_i is less than the melting point of A, then A can be a mould material for this situation. So, which means, we want to then see whether this interface temperature, when we estimate, can be useful or not. So, let us do that in a particular manner. What we do is that, we see that in this situation the way we have drawn these two curves, we can imagine the error function solution available here, in these domains with these scaling factors.

So, the scaling factor one, scaling factor comes out as T_i - T_A here and another scaling factor comes as T_B - T_i here and other than that, the error function is straightaway applicable. So, we go ahead and apply error function to these temperature profiles and see what can be reduced.

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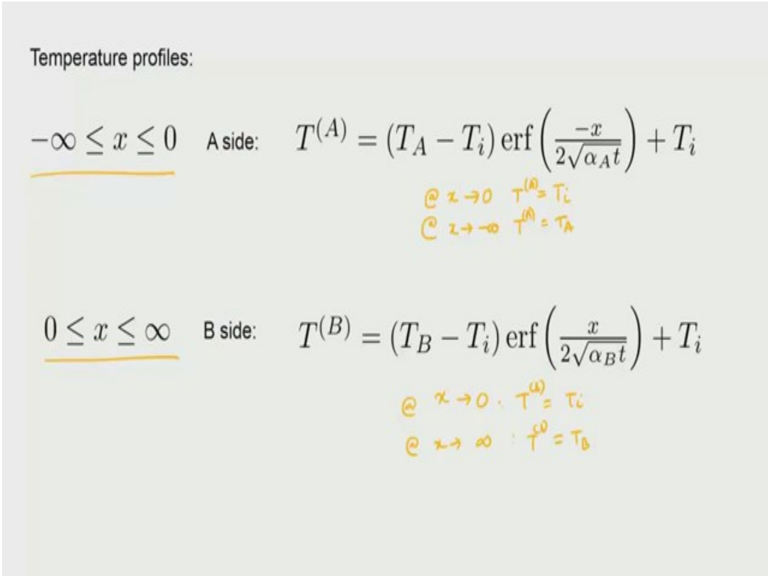

So, we write the error function solution here of the left hand side of the domain, we write it in this manner. So, verify that this function shows that at x goes to 0. So, you see that error function could be zero. So, the temperature in the A side is T_i and at x goes to -∞, then you see that minus and minus cancel, that will be 1 T_i cancels and T(A) in A.

Now, look at the problem, you see that at x = 0 and x is going to -∞, the temperatures had T_i and T_i respectively. So that we got that character. Now, we look at the other side of the domain and at x goes to zero, we see that the temperature is in the B side, temperature is T_i and at x goes to ∞, then temperature on the B side is basically T_B. So, you could see that this is one. So, T_i cancels it is T_B.

So, you can immediately see that there is the same situation at x_0 is T_i and at x = T_B. So, these two solutions that way are valid. So, we can then use them or to analyze what would be the T_i. So, we need a condition for T_i, the way we derive the condition is by saying that.

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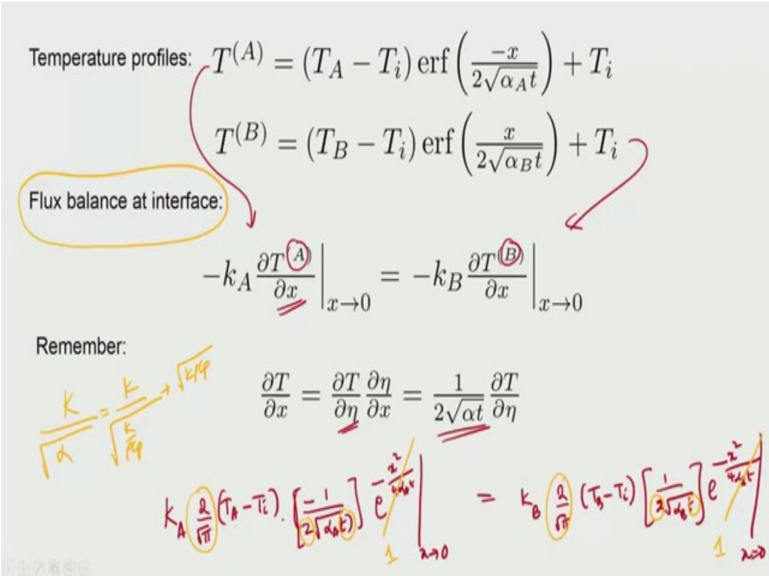

There is a flux balance at the interface that is whatever heat is coming from B side onto the interface is lost. So, whatever heat flux is coming from the B side, whatever is coming from the B side or to the interface is then going away into the A side.

So, there is no special thing happening at the interface like for example, release of heat or absorption of heat. So, flux balance can then be applied there and that is what we are trying to do here and then of course, we know the flux expression from the Fourier heat conduction. So, we write it on the left hand side with the expression for A on the right hand side with the expression for B. So, we note that the solutions are to be used appropriately ok?

Now, we can then forget to these minus signs, because they are on the same side, same sign on both. So, we can then substitute. So, what we do is that, we also remember that the slope that we are taking is with respect to x, but for error function, we have got actually as a function of η and therefore, we have to also not forget, this $1/(2\sqrt{\alpha t}$ ¿ as a term that is coming in very often. Students make a mistake by differentiating and forgetting that there is a multiplicative factor in factor.

So, you can see, on the left hand side for example, it is k × ∂T/∂x is nothing, but $2/\sqrt{\pi}$ into this expression, which is basically T_A - T_i into this one, which is basically $-1/(2\sqrt{\alpha t}$ ¿. So, this is coming on the left hand side and into $\exp(-x^2/4\alpha t)$, but evaluated at x = 0. So, which means that becomes one and on the right hand side, you could see that it will be k_B. I am removing the minus sign $k_B \times 2/\sqrt{\pi}$ into this expression, that is T_B - T_i into, then we have got $1/(2\sqrt{\alpha t}$ ¿ $\exp(-x^2/4\alpha t)$, evaluated x = 0.

So, you could immediately see that we can make some simplifications, this is gone because that is one and this also is gone, that is because one and then, the time also will be gone because it is the same on both sides and then, you have got multiplicative factors that will be all gone. So, we could then see that, you are seeing $k/\sqrt{\alpha}$. So, let us look at how that expression looks like, $k/\sqrt{\alpha}$ is basically $k/\sqrt{k/\rho C_p}$. So, that becomes basically $\sqrt{k\rho C_p}$. So, we can cancel $\sqrt{k}$ from the denominator and numerator.

So, which means that on the left hand side, you have got $\sqrt{k_A \rho_A C_{pA}} \times (T_A - T_i)$. Similarly, on the right hand side.

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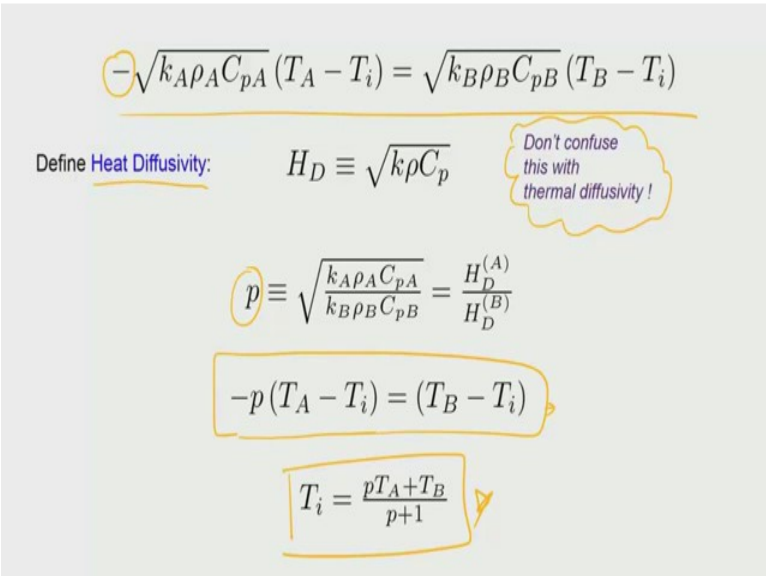

So, we do that here and then, use it expression and the minus sign is coming because, this fellow is still there. So, we write that expression with the minus sign, without forgetting. Now, this quantity $\sqrt{k_A \rho_A C_{pA}}$ with the root has a name in the casting literature, it is called the Heat Diffusivity. So, we should not confuse this heat diffusivity with thermal diffusivity. Thermal Diffusivity have the units of meter square per second, but this will have very crazy units, but it is actually mentioned here, as a quantity because it is coming together and the ratio of these two Heat Diffusivities can then be given a symbol p which is

$$p \equiv \frac{\sqrt{k_A \rho_A C_{pA}}}{\sqrt{k_B \rho_B C_{pB}}}$$

Now, if you do that, then –p(T_A-T_i) will be on the left hand side. Right hand side will be; this have gone all the on the other side's or T_B -T_i . So, which means that when we now try to solve this, we are to be just appearing to be very nice elegant expression, which comes as

$$T_i = \frac{p T_A + T_B}{p+1}$$

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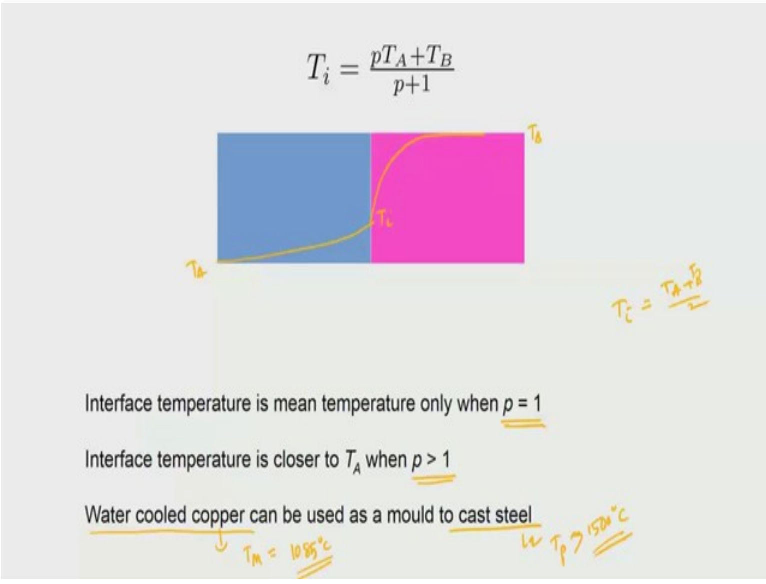

Now, look at this expression carefully. What this implies is that, when p is equal to 1, what happens, it means that T_i is nothing, but (T_A + T_B)/2, which means when the two materials that are being brought into contact are the same materials, then the interface temperature will be the average of the two temperatures, which is something that we expect, but if p, for example, the p is greater than one, it implies that the T_i would be actually weighed in favor of T_A and it will be closer to A.

Now, what implies that, so you have got T_i, T_A and T_B. You see there it to be closer and that is the situation when p is greater than 1. So, heat diffusivity will then dictate whether or not the interface temperature is within some limits or not, which means that we can actually evaluate and estimate to see what will be the T_i and when we do that with copper and liquid steel. So, we know that the copper melting point is 1085 ^{0}C and steel in the liquid form would have for example, a pouring temperature of 1500 ^{0}C or more. So, it will be greater than that.

So, which means that when we substitute those values with respect to the properties of both copper and steel, then we get the value of p and then, we can evaluate and we will see that the T_i is actually coming below the melting point of copper, which means that if you want to, then do it sustainably, you will do it with water cooled copper and water cooled copper can be a mould for casting steel. This is a very different conclusion than what we would do if you did not do the heat transfer. You are pouring a liquid metal at a higher temperature into a body which will melt at lower temperature, but it will still work, the reason is that the heat transfer in the copper is way faster because p is greater than one and therefore, it would have the interface temperature less than the melting point of copper, which means copper will not melt when you pour liquid steel on top of it.

So, good news because we can then use you copper as the mould for our metallurgical casting processes. So now, we then go ahead and use this idea that is, you are actually having a thick mould and we are having a little metal and when the solidification is happening in a casting, then how would we then, characterize the rate at which the casting is happening.

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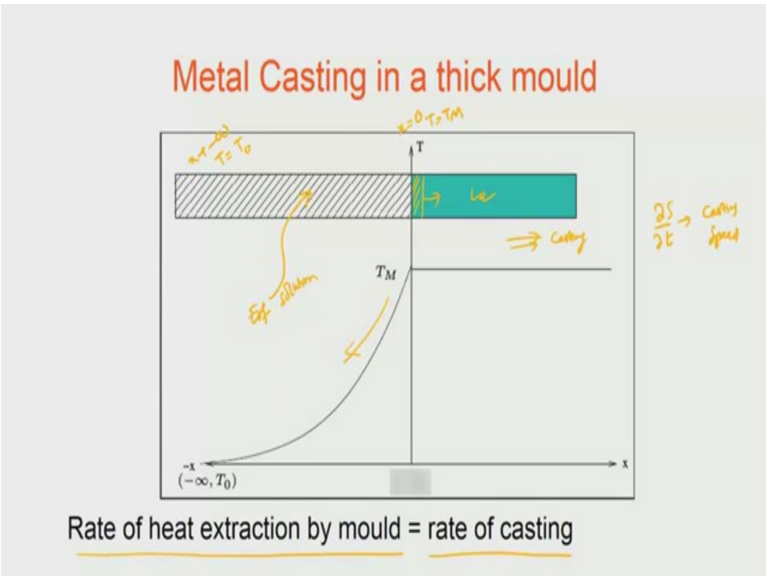

And dictate that by the heat flow. So, you could see that the casting is happening in this direction, the solid is actually growing. So, it would actually grow in this manner, this is all solid and this is all liquid and the casting speed is in related to the ability of a heat removal through the mould and we want to then see, whether we can use some of the solutions we have obtained till now to obtain the expression for S which is the casting speed. Divide ∂S/∂T. So, ∂S/∂T, this is the casting speed. We want to see whether we can get that in terms of meter per second.

So, we want to then make a supposition. Here, we want to say that, we want to simplify the problem. We do not want to have different temperatures for the liquid and solid and we do not want to have the liquid temperature more than the melting point and so on. So, you want to make it very simple and say that, the rate of heat extraction by the mould is giving you a rate of casting. So, strictly speaking this is not valid because, how would heat latent heat released in the liquid come on to the interface, unless there is a temperature gradient?

So, strictly speaking it is not valid, but we say that the gradient in the metal solid and the metallic liquid is so small that it is negligible. So, with that assumption we can proceed to solve this problem.

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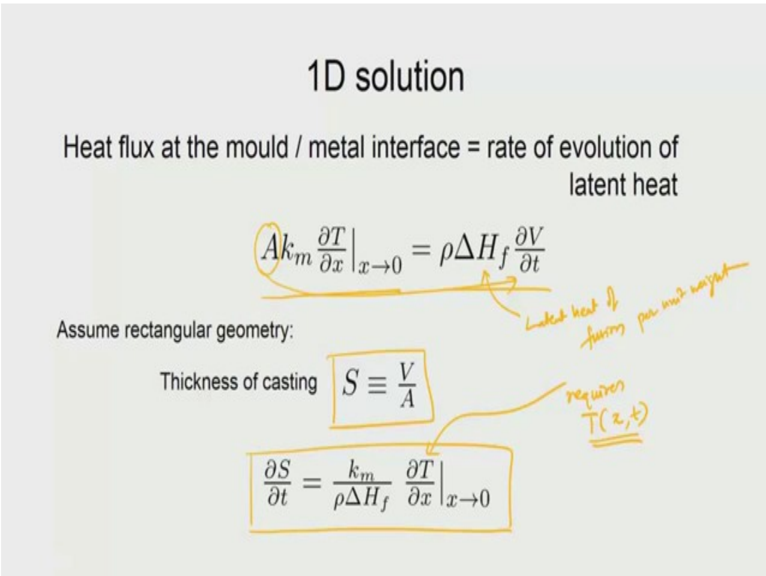

So, heat flux at the mould is then, given by this expression which is Fourier Heat Conduction Equation. We are dropping the minus sign in the front, because heat is actually going in the minus x direction. So, the casting is positive, which means it is going the plus x direction. So, we have a sign change that is happening. So, minus is dropped there to account for this particular difference.

So, the rate at which the heat is evolved is given by the latent heat. So, this is nothing, but latent heat of fusion and that is given with per kilo. So, because it is per kilo, you need to have mass there. So, the mass is then called as $\rho \times V$ and then therefore, we are actually calling as a $\partial V/\partial T$. So, basically there was a heat per unit weight that is what we are actually using here.

Now, the thickness of the casting is S, that is given as basically volume of casting there by the area through which heat is being extracted. So, A is area through which heat is extracted. Now, in the case of planar cases, you do not have to worry or in the case of cylindrical spherical cases, we need to watch out. Now, we can see this expression and manipulate to see that, you take A to the denominator there and you can see $\frac{\partial S}{\partial T}=\frac{k_m}{\rho}\times \Delta H_f \times \frac{\partial T}{\partial x}$¿$_{x \to 0}$which means that this requires T as a function of x and maybe also t, but we need this function form and for that, we basically lean onto the solution that we just now obtained. We already have a solution in the form of error function. So, we go ahead and use that, the error function solution is available here. We have written it in this form.

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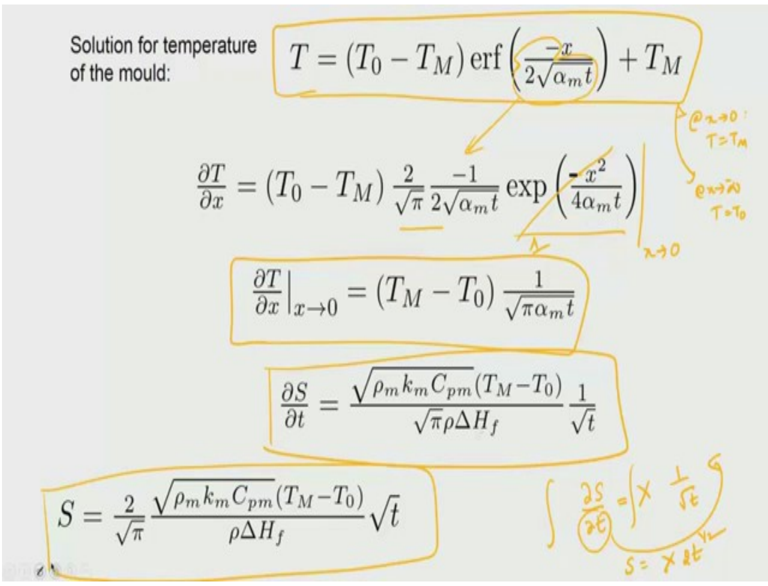

So, again verify that the solution is written correctly. So, we see that this solution. How does it look like at x tends to zero. So, you see this is zero. So, then $T = T_M$ and at x tends to ∞, then you will see that this goes as one minus one and therefore, you would get $T = T_0$. Now, you see that x tends to $-\infty$ will give you T_0. So, this means that, this solution is written in the mould here, in this sphere. So, this is our domain. So, Error Function solution is written in this.

So, that at x is equal to zero, you have got $T = T_M$ at x tends to - ∞. You have got $T = T_0$. So, having a calibrated that way, we can go ahead and then look at what will be the ∂T/∂x, which will be 2/√π into this quantity that is coming in front. So, that is used as it is and here now, we have to put a minus sign there. Now, we evaluate this at x tends to zero. So, that goes off as one and then, you could already see that the solution of the slope is coming out quite elegantly, $T_M - T_0$ which is $\Delta T/\sqrt{\pi\alpha t}$

So, then its substitute, this into the ∂x/∂T expression and we get that expression here and we then see that, you can integrate this expression. So, you could treat the entire thing as a constant. So, you could pretend that is this is appearing like this is equal to some constant, say X ×1/√t and which means that, you can take this to the other side and then, when you integrate, then you could see that S is given by x in T. So, it comes out like that.

So, you could then use that same expression here and we write the expression here. So, S is going as √t.

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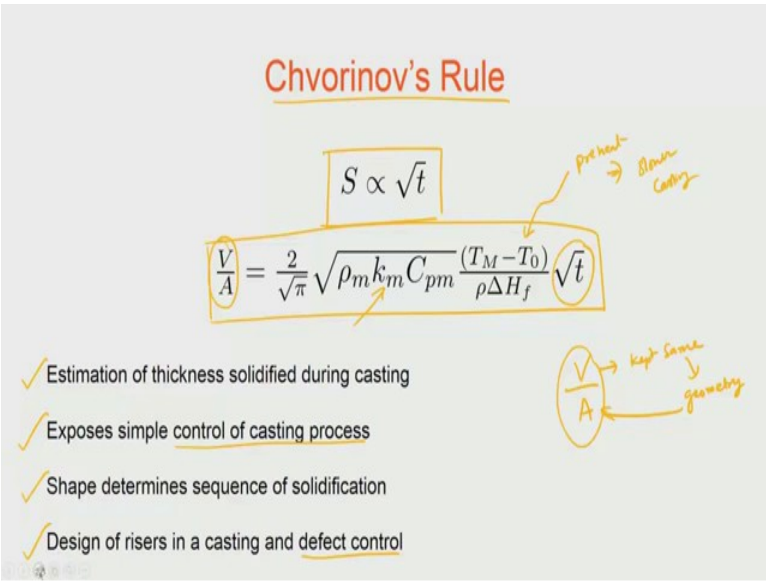

So, that basically is also known many - many years back Chvorinov's Rule. So, that is, the casting speed is going parabolically. That is, it goes as a square root of time so; that means, as you proceed solidification, the spirit which solidification happening is happening slower and this, when you expand, what is the definition of S we already know that V/A. So, we can then see the entire expression that is coming in front like this.

So, this actually is a direct usage of error function solution and the idea, that the flux balance is happening at the interface and the way we can use this expression to understand metallurgical processes, such as you know casting as follows, you can do what is called estimation of thickness as a function of time during casting. So, you can already and directly see that you substitute all the values, put the amount of time spent, then you immediately see how much of volume of liquid metal has solidified.

Now, in hollow casting, you may want to solidify for some time and then pour the liquid out and get the hollow casting. So, this is again a direct application of this particular expression. You can also see this expression exposes the control of casting processes. You could see that for example, this T_0, that is the temperature of the mould, if T_0 is less; that means, if you have a water cooled mould, it implies that this entire number is large, which means that your castings speed will be high and if T_0 is high, which means if you have a pre heat, it implies that you would have slower casting.

So, you could see that you can actually avoid cracking in alloys, where it is very much possible because of high cooling rates or high speeds of casting. If you pre heat the mould, then you can avoid that problem. So, you could only see that the effect of parameters is available. You could also see that, when you use a material of high thermal conductivity, then, the casting speed is high which means that, if you use copper mould instead of cast iron mould casting speeds are higher. So, you could see that this expression exposes the control of casting processes by which we can go ahead and design the kind of mould you want, the pre heats etc. You also see that, from this expression we also have the shape determines the sequence of solidification. What do we mean by that, on the left hand side we have got V/A?

So, let us say the volume is kept same and then, you see that when you change the geometry it changes the A. So, when you see that for a sphere, you would have the smallest surface area. When you go to cylinder, then you have surface area increased, which means that for the same volume of liquid that has to solidify, depending on the geometry, this quantity is going to change. Which means, the time it takes to solidify will also change. Which means that you would have different shapes, solidifying in different sequences, which actually will help you in designing a casting setup and the risers are actually designed in the same manner, so using Chvorinovs Rule?

And we also can find out defects because we know that the solidification length, if it is same, then the time is same. Which means that we can estimate by using geometrical constructions, where are the likely defects that can come. So, I will just give you one example by choosing a sample case here ok?

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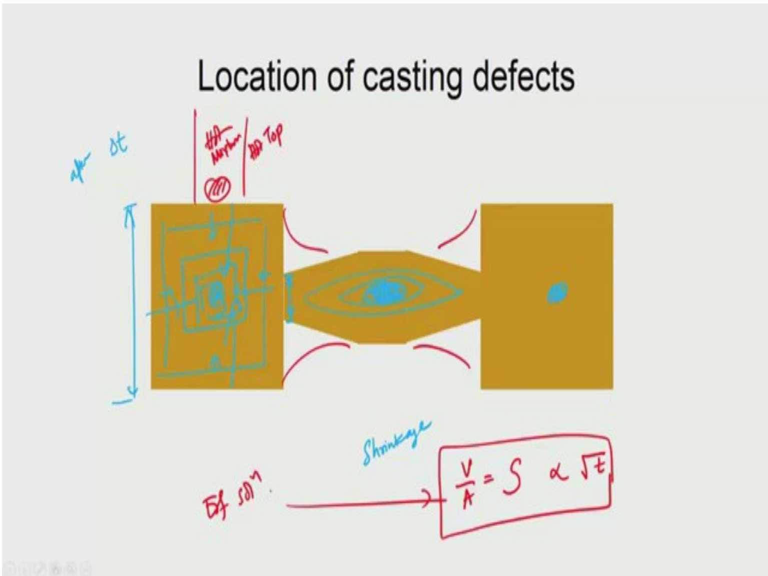

So, let us see that this is a cast that we want to make. So, it is a poor design of casting. The reason is evident here. Now, let us say, the casting actually proceeding from all the surfaces which means that after a particular amount of time, the solidification is happening, such that the thickness is same in all directions.

So, you could see that if this is going in all directions, then where would be the last to solidify liquid that is present. It will be here, similarly it will be here and it will be here, you would also see that this distance is small compared to the entire thickness, which means that you would actually block this direction for liquid flow very early, which means that the last to solidify liquid here, has to solidify and if there is a shrinkage, then that shrinkage will be visible here, because there is no feeding of liquid from elsewhere. The same thing is a problem here and problem here.

So, like this you can actually identify by geometrical constructions. So, as you keep shrinking it. So, you see that, for the same amount, same thickness has to solidify. So, the last tool solidifies liquid, you can identify using geometrical construction. So here also, you just follow the contour and you would see that last solidify liquid will be in the center. So, you can identify locations of casting defects by using the Chvorinovs Rule and then of course, you can go ahead and then fix them up. So, the way to fix for example, in this situation, would be first to avoid this kind of a curvature. So, you have curvature this way and later on, you can machine it. That is the first thing that you would do and of course, you would have for example, a riser with some hot mixture, hot top.

So, that the last solidify liquid would be somewhere here and not here and therefore, you can avoid the porosity or shrinkage defects at the center. So like this, you can actually go ahead and solve, but we will not discuss them here further, but we would actually see that, the very fact that $S \alpha V/A$ is equal to $S \alpha \sqrt{t}$ is already. So, valuable and from where does it come? It comes from Error Function Solution. So, this is actually showing for example, how very simple analytical solutions that are available from transport phenomena can have very important effect on the design of metallurgical processes. We will take up some more examples as we go along. So, with that we come to conclusion of this session. We will have some numerical problems available for you to practice in the course website.