Transport Phenomena in Materials
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Lecture - 10
Navier Stokes Equation - Part 2

Welcome to the session on Navier Stokes equation as part of NPETL MOOC on Transport Phenomena in Materials. This is the second session on Navier Stokes equation. In the previous session we have derived the Navier Stokes equation and in this session we are going to do to a little more analysis of this equation and special cases of this equation.

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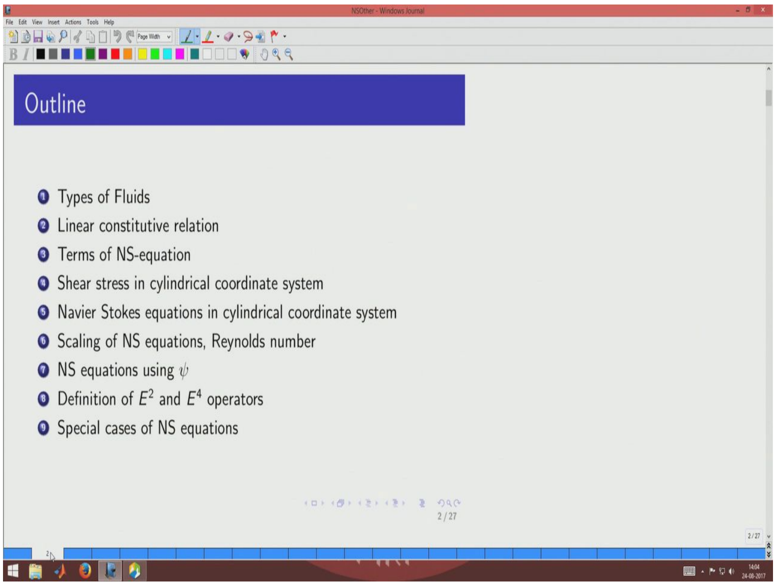

So, this is the outline of the lecture in this session. So, we are going to look at the types of fluids based on the linear constitutive equation we started off and see how this can be violated in some of the fluids and then we will see different terms of Navier Stokes equation and what those mean. So, that for a given problem at hand then which terms can be dropped will be evident.

We will also see how the shear stress will be expressed in other coordinate systems and how the Navier Stokes equation also will appear in other coordinate system. We will allot the attention to some terms that are coming in because of the nature of the coordinate system being cylindrical or spherical very different from the rectangular. And we will also then scale the Navier Stokes equation. So, that we do not use the absolute velocity or distances and in the process we will discover that there is a non-dimensional number that is coming up which called as a Reynolds number. And then we will also convert the Navier Stokes equation to use not the velocities, but the stream function and then we will discover that we will have two new operators that are needed for axisymmetric flow and we will list the special cases in the end and then close this session.

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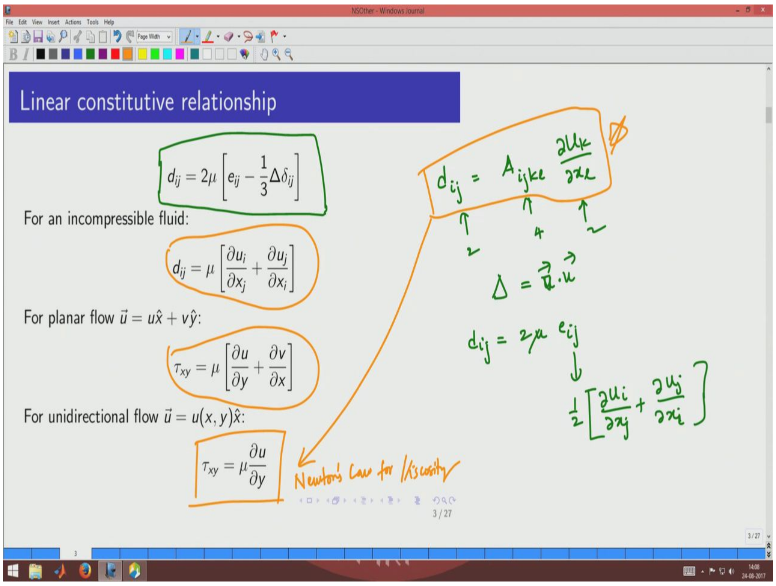

So, here is the linear constitutive equation that we came about the equation originally was written in this fashion $d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l}$ and then we said that this is a tensor of order 2 strain rate tensor of order 2. And this is stress deviatoric stress of tensor of order 2 and we said that the most general form should be the this must be tensor of order 4 it should be isotropic and then we said that the symmetry over the two indices because stress is symmetric will implies symmetry over the same indices on the right hand side and then we expressed as a combination of the δ and then after some manipulation we arrived at this particular expression.

So, please refer to the previous session to identify how this particular expression has been derived. So, here the delta is the same as what we have come across which is nothing but the rate of dilation and this is coming here and you could see that this constitutive relationship does not distinguish between incompressible and the compressible fluid flow. If it is incompressible then the second term can be dropped. So, you could see that the first term would then be looking like this you can see that $2\mu e_{ij}$ for incompressible fluids because the second term gets dropped off. And we already saw that e_{ij} was defined as the symmetric portion of the strain rate tensor which means that it must be this way. So, this is the symmetric part and therefore, we can see that this happened to go away and we see that the equation linear constitutive equation we have derived for an incompressible fluid would look like that.

And if we were to take the velocities as u and v in x and y directions respectively then this is the expression that is normally seen in many of the text books where we use τ for the symbol, deviatoric stress in x y coordinate system we call it as τ_{xy} and it is expressed in combination of two off diagonal terms of the strain rate tensor and they are written like this. And if the flow happened to be unidirectional then you have only one velocity component which means that this is the expression that we come about which means that you can see the most general form happens to be here and this is a 1D version of it and it is this version which actually is called as the Newton's law for viscosity. So, which means that basically the most general form where the fluid can be called as the Newtonian fluid is here and for 1D flow it is what is normally coming across in the text books namely τ is proportional to the strain rate and the proportionality constant is defined as viscosity. So, we have chosen the symbol as μ because the typical symbol used for viscosity.

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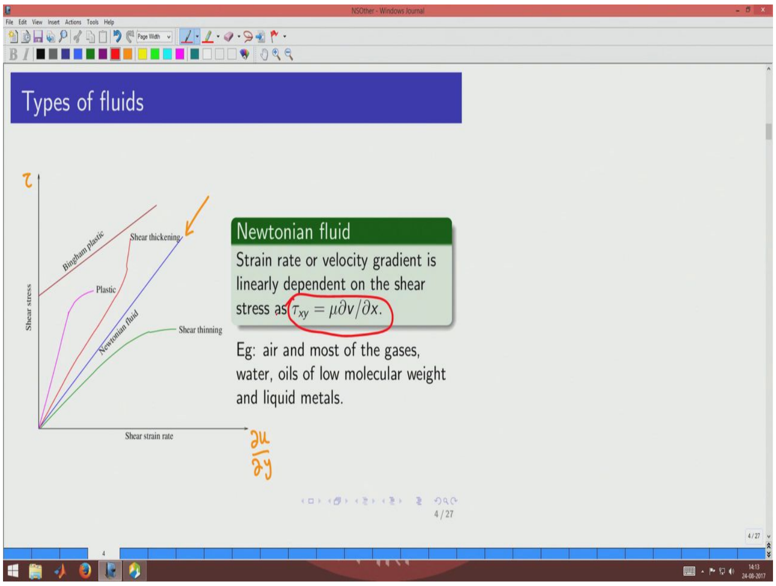

So, now we see what are the different types of fluids that we encounter in engineering and which of these can be called as Newtonian fluid. So, you could see that from the expression that we have come across here $\tau_{xy} = \mu\frac{\partial u}{\partial y}$ you could see that it is this expression when we plot with a τ here and this way. So, you could see that it should be a straight line going through the origin and which means that for infinitesimally small velocity gradient you would require a very small amount of shear stress to cause and vice versa. And infinitesimally small shear stress will immediately lead to velocity gradients being set up. And this kind of a behavior is called the Newtonian behavior and the fluid that exhibit this are called the Newtonian fluids.

And luckily for us most of the fluids that we encounter in engineering problems are Newtonian fluids. So, air, all the gases and water and oil etcetera these are all Newtonian fluids then in metallurgy particularly liquid metals are known to be Newtonian fluids for most of the strain rates that we encounter. And we see that there are other kinds of behavior that are possible and we will look at them one after other.

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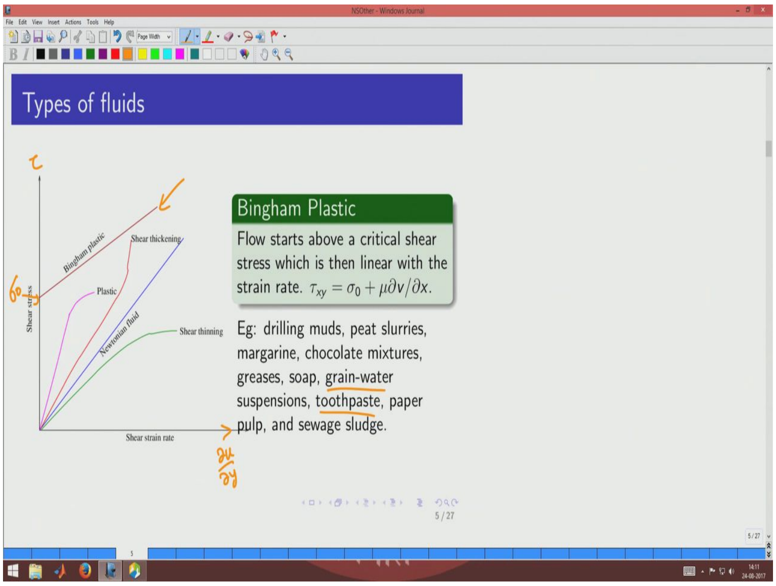

So, we see that the second plot here it corresponds to Bingham plastic which means that the behavior is such that until you apply a certain amount of stress in the σ_0 here. So, you could see that this is σ_0 . So, until you apply a minimum amount of a shear stress σ_0 the material the fluid would not flow and once you exceed that then the flow would take place and you would see that the relationship is given in this form which means that this behavior cannot be directly used for us in the Navier Stokes equation you need to modify. And you already know at what point we introduce the linear constitutive equation and therefore, you see that this particular material should not be modeled using in Navier Stokes equation. We need some other equations to do that.

And the materials that we can encounter in daily life which behave in this manner like a Bingham plastic or drilling muds, slurries, margarine, most important toothpaste paper pulp etcetera. So, it would be very sad if you open the toothpaste and it starts to flow immediately and luckily for us it is a Bingham plastic. So, apply that mush of flow and you get that much of flow out on to your brush, same thing for sludges and mixture of grain and water etcetera. You could make these at home and look at the behavior yourself.

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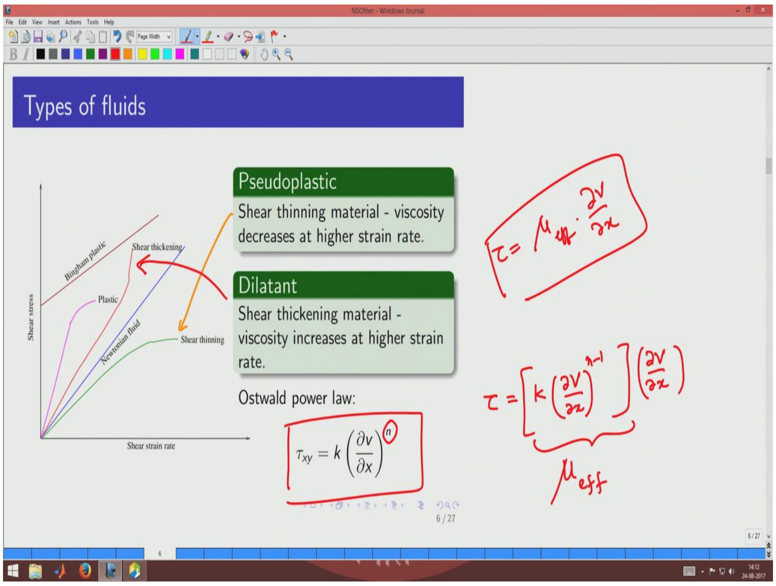

So there are other kinds of behaviors also for example, pseudo plastic, which is shear thinning. So, I indicate here and dilatant material which are shear thickening. So, these materials are such that at low strain rates you may see the proportionality, but at high strain rate you would see that the shear stress required is either low or high compared to the proportionality which means that it is going to be model to using a power law. So, you could see that no longer you could use the linear law and you need to use a power law and here is the term that makes this particular type of fluid different from the Newtonian fluid which means that these fluids at low strain rate could perhaps the modeled using Newtonian fluid, but at high strain rates you need to modify and the power law will come in.

So, if this is the way then and we are actually force to use Navier Stokes equation to solve the problem then what do we do. So, there is a quick trick available in case the exponent is not very high then you could actually like this, you could write this expression in this form. So, you could write in this from so that I have just not done anything except split the term the power n as two terms one with power n -1 another as just power 1 and now you could think of this as effective viscosity. So, which means that the effective viscosity is now a function of the strain rate and then you could then think of this τ as $\mu_{eff}\frac{\partial v}{\partial x}$. So, again we are back to sort of I would say modified Newtonian behavior and then you can go ahead and solve the problems, but please be warned that this is only an approximation when the deviation from Newtonian behavior is not too much.

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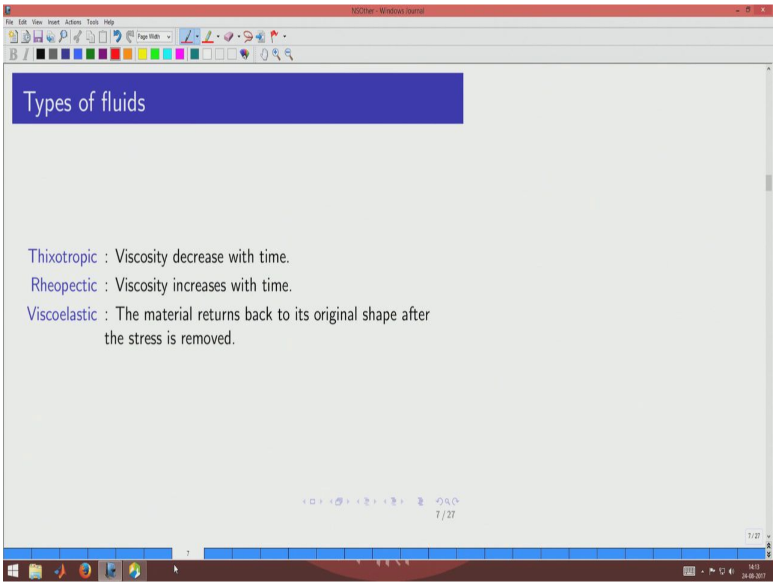

So, there are also other behaviors where it is not the strain rate that actually dictates how the viscosity changing, but it is the time. So, you could see that thixotropic materials, rheopectic materials are materials in which giving the time for settling the particles that are dispersed you could have actually the viscosity either decrease with time or increase with that time respectively. We also have behaviors such as viscoelastic materials where the material would return to its original shape after the stress is removed.

So, this is for example, the jellies that we use for the ice cream deserts where you push the jelly and remove the hand and then the jelly comes back to its shape. So, such kind of behaviors also encountered in the daily life. So, you could see that there are so many different types of fluids, so it will be very sure that the fluid we are having at hand is Newtonian so that we can then go ahead and use the Navier Stokes equation for that ok.

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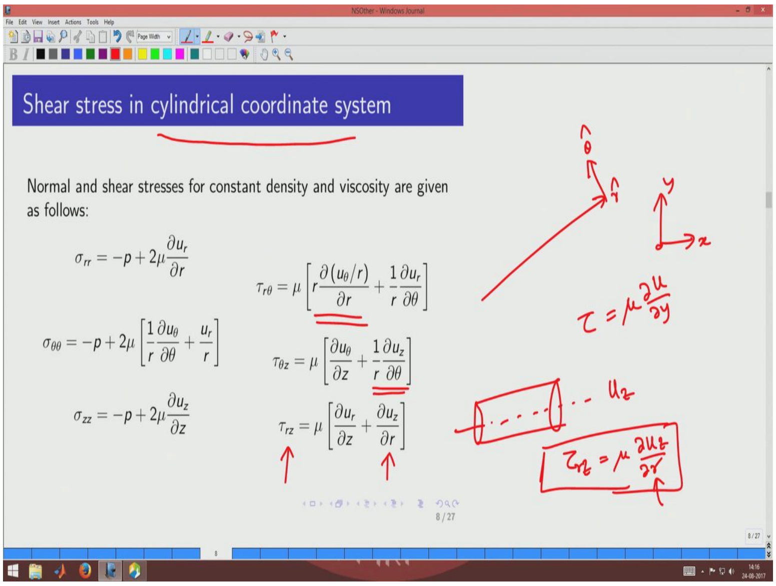

So, now let us look at how the shear stress would look like in other coordinate systems. So, we saw that in Newtonian case the shear stress looks in this manner very simple in 1D and in other coordinate systems how does it look like. So, I want to just bring to your attention that the additional terms that will be involved and so for cylindrical coordinate system this is how it looks like.

So, here the pressure is actually put in for the diagonal terms mainly because we would like to see the diagonal terms as together with the pressure components also put in. If you want you actually drop that and then look at the remaining part as just only the deviatoric stress components that is coming in. So, for example, if you have a cylinder and you are looking at how the shear stress would look like when the velocity is in the z direction then you know that τ_{rz} would have this terms. So, it look very much like a rectangular coordinate system where you would write in this from $\tau_{rz} = \mu \frac{\partial u_z}{\partial r}$.

However, this is when u_r component is 0 only actual velocity is there and you may see that the way we write is very similar to the rectangular coordinate system, but this kind of a very simplistic you know extension should be warned because you have terms that are coming with various other quantities. So, mainly because the coordinate system is cylindrical and you know that in cylindrical coordinate system the axes are defined a little different in rectangular coordinate system they are defined in this manner and the stationary. Whereas, in the case for example, you know x and y, the y is stationary in the cylindrical coordinate system the y axes which is the theta axes is actually pegged up the end of the r axes and it would move around as you relocate the r. So, that is how you see that the additional term that will be coming in and we have to pay attention to that when we use these expressions.

So, whenever we have a velocity distribution and we are asked to find out the shear stress then simply plug in the velocity distribution into these equations and then the stresses are available. Then when you plot those stresses as functions of r or z or θ then we see how the velocity distribution can be explained as a function of distances looking at how the stresses are actually causing that.

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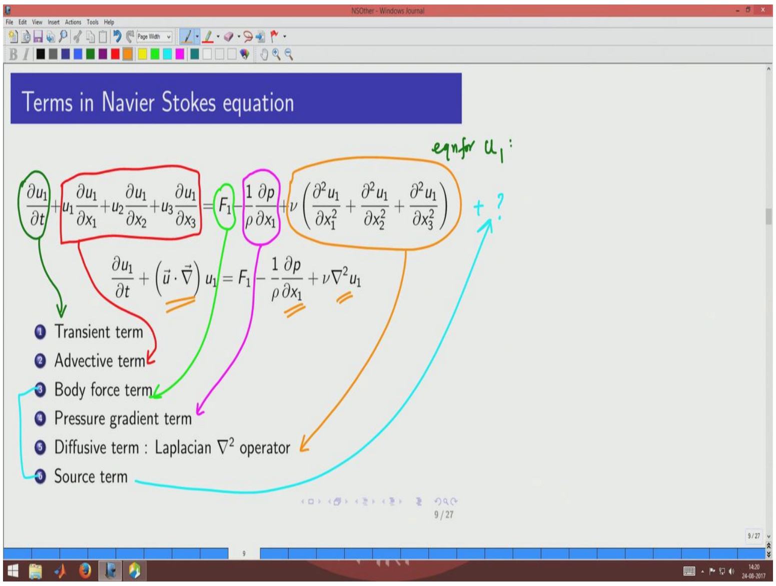

Now, we now being our attention to the different terms of the Navier Stokes equation we want to call different terms by names, so that later on when we solve some problem we want to see which terms can be knocked off and we can make a simpler case of Navier Stokes equation to obtain the exact solutions of that for a given problem.

So, we will give names and this is how it is, you could see that this equation is written for the u_1 component. So, this is the Navier Stokes equation for u_1 component and for u_2 and u_3 components you need to then change the variable here. So, I am going to actually just only show you here. So, u_1 here you one here, u_1 here, u_1 here should be changed to u_2 and then the

u_1 here and here and here should changed to u_2 and that will give you the Navier Stokes equation for the u_2 component. And you see that these are actually coming from the term which is basically from the material derivative and these three will stay the same whichever component you are going to write for.

And if you are going to write for u_2 equation then this will become the f_2 component of the body force and this will become x_2 which is basically the pressure gradient in the second direction and so on. So, you could just extent this equation for other two components. So, essentially we have Navier Stokes equation ready for all 3 components. So, for one component we have written. So, let us look at the terms.

So, the transient term is this. So, the first term which is the partial derivative of the velocity with respect to time that is called the transient term. And then the second term is where the velocities effecting the momentum the momentum is effecting itself and this is called as a advective term and this is basically coming as part of the material derivative. And the third term here is basically the gravity direction or electromagnetic forces causing the flow and this is called the body force term. And the pressure gradient term is here and you could see the pressure gradient term comes with the minus sign because we define pressure as a stress that is acting to compress whereas, σ_{11} would be acting to expand, so the minus sign is to preserve the sense of how we define the pressure. So, the minus is coming the pressure gradient term is here. And the last term is basically the Laplacian. So, this actually Laplacian operator is there and this is called as a diffusive term.

Now, what happens is that there are there are situations where you may want to add some more terms to this equation you may want to add some more terms to a handle more behavior of the fluid flow. For example, to take into account the enthalpy porosity formulation etcetera and in such situations those terms would be called as source terms and strictly speaking source term and body force term are actually the same. So, it just to look at them as you know in addition to what was there in a default form of the Navier Stokes equation you want to give a name called source term, but otherwise source tem is same as the body force term.

We know that this equation as come about from the integration of all these quantities over the moving control volume. So, therefore, whatever you are doing for f_1 you are also doing for source term integration over the entire volume and therefore, they are also called the body force terms ok.

And if you want to look at this equation in the vectorial form then you would see that there the there is a material derivative term here, there is a ∇ term here and there is a Laplacian term here. So, having derived the Navier Stokes equation if you now want to look at how this equation would look like in other coordinate systems perhaps we can just look up these operators in other coordinate systems substitute and proceed. But there is a warning because we have the curvilinear coordinate systems like cylindrical or spherical having additional terms coming in and therefore, we have to watch out what additional terms are required, but otherwise it is just the same concept and therefore, once you have understood how these terms have come about then you have understood how the Navier Stokes equation has come about.

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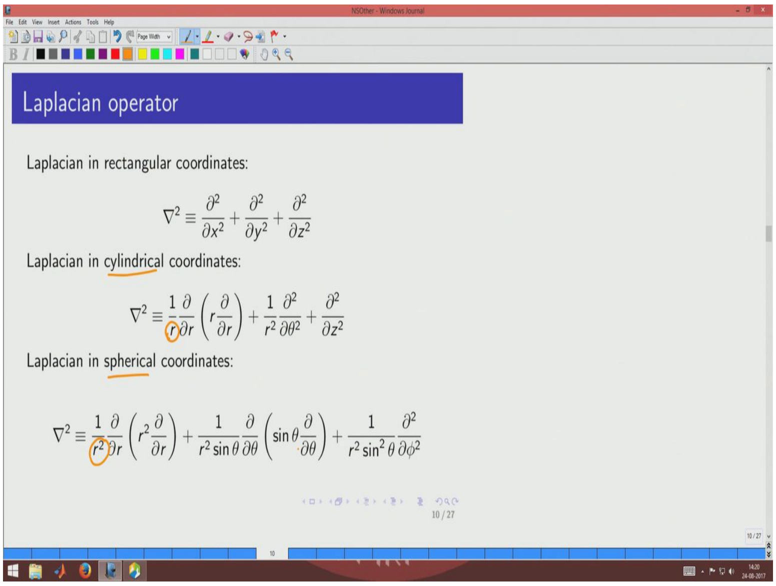

Now, the Laplacian in different coordinate systems is going to look like this you notice that in the cylindrical situation you have r and in the spherical situation you have got r^2 and the remaining terms as we have already discussed in the previous sessions. So, you could see that for the last term here. So, you could substitute these kind of expressions and proceed;

however, on the left hand side for the advected term you will have additional terms because of the nature of the coordinate system being different from the rectangular ones.

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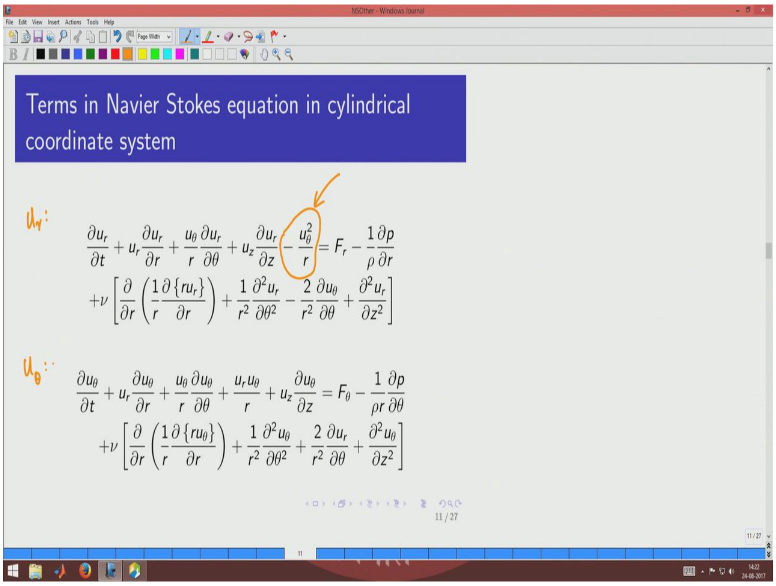

So, how does it look like, how does a Navier Stokes equation look like in cylindrical coordinate system? So, for this you look up the equation from any of the text books, but you can also derive it and we will have in hand out uploaded onto the course website to help you make this derivation it will be quite tedious, but it is possible to derive just knowing how the three vectors in cylindrical coordinate system are related to the three vectors in rectangular coordinate system. That is all the information that you need rest of it whatever you have about the Navier Stokes equation from the rectangular coordinate system you can directly go apply those transformations and you can arrive at these equations.

So, one thing that I was alerting you to pay attention to is like this these terms for example. So, the Navier Stokes equations for the u_r component is given in the first equation and you see that the absolute velocity along the θ direction is coming on the left hand side. So, if you have noticed the Navier Stokes equation you would see that no term will have velocity in absolute form. So, everywhere you have only differences that are coming in, only the derivatives are coming in which means that absolute value of the velocity does not coming to the Navier Stokes equation in a rectangular coordinate system.

However, in cylindrical coordinate system you have the absolute velocity coming in on the left hand side and you could already guess why that comes in. You can see that u_θ means basically the centrifugal force that is coming in for the u_r component. So, these kind of terms have to be paid attention to. So, the Navier Stokes equation to u_r is written here, Navier Stokes equation for u_θ is written here and the Navier Stokes equation for the u_z component is written here.

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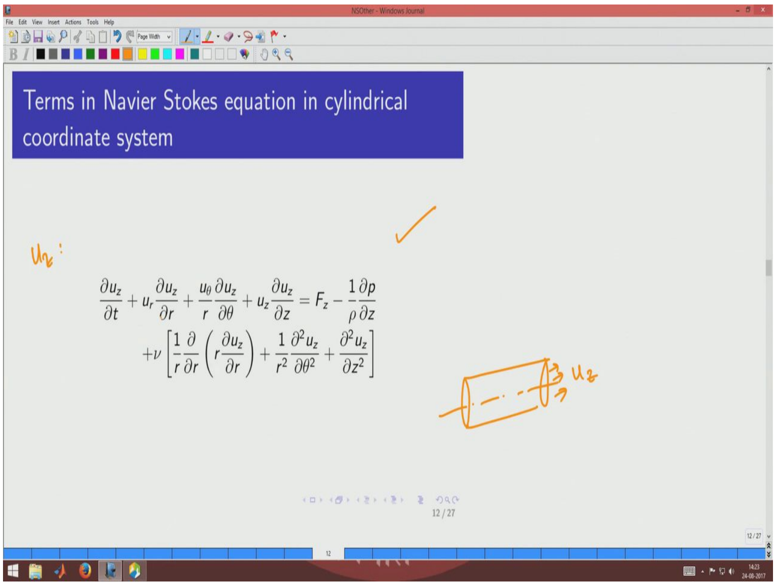

So, you can see that Navier Stokes equations are basically sets of three equations in the case of rectangular coordinate system all the three terms look same whereas, in the cylindrical and spherical coordinate system they do have small variations here.

Now, let us say for the same actual flow we talked about just a while back. So, let us say this is actual flow we are talking about, so which equation should I use then naturally this is the equation I must use. And by making various assumptions we will go ahead and solve how this look like. So, this is the equation that is applicable for the situation.

And for each problem we need to observe which component I want to solve the velocity for and then bring those equations on to the board. Yes, sometimes you may have to have multiple velocity components required, but very often if you can simplify the problem using the symmetry arguments and aligning the coordinate system appropriately if you can reduce the number of velocity components 1 then analytical solutions are available.

Now these equations are written with u and t and r which are basically with absolute scales and very often it will be very useful to have the Navier Stokes equations in non-dimensional form.

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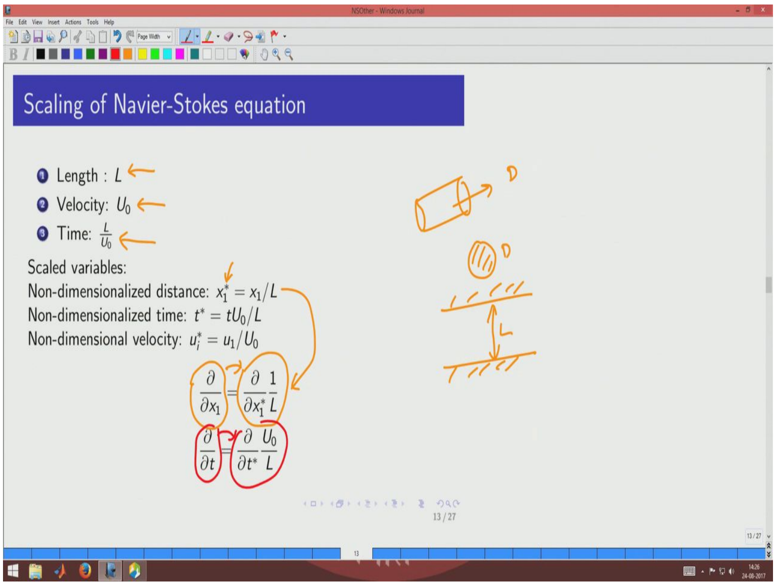

The reason is that when you change one of the parameters by a small number then how does solution behave. So, we should expect that the solution should change only by a small amount and that similarity should then be evident from the equation itself. So, in order to have a solution of Navier Stokes equation for one problem applicable to another problem then non-dimensionalization will be very essential and the way we do it as follows. So, we scale all the variables that are there in the problem. So, we scale the way we know the problem.

So, usually you have a length scale available for a problem. So, we let us say the length scale that is in the problem is L. So, in the case for example, actual flow, we say that the diameter is my length scale. So, in the case of for example, a flow over a sphere then I would take the diameter of the sphere as a length scale typical length scale that is playing a role in the capital. Let us say the flow is in between two plates in a channel then I may choose the distance between the two plates as the length scale. So, we can always identify a length scale that is characteristics of the problem and then choose that as L.

Now the velocity we need to choose a velocity also to scale and U_0 is a reference velocity. So, it could be for example, the maximum velocity that is the encountered it could be the input velocity inlet velocity or exit velocity average velocity and so on. So, whichever velocity we are able to measure in experiment and are able to characterize the problem with then that velocity we could choose as the scaling factor U_0 . And for time we note in now we do not have to now cook up one more quantity we know that the length divided by the velocity will give you the time scale and that time scale is adequate for us to scale the time variable.

So, now what we do is, each of the terms the distances we will use the same length scale. So, once we have identified the length scale the x y z or x_1, x_2, x_3 all of them we will non-dimensional with the same length and all the velocity components we will non-dimensionalized using the same quantity U_0 and the time will be non-dimensionalized using $\frac{L}{U_0}$. So, what we do is that we use the star to indicate that it is a non-dimensional quantity. So, the distance is non-dimensionalized by dividing with L, so $x_1^* = \frac{x_1}{L}$ what we use it for is like this.

So, in our equations we have this quantity. So, wherever what is there we are going to replace with this quantity so that the derivative is now with respect to the non-dimensional distance and the scaling factor has come out. So, we are going to do this for all the terms. Similarly we are also going to do it for the time variable. So, we will replace this with this and $\frac{U_0}{L}$ will come out. So, what we do is we take the Navier Stokes equation and make the substitutions blindly. We do not have to worry about the differentiation of this quantities because we have already chosen these to be characteristics of the problem, so L U_0 are not dependent on anything they just constants. So, you could differentiate take them out as constants no problem.

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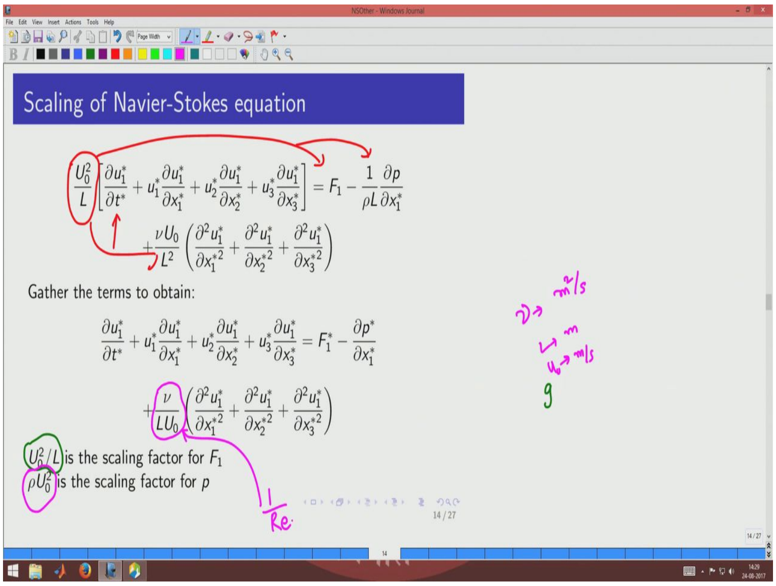

So, we take that and do the substitution and when we do that the Navier Stokes equation will appear like this. So, on the left hand side you would have $\frac{U_0^2}{L}$ that is coming out ok, as a non-dimensional number on the right hand side you have got those other quantities that are coming up.

So, you already know that if you see this quantity it is basically the acceleration and you could see that velocity square by L would also look like acceleration units and therefore, is not surprising that it should be the same for all the three because after all these 4 terms are nothing but they are parts of the material derivative to tell about the acceleration. So, when you do this and correct the terms what we do is we take the $\frac{U_0^2}{L}$ on to the right hand side and see the how the equation would look like. So, we take to the right hand side. So, we take here and here and here. So, when we do that. So, you would see that we are able to now define the non-dimensionalization for the body force and that quantity turns out to be $\frac{U_0^2}{L}$ because when we bring it on the right hand side everything on the left hand side is non-dimensional. So, everything on the right hand side also should be non-dimensional. So, by that argument you can immediately see that the body force can be non-dimensionalized using $\frac{U_0^2}{L}$.

By the way body force term F_1 is nothing, but specific force which we already saw that in the case of gravity would be just g, which is acceleration and you already see that $\frac{U_0^2}{L}$ is acceleration. So, we are choosing the acceleration as units to non-dimensionalised the specific force body force that is coming in here. And for the pressure then we also discover that there is a scaling factor that is available. So, ρU_0^2 is the scaling factor for the pressure and once we do that then you see that every term is non-dimesionalized. And there is a non-dimensional quantity that is sticking in front of the Laplacian term. So, this is the only quantity that is sticking around other than that every term is non-dimensional. So, this quantity also when you plug in it would be also non-dimensional you know that ϑ is m^2/s and L is meter and U_0 is m/s. So, this also is non-dimensional number.

So, now, you see that this entire Navier Stokes equation is characterized by the problem using only one quantity. So, this quantity is what we would like to give a name. So, you could just give the name for this entire quantity as it is and or the 1 by something. So, you could to the inverse or not as you like. So, we choose the inverse of that quantity as Reynolds number.

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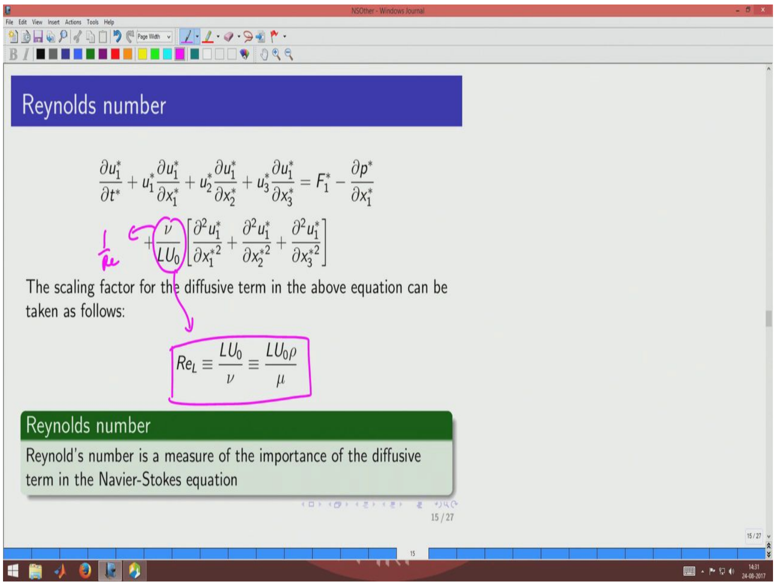

So, we define Reynolds number like that. So, the inverse of this quantity is defined as the Reynolds number. So, you could see that the Reynolds number is naturally coming out of the Navier Stokes equation as a scaling factor coming in front of the diffusive term and the units are wiped off because the non-dimensional is done for all the terms.

Now, it is also basically characterizing the problem because we saw that it is coming out of choosing the length scales and time scales and velocity scales to characterize the problems. So, Reynolds number actually is characterizing the Navier Stokes equation for a given problem. So, which means that the solution of this equation as long as Reynolds number is same is going to be the same which is a very great relief because we do not have to then solve this equation every time. So, we need to first only non-dimensionalize and check the Reynolds number for the given problem the number is same then the solution must look like the same of course, the initial unbound condition also must be same.

So, you could already see that this is nothing, but $\frac{1}{R_e}$. So, which means that if the Reynolds number is small if it is close to 0 then numerically the diffusive term is going to be heavy which means that the solution of this particular partial differential equation is going to be governed mainly by the diffusive term. And if the Reynolds number is very large for example, very close to 10^8, 10^9, then $\frac{1}{R_e}$ would just like 10^{-9} into the diffusive term which means that diffusive term does not play much role. So, this is the meaning of how the Reynolds number is going to help us with this and this is how we say.

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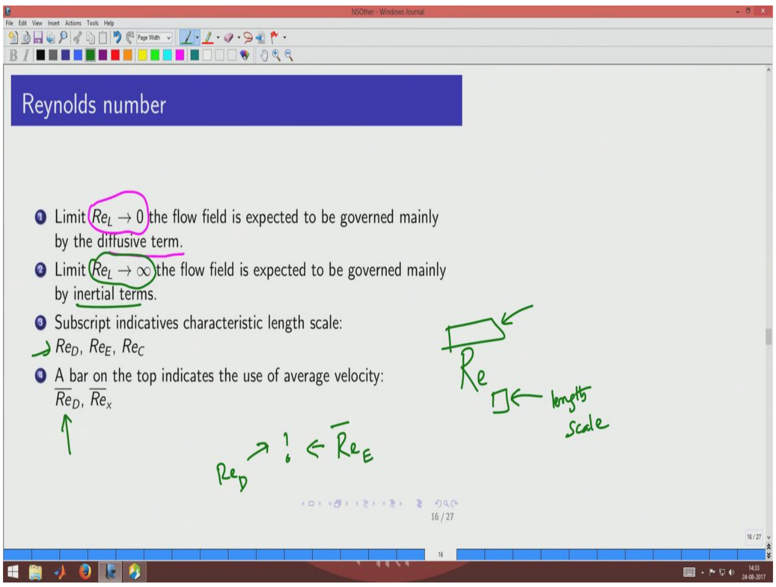

So, in the case when Reynolds number is very small then the solution of the flow field governed mainly by the diffusive term and in the case where the Reynolds number is very large then it is governed by initial terms that is remaining terms that are there on the left hand side. So, this is we can actually see how the solution of the Navier Stokes equation is going to be evolve.

Now, it also means that in situations where we do not have ability to solve the problem completely we can simplify it by choosing the Reynolds number range and dropping some of the terms which we know that are not playing a big role. So, it helps us. So, we can say that in the limit of very small Reynolds number I will keep only the diffusive term and I will drop all other terms and that would be quite acceptable because from the scaling that we have seen that is how the behavior is going to be.

Now, whenever you encounter Reynolds number in any book or any problem do pay attention to what is written at its subscript and at the top. So, at the subscript there will be things that will hint what is the length scale that is being used. So, here if you see the quantities like a R_{e_D} ,where D is the subscript then it means that the diameter is the length scale that has been chosen. So, you could choose the radius or diameter, but diameter is normally chosen because that is what you can measure easily for a sphere for example, you can measure the diameter directly using a caliper. So, like that you know you can observe and see what was the quantity that is used to define that quantity for scaling the length. And on the top if you have a bar which means that some averaging has been done. So, velocity has been averaged or the problem solution being sought is for an average quantity etcetera.

So, do pay attention and the way Reynolds number is defined could be different if any of these two are different. So, do not compare an expression which is written like this and written like this. So, these two can be a very very different. So, pay attention and then if it is looking different at the subscript then look at how it has been defined and it could just mean that the way we define the Reynolds number may be explaining the differences that we have seen the expressions. And we will come to some of these nuances when we come to the friction fractions discussions.

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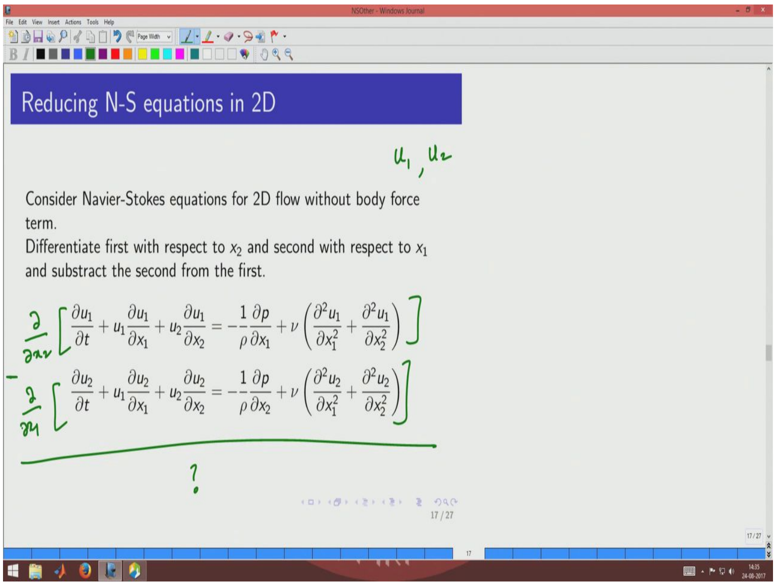

So, now, what we have done in the session on planar flows is that if you have 2D flow you have got two components of velocity and if you want to reduce the number of unknowns to just 1, then we have come up with stream function which can generate those two components and then we saw that the equation just comes down by 1.

So, you could also then write the evolution of velocities in the same manner you could actually see that in situations where you have 2D flows you can also describe the Navier Stokes equation using the stream function. So, the way we do it as follows. So, what we do is that those two components of velocities for the 2D flow we choose them as u_1 and u_2 respectively. So, the Navier Stokes equation in 2D for the u_1 component and u_2 component are written. So, do observe that on the right hand side we have not written the body force term. So, we are this moment we are in neglecting that term otherwise ideally must have body force term on the right hand side so we are neglecting that, other than that this is a 2D version of Navier Stokes equation for u_1 and u_2 respectively.

So, what we do is that the first equation we differentiate with respect to x_2 and the second equation we differentiate with respect to x_1 and we subtract and then see what is coming out ok. So, you could already see why I am a doing a subtraction there because when we have the stream function its derivative one velocity component is positive, another is negative. So, then we subtract we are then able to bring it out. So, we just blindly do this and it is a lot of algebra. So, when you subtract this is how the equation looks like. Just straight forward you could do that.

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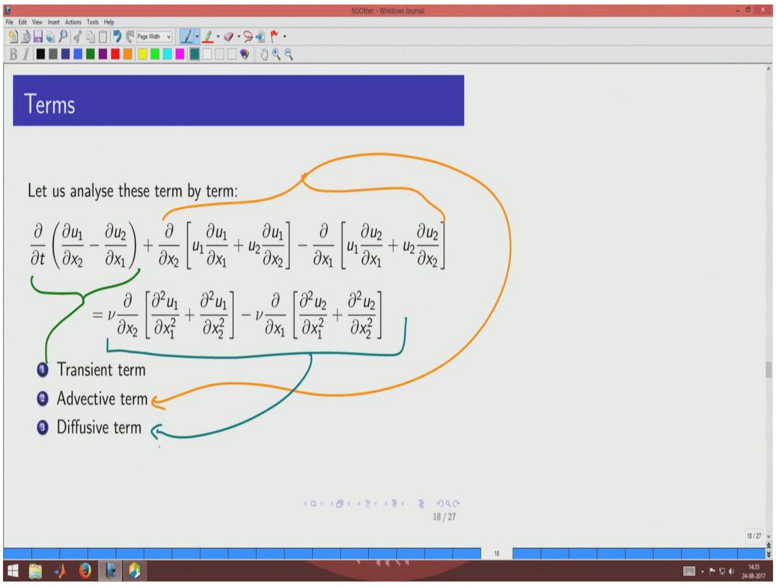

What we do is that we already know the names of these terms. So, the first term was transient term, so we pick this term. And the second term is the advective term, so we pick that separately. And the third term is the diffusive term and we pick it separately. So, term by term we will see how the equation is going to be modified by using the definition of the stream function.

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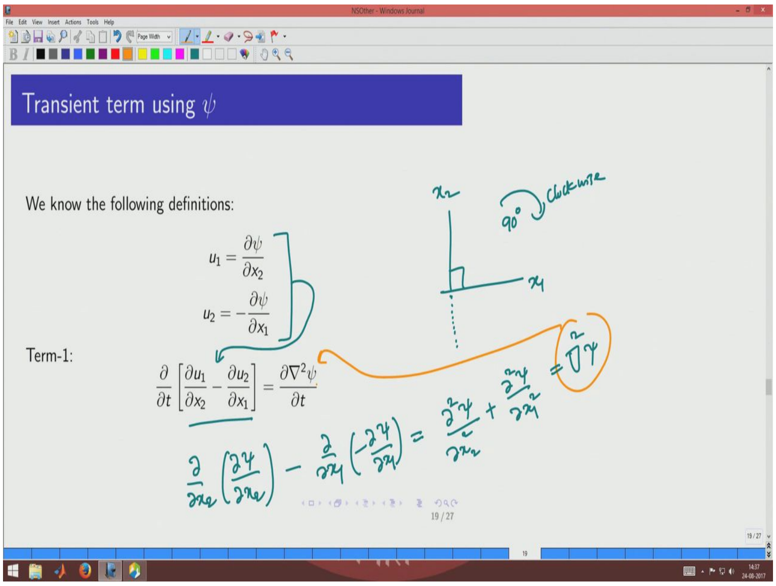

Let us take the transient term first. So, we already know the definition of stream function we see that this is how they are written and we know the convention if you have x_1 and x_2 the 90° clockwise. So, differentiation of a phi along x_2 would give the velocity component along x_1 and differentiation of ϕ along the x_1 we will give you the velocity component 90° clockwise, which means it is minus x_2. So, that is how the velocity component is u_2 for the differentiation along x_1 direction.

So, now, what we do is we plug these two expressions here and immediately we could see that this term is coming out as Laplacian of ψ. So, you could see that this is nothing but the first term if you see $\frac{\partial}{\partial x_2}$ of u_1 which is nothing but $\frac{\partial \psi}{\partial x_2}$ - $\frac{\partial}{\partial x_1}$ u_2 which is nothing but $-\frac{\partial \psi}{\partial x_1}$. So, these two will cancel and therefore, you see that $\frac{\partial \psi}{\partial x_2^2} + \frac{\partial \psi}{\partial x_1^2}$ which is nothing but this, which is what we have then plugged in. So, we have then plugged in here and which means the initial term they for the transient term is nothing but $\frac{\partial}{\partial t}$ of Laplacian of the stream function.

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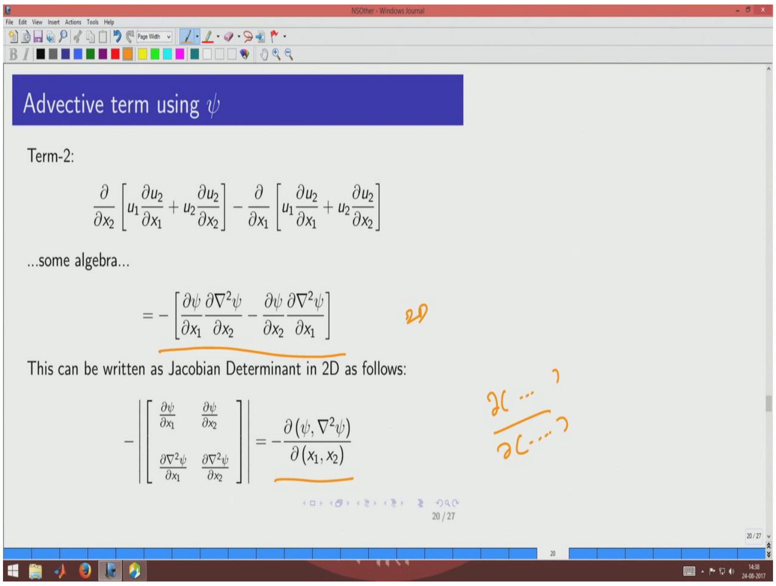

So, similarly we do the manipulation for the advective term also and because there are no terms there will be algebra so I will not do that it is just straightforward differentiation there is no trick or no new concept that has to be brought in its just differentiation by parts. So, you differentiate one at a time and you have got basically 4 terms and when you differentiate we are getting 8 terms. So, all the 8 terms you then collate and you would see that it will come in this form, so minus of this particular combination.

Now if you see this combination you will see the there are some cross terms that are coming in you know the differentiation of ψ with 1 and here with 2 but then here with 2 and here with 1. So, then you could see that may be this is a 2D version of the Jacobian and you could write it as the Jacobian itself. So, we have already seen this particular notation in earlier session. So, you could just basically convert the advective term using the stream function as a Jacobian. So, it is nothing but basically this into this minus this into this that is about it there is nothing more than that. But this notation of something something by something something this kind of a notation is very short to write and that is why we are using that otherwise you will have so many different terms that are coming in.

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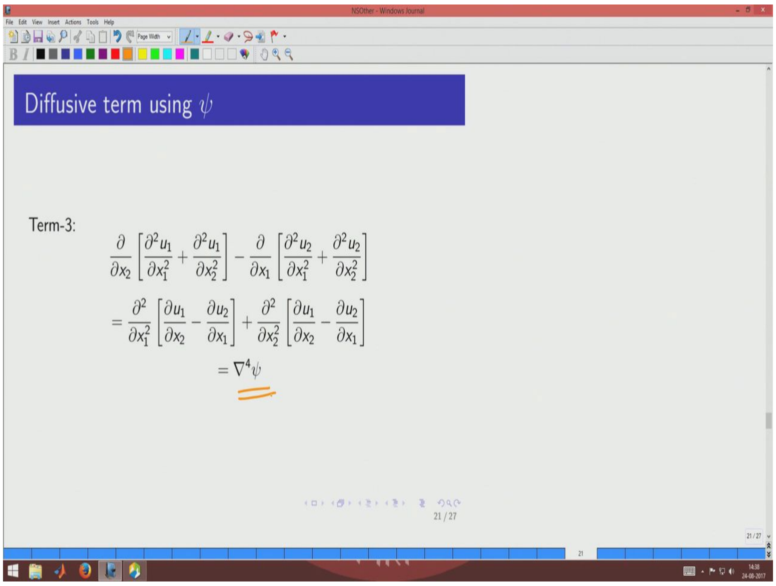

And the last term diffusive term that also can be then modified. So, you have got diffusive term here. So, you just simply directly multiply it in and then take the x_1s out and then collate you would see that that comes as $\nabla^4\psi$ that is the Laplacian operator coming twice with respect to ψ. So, when you put all these three terms together you see how the Navier Stokes equation has been transformed and you could see that it has been transformed with respect to the Laplacian of ψ saying that the local change of the Laplacian of ψ at any location with respect to time minus the Jacobian term is equal to the ϑ that is kinematic viscosity times twice ψ which is basically $\nabla^4\psi$.

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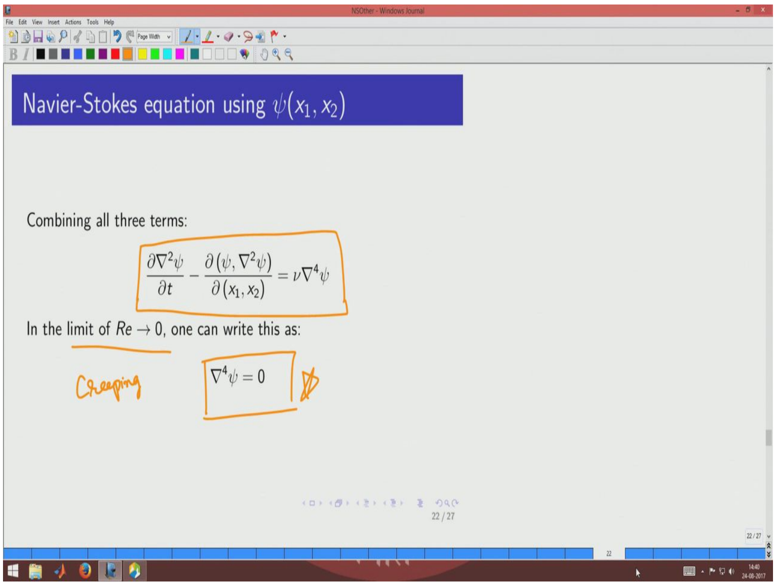

So, you can see that this particular equation we then you know for stream function and we have only one equation. Now if you solve this equation and get the solution for ψ and then differentiate ψ with respect to x_1 and x_2 then you have got the velocity components in 2D. So, with one equation you have got two velocity distributions that are available. So, that is the simplification we were seeking when we introduced the concept of stream function at all.

Now, we can also see that from the discussion on Reynolds number that the diffusive term is going to be governed by the Reynolds number and if the Reynolds number is small the diffusive term is going to be play more role. So, in a situation where Reynolds number is very small, then how I can approximate this particular equation. You could see that the Navier Stokes equation is approximated to such a simple form. So, this also will be given a name stokes equation and we will see that Navier Stokes equation comes out to be a very simple form in the limiting case of Reynolds number being very close to 0 which means that basically creeping regime. So, where the liquid is flowing so slowly that it is able to fall follow all the nooks and corners of the particular you know wall and it is like a creeper going along a tree. So, the flow is very slow and going at a very low Reynolds number and such a flow can be described by ignoring all the terms on the left hand side and writing the $\nabla^4\psi = 0$ as the only equation. So, this is the benefit of writing it in the form of stream function.

So, the same thing can also be then written for other coordinate systems.

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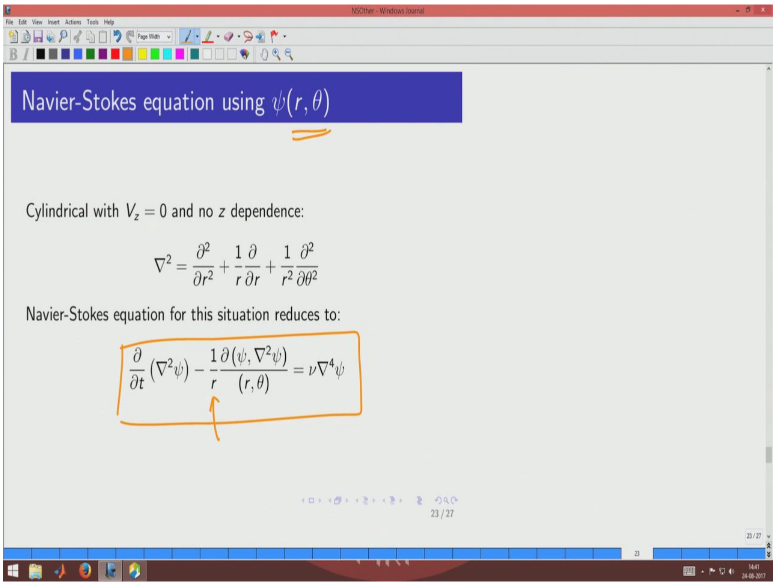

So, if you take cylindrical coordinate system for example, r_θ, r_θ that is the planar flow in a cylindrical coordinate system the z direction velocity is not playing the role then it looks the same way. Only thing is that there is a additional term that is coming in because of the curvilinear coordinate system that we have used, but the nature of the form is the same and in case you are looking at axis symmetry.

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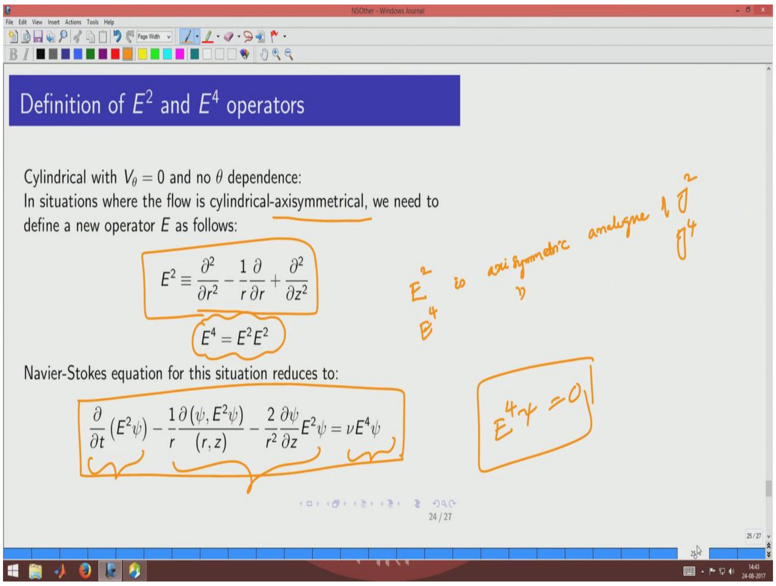

So, axisymmetric flow would then not be described by the same manner. So, you see that the Laplacian operator can be modified to reduce the one of the terms which is causing the axisymmetry come in and therefore, you now need to have the operator, new operator defined E^2 . So, look up the Laplacian operator, we have the Laplacian operator here and you could see that it is not the same, so this and the Laplacian operator and e square operator are different. So, we could see that we now need to give a new name for the operator we have at hand and that is why we call it as E^2. So, we see that the operator is different and so we call it as E^2.

So, this operator is then also use to define one more operator E^4 which is basically the E^2 operator on itself. So, you could see that this operator is basically axisymmetric analog of ∇^2 operator and of course, E^4 is same thing for ∇^4 . So, which means that you can write the Navier Stokes equation in the same manner and instead of ∇^2 operator you put E^2 operator and then look at the terms and you see that a very similar kind of a form will come out as we can look up from the books.

So, these can also be derived by the same process that we have used to derive it for the rectangular coordinate system only thing is that there is an algebra that is involved. But we already know how each of these terms have come about. So, this term is like the Jacobian term and this term and this term we already are familiar how they came about. So, which means that in a cylindrical axisymmetric case if the Reynolds number is very small then how do I approximate the Navier Stokes equation. So, you could already guess that it is going to look like that. So, this is how the simplicity is coming in.

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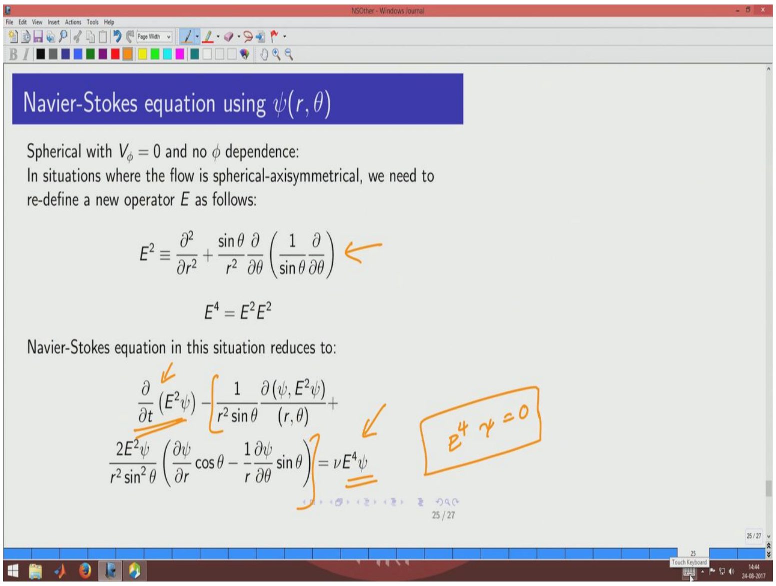

So, the same thing for the spherical case also, so we define it and you see that the E^2 definition is different. So, E^2 operator definition is different for spherical axisymmetric and cylindrical axisymmetric, but once you have defined that way then rest of the equations are derived in the same manner and the Navier Stokes equation looks like this. And again you could see that the terms are very familiar to us. So, this part is like Jacobian and this part we know how it came about and this part being the diffusive term on the right hand side and if the Reynolds number is very small then again we could approximate our Navier Stokes equation using the stream function to be like that.

So, what happens is that we now can give names to such equation which are special cases with restrictions such as for example, it must be axisymmetric case etcetera. So, we have special cases that are coming in.

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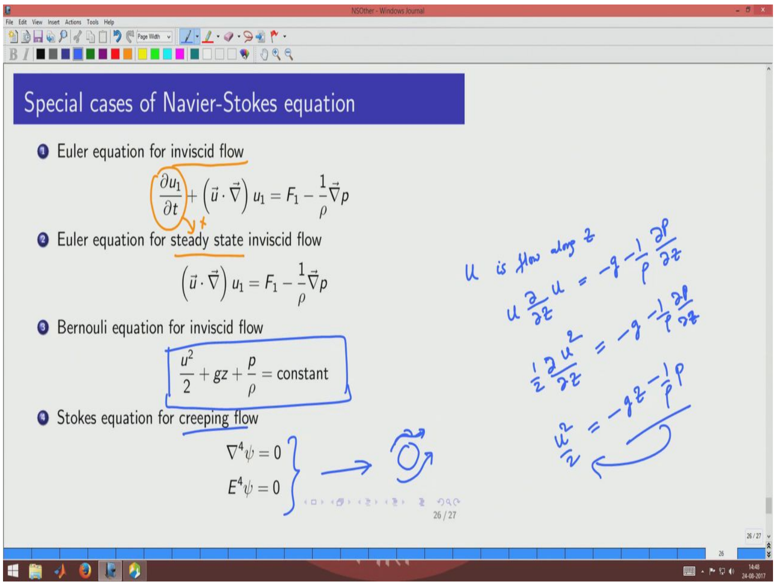

So, there are some special case that have names, so we are only using those now. So, we have what is called Euler equation. So, Euler equation is basically a special case of Navier Stokes equation by setting the viscosity to be 0. So, it does not mean that Euler equation cannot be used at all. So, it should be used in situations where the viscous effects are negligible which means that the liquid viscosity is not playing a role. So, we already saw that viscosity does not play a role when Reynolds number is large, which means that whenever the Reynolds number is large you may get the solution in the form of Euler equation solutions and then go ahead and use. So, you could see that this is nothing, but the Navier Stokes equation without the term on the right hand side corresponding to the diffusive term.

Now, inviscid flow is basically flow where the viscosity is negligible. So, that is a definition again we can tell. So, whenever we say that the problem domain is can be approximate as a inviscid which means that viscosity can be approximated as 0 and then substitute that and then see the rest of the terms.

Now, the same equation for steady state would mean that this term then can be dropped. So, if you drop the first term then you can see that the rest of the equation is staying the same, which means that compared to the Navier Stokes equation we have dropped two terms one term corresponding to the viscosity multiplied the Laplacian operator there and another term on the left hand side with respect to the transient term. So, if you drop two terms on the

Navier Stokes equation we get the Euler equation for steady state inviscid flow which is quite popularly used in many situations.

Now, this equation, when you integrate along the path of the fluid then you get Bernoulli equation, so that can be actually seen in very quick manner. Let us pretend that u is flow along z and there is no other direction that is important and we are just trying to look at how the Euler equation can be modified for flow along z. So, you could see that because of other flow components not being there the advective term can just be only $u\frac{\partial}{\partial z}u$ and u here is equal to this is f, f in the z direction let me take the gravity that is acting and $-\frac{1}{\rho}$ and then the gradient along the z direction that is this.

So, what we do is now we integrate this. So, if you take u in then you could write it as $\frac{1}{2}\frac{\partial^2 u}{\partial z}$ because sorry $\frac{\partial u^2}{\partial z}$ and because if you differentiate you get 2u, u 2 and therefore, you can also then write the rest of it and you now see that when you integrate you get $\frac{u^2}{2} =- gz - \frac{1}{\rho}p$. So, you then take these two terms to the left hand side and then we see that the integration constant has to be kept and the integration constant is coming on the right hand side on the left hand side you have got $\frac{u^2}{2} + gz + \frac{P}{\rho}$. So, this is the Bernoulli equation which is very popular in situations where for example, the height difference is causing a flow or pressure differences are causing a flow, but the viscous effects are not present.

So, for example, in situations like filling the cavity for a casting situation you could use Bernoulli equation and then make some approximation. So, for example, liquid pouring out of a ladle into a caster you do also use Bernoulli equation to make the first guess of how the velocity is coming about.

Now, we have already come across these equations just a while back stokes equation for creeping flow, it is nothing but the diffusive term alone being present. So, in situations like rectangular coordinate system you write as $\nabla^4\psi = 0$ in the case of cylindrical or spherical axisymmetric situation you write $E^4\psi = 0$. So, situations like flow around a spherical or cylindrical body you would use this kind of a situation. So, for each is equation the solutions are applicable in metallurgy we will go through them as we do some of the problems in the coming sessions.

So, we could see that all these equations are basically special cases of Navier Stokes equation. So, it is a very good idea to see how the Navier Stokes equation has been derived and how different terms are dropped when some assumptions are made to arrive at these equations that would be way more generalizing an approach than for example, having to derive each of these equations separately and still not knowing how all of them can be generalized. So, we must always remember that the Navier Stokes equation is the most general form that encompasses the physics that is covered by all these equations that I have listed here.

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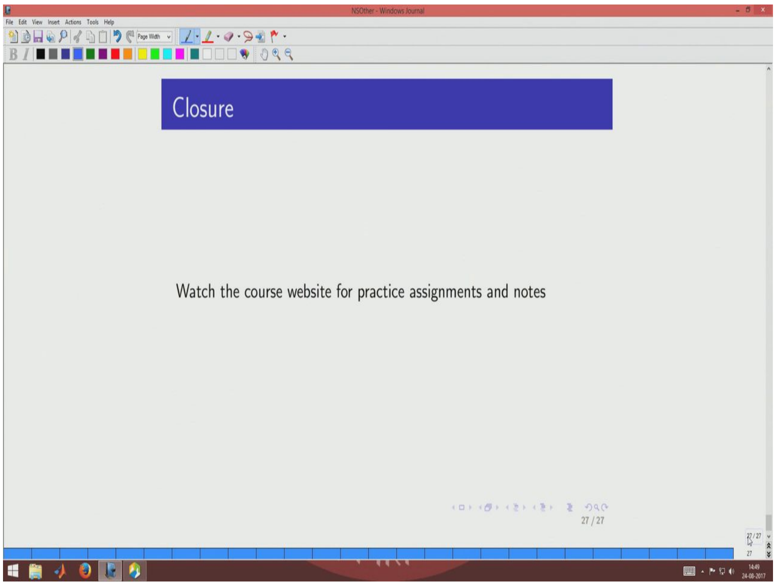

So, with this we close the session and in the course website we will put some problems and we will take up now, in the next sessions problems one by one and we will have the starting point as Navier Stokes equation and then we will go and integrate the terms and see how the solutions would come about.