Transport Phenomena in Materials
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Lecture - 1
Subscript Notation Part 1

So, welcome to the lesson on subscript notation. So, the objective of this lesson is to be familiar with different forms of notations and subscript notation provides us with very succinct and brief way of writing expressions. During the course of this NPTEL move on transport phenomena and materials, we will be doing quite a bit of algebra and the number of expressions, we use can be reduced if we use subscript notations.

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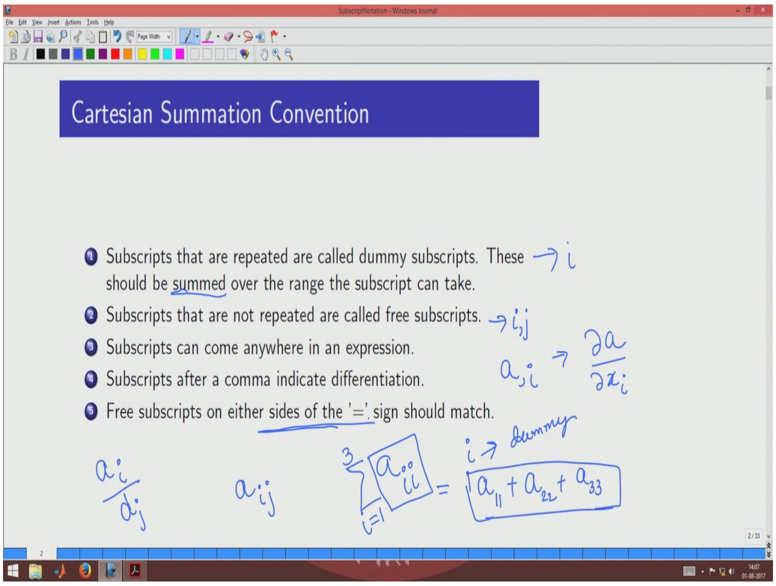

So, here are the simple rules about the subscript notations, it also refer to as the Cartesian summation convention and this has been proposed by Richie long back a mathematician, and it has been popularized by Einstein because he used it in many of his papers later on, and the number of rules are quite few. So, you should be able to commit them to your memory and then use them as we do the derivations later on. So, the rules are as follows subscripts which come below the term that we are using are called dummy subscripts, if they are repeated. For example, you write an expression and if we happen to write 2 subscripts that are same then here i is a dummy subscript.

And in case I have subscript that are not repeated let us a and, then I write 2 subscripts i and j, then both i and j are free subscript. So, here in this example for example, i and j are both free subscripts and in the first example we had i as the dummy subscript and these can come anywhere in the expression, what we mean by that is they could come in the numerator or denominator and the terms can be written as follows for example, you could write an expression like this. So, this is an expression which is valid because it does have the subscripts that are different and numerator or denominator does not matter and we can see that there are 2 free subscript**s** in this notation, we have i and j as free subscripts.

And what we mean by the summation here we mentioned summed what we mean by that is whenever you see an expression of this type what we meant was that i has to be summed and usually it is summed over 1, 2, 3; the reason is we live in a 3 dimensional world, most of us in engineer and therefore, we use the summation over 3 dimensional space. So, this expression what we wrote just now is nothing but, summation as follows. So, it would be $a_{11} + a_{22} + a_{33}$. So, as you can see immediately that when we expand we have several terms, but then when we write it using the summation convention the expression is quite small. So, this much will suffice to write what we otherwise write in this manner. So, this is way the savings comes as making the expression brief.

And the fourth rule is about the comma. So, what we mean by that is whenever there is a comma the differentiation is implied and it is also implied that the differentiation with respect to the location. So, for example, when you write a, i what we mean by that is $\frac{\partial a}{\partial x_i}$. So, it is implied that the differentiation is with respect to x where x is the distance variable and i is the subscript here which is free subscript because there is no repetition there. And whenever we write any expression, we would like to have the subscripts free subscripts matching for every term and including for the left hand side and the right hand side. So, that is what it meant by the 5^{th} role, we say that on both sides left hand side and right hand side of the equal 2 sign the number of subscripts should be matching the free subscript should match, it does not matter the dummy subscript which one we use in different terms. So, technically we could use different dummy subscripts in different terms on the same side of the = sign. So, that is allowed.

So, let us take some examples.

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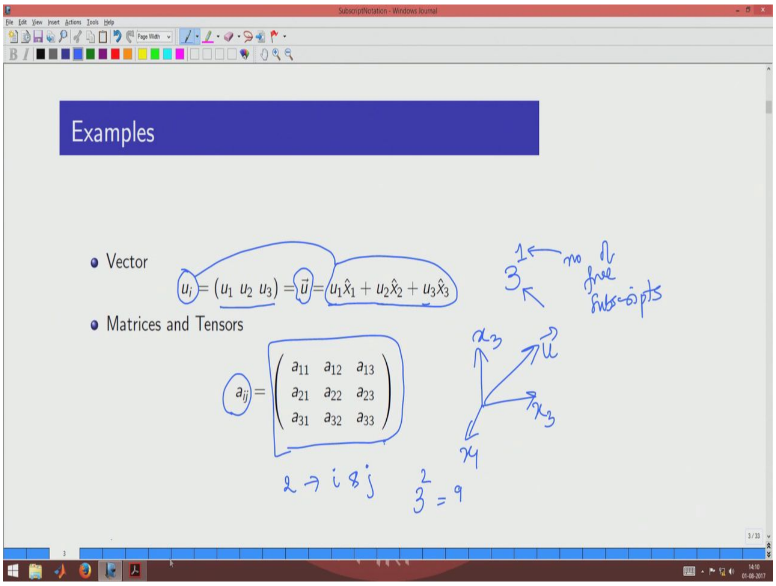

So, here is where we see the savings in the notation. So, what have we written here, we have written a vector, $\vec{u}$ and we normally use this kind of a notation in our engineering classes a $\vec{u}$ is denoted like that, and if you want to express $\vec{u}$ as its components u_1, u_2, u_3 along the 3 axes, then you do have some $\vec{u}$ and the 3 components are written as u_1 u_2 u_3 and. So, the vector can also be written as a set of 3 numbers u_1, u_2, u_3 and here is where the savings will come in terms of the subscript notation u_i is adequate to indicate the entire expression that we are showing here. So, this is adequate to imply the entire expression that we have shown here. So, you can see that the amount of writing is reduced and one subscript is there to indicate that it is a vector with 3 terms. Note that the number of free subscript is here is 1 and the number of terms is 3. So, the number of terms is 3^1 and here this is basically the number of free subscripts and this 3 is because we are talking about this quantities in 3 dimensional world.

Now, there are many quantities in engineering which require more than 3 components. So, vectors require 3 components, but there are many things that require more than 3 tensors second order tensors require 9 components and we do have matrices also require in 9 components, when we look at many of the transformation that we will be doing in the classes further on and So, if you take a matrix like this a for example, the matrix a then the 9 components have to be written tediously in this fashion and you could see that the entire set of 9 elements is indicated by a very simple expression a_{ij}. You can see that the 2 subscripts are not repeating. So, both i and j are free subscripts. So, there are 2 free subscripts they are namely i and j and you can see that the number of elements on the right hand side are 3^2 that is 9 elements. So, this is how we could save the writing 9 elements are written with just one symbol, a_{ij} where i and j are the free subscripts.

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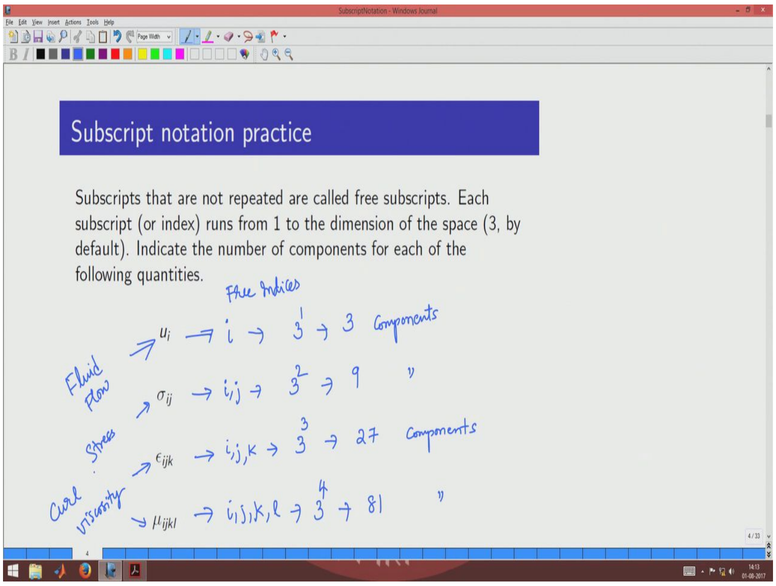

So, here we just practice what we have seen till now, let us say; we are restricting ourselves to the 3 dimensional space and I am asking how many components are there for each of these quantities. So, you could see that the first quantity is u_i. So, you have basically only one let us indicate the free indices, we have only one free index i and because it is only one. So, 3^1 and. So, therefore, there must be 3 components. So, u is a vector and there are 3 components subscript i indicates immediately that there are 3 components.

So, here we have σ_{ij} and there are 2 subscripts and which means that we must have 9 components, and the savings come even more when we go higher dimensional quantities.

So, here we have what ϵ_{ijk} and the free subscripts are all the 3 i, j, k all are free subscripts. So, we are suppose to have 27 components, and imagine having to write 27 components with ϵ; ϵ_{111}, ϵ_{112}, etcetera. So, that is going to be very tedious. So, writing it as ϵ_{ijk} would be very simple, and μ_{ijkl} is a quantity which has 4 free indices and it would mean there are 81 components. And you could imagine that doing algebra with 81 components would be quite tedious. So, using the subscript notation would be quite useful when we take quantities which have many subscripts like this.

So, what do these mean how are they useful to us, do we encounter them in our subject namely the transport phenomena. These things become clear as we go along, but let me preempt myself by indicating that these things are going to come in the following manner. For example, we are going to see u because we are going to look at the fluid flow and therefore, the fluid flow represent it as a velocity will be represented with this quantity and σ will be coming in because we are going to look at the stress shear stress causing the fluid flow will be represented with this quantity and therefore, are going to use that some point.

ϵ is going to use because we want to convert the expressions that will be using the curl operator and μ is basically going to be for us the viscosity, and in the most general form and it is going to require 81 components, but then later on we will see how we were able to reduce all the 81 to just one component for an isotropic material like liquids.

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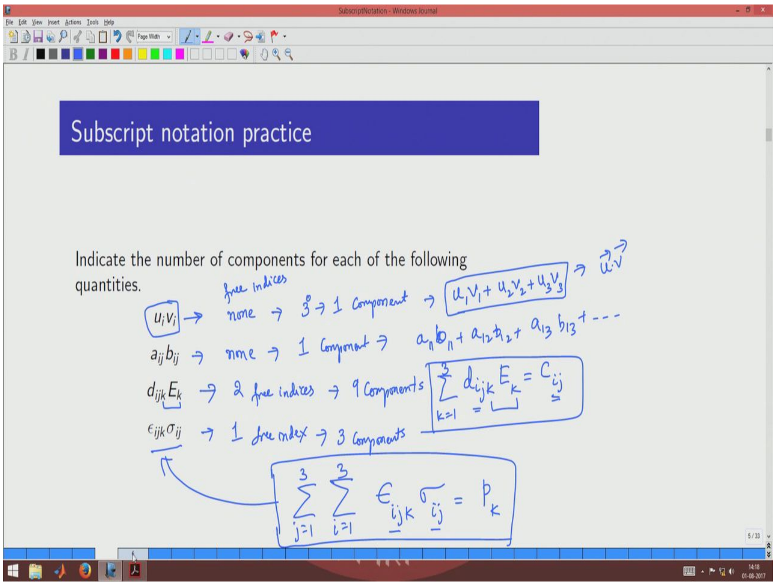

So, we now go further to practice further, now we have not one parameter, but we have 2 parameters. So, look at the first expression and see analyze what is required. So, we got u_i and v_i. So, how many free indices or free subscripts are there? None, because both are repeated. So, it implies that 3^0, there is only one component here and what is that component? It is nothing, but $u_1v_1+u_2v_2+u_3v_3$. And you might already been familiar this is nothing, but $\vec{u} \cdot \vec{v}$ and that is just one number that you get; you can see the savings in terms of the writing.

So, such a large expression is going to be represented by a very brief expression u_iv_i. Take a expression like u a_{ij} and b_{ij} it is a same thing here also. So, number of free indices are none, so which means that there is just one component and when we try to expand this then we would see the savings you can see that they are going to be 9 elements. So, you would have $a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13}+$ so on. So, you could see that all the indices are repeated and you get just one number and we multiply all these quantities.

And here, for example, you can see that the k is repeating and i, j are not repeating. So, which means that there are 2 free indices and this would mean that there will be 9 components. So, you could then write for example, this expression as follows d_{ijk} E_k as some other quantity may be C_{ij}. So, that the free indices are matching on both sides and the summation is over the index k and that you could write it as $\sum_{k=1}^{3} d_{ijk}E_k = C_{ij}$. So, this is the expression that we meant by this. So, you could see that such an expression is written very simply. So, what we have done here is basically 2 things k is the dummy index. So, we are summing up over that dummy index, and i and j are the free indices. So, we are then mapping over to another quantity c, which has the same free indices i and j. So, here the last quantity ϵ_{ijk} and σ_{ij} you can see that i and j are repeating.

So, therefore, the only index that is free is one free index, which is basically a k and therefore, there will be 3 components. So, if there is only one free index k then you could write this expression as follows you could write it as for example, for example, you could write it as follows $\epsilon_{ijk}\sigma_{ij} = p_k$. So, here you can see that i and j and i and j are repeated. So, they both are dummy indices, k is a free index and because its only one index then it means a vector we are talking about and the expression really would be like this $\sum_{j=1}^{3}\sum_{i=1}^{3} \epsilon_{ijk}\sigma_{ij} = p_k$, and when we sum it up and express left and side and we would get the elements of p for each value of k that we are going to write. So, such a large expression is then written in a very simple form here, and that exactly is the purpose of subscript notation. So, you can see that this is what is mapping up here ok.

So, this are; these are the benefits of subscript notation, and every rule of the subscript notation that we have seen must be kept in mind while manipulating these expressions.

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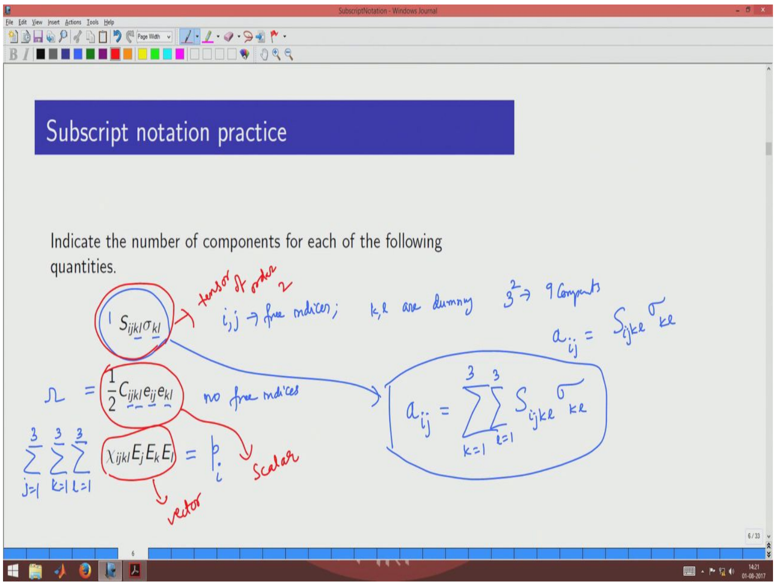

So, we can go further here and look at what are the free indices and what are the dummy indices. So, we can see that k and l are repeated. So, i and j are the free indices here and k and l are the dummy. So, which means that this is a quantity which is like a second order tensor, and it would have 9 components because there are 2 free indices. So, 9 components. So, which means that you could also write this quantity as like this $a_{ij} = S_{ijkl}\sigma_{kl}$. So, this is what you could write and what we meant by the dummy indices as follows, what we meant was this $a_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} S_{ijkl}\sigma_{kl}$. So, this is the expression what we meant by this similarly here also we could analyze what is happening.

The second expression is C_{ijkl} e_{ij} and e_{kl} you could see that i,j are repeated, k, l also are repeated, which means that there are no free indices which means that this must be just a number. In fact, it is n d a scalar and this incidentally happens to be an expression that will give you the energy and you could then write it to be some quantity and you may want to just put something like that. We will not put any subscript there because there are no free subscripts on the right hand side and therefore, there should none on the left hand side as well similarly here in the last expression we have on this slide i,j,k,l are the subscripts for χ , and then there is a vector that is coming 3 times with subscripts j and k and l i is the free subscript. So, you could write this as for example, p_i and what we mean by the matching of the indices j, k, l in 2 different terms, is that there is a summation that is being done over each of them j, k and l. So, this is what we meant by that expression.

So, it is a vector by the way. So, you could see that here is a situation where $\chi_{ijkl}E_jE_kE_l$ is actually a vector, $\frac{1}{2}\ C_{ijkl}e_{ij}e_{kl}$ is a scalar, and $S_{ijkl}\sigma_{kl}$ is a tensor of order two. So, you can generate vectors scalars and tensors of any order by using subscript notation, which takes into account multiple components and choosing some as free indices and some as dummy indices you could generate these expressions.

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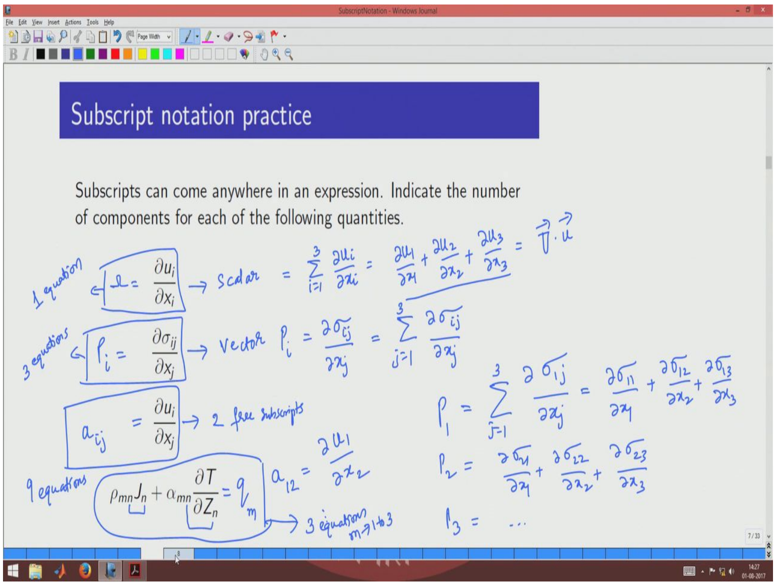

So, here we can see that we are now making use of one more rule of the subscript notation, namely the subscripts can come anywhere. So, in this case we are getting the subscripts in the denominator. So, let us see what we meant by this expression. So, the first expression does not have any free subscript both the indices are i. So, which means that this must be a scalar, and when we expand this what we meant by this expression is nothing but $\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$ which is nothing but $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$ which is nothing but $\nabla.\vec{u}$ which is actually a scalar. So, you could see that this expression is giving you a very brief way of writing the divergences expression that is here. Similarly here the same thing, but you have got one free index the j index is dummy i is a free index.

So, this must be a vector and let us choose this vector to be P and the free index is i. So, P_i, $p_i = \frac{\partial \sigma_{ij}}{\partial x_j}$ and because j is the dummy index this would mean that is this is what we meant by this expression, which means that we could then sum up over the j index and get each value of p. So, you could that ask what would be the value of P_1? There are 3 values of P P_1, P_2, P_3 for each values of i going from 123. So, what is P_1? P_1 is nothing, but this this $\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$ is what is P_1, and you could then expand this with j and you can get what that expression that is nothing, but $\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$.

Similarly, we could write for P_2 which would $\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}$ and so on for P_3. So, you could see that the vector P can be written in a very very succinct form in this manner by using the indices both in the numerator and denominator here. What would be this quantity? This quantity there are 2 subscripts both i and j are free and there are no dummy subscripts. So, this must be something of the order two. So, it has 2 free subscripts. So, it must be a tensor of order two. So, you could write this as a_{ij}. So, for each value of i and j you could write what would be the expression. So, let me write here what would be a_{12} is nothing, but $\frac{\partial u_1}{\partial x_2}$ and so on.

So, you have got basically this equation what I wrote just now is nothing, but 9 equations because for each element for the matrix a; we have got an expression for the right hand side. So, there are 9 expressions that are written. So, what we wrote here is actually 9 equations when we wrote here P, P_i is equal to if write this a equation, then these are basically 3 equations and when we wrote here for example, $\Omega = \frac{\partial u_i}{\partial x_i}$ this is just one equation. So, in subscript notation when we write an equation though we may see, that there is only one expression depending upon the number of free subscripts actually there will be large number of expression that will be coming up you could be see that the same equation very simply written would actually mean whether one equation or 3 or 9 depending upon the number of free subscripts ok.

So, what about the last quantity? In the last quantity we have expanded the complex a little bit by putting 2 quantity side by side, you could see that in the first term we have got n that is matching which means that is the dummy index, the second also the same thing that is matching the free index is m. So, this must be a quantity which is having one free subscript which is m and I may want to choose the quantity to be like this. So, this must be the equation we can write and for each value of m we can write what would be expression on the left hand side by summing up over n, n going from 1 2 3 and how many equations do we have here? We have got 3 equations which basically are for values of m going from 1 to 3 and the 2 terms were evaluated and them summed up. So, the plus sign does not have any significance apart from adding 2 terms just like you add 2 numbers.

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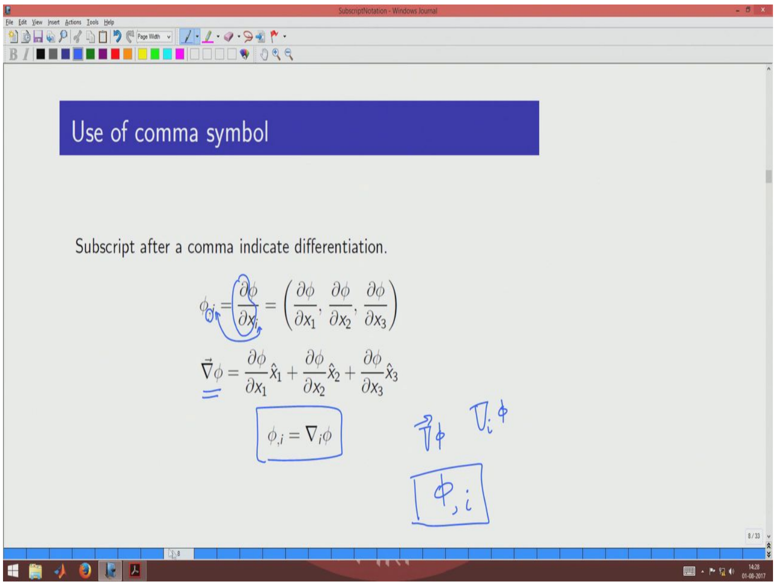

So, the comma symbol will then have to be practiced. So, here we see; how is it used. The comma is basically an implicit assumption that we are going to use it were differentiation. So, here is where it is indicated. So, the comma is coming here it means that we are going to differentiate with respect to the space variable, the i index has to match with respect to the distance variable we are using for the differentiation. So, you could see that the 1 comma there is actually substituting for $\frac{\partial}{\partial x}$ which is a you know there are 3 symbols there are sitting there and comma is just one symbol. So, we have got savings in amount of writing supposed to do there.

So, what we mean by that is you could write expressions like this. So, whatever you have written here which is nothing, but gradient of ϕ is written in this form. So, you could write what is a gradient? you would write this way $\nabla\varphi$ or you would write it in this way

$\nabla_i \phi$, but you could write it with any other symbol the ∇ symbol that is here is not used you could actually just put ϕ, i and this enough to just to say that it is a gradient with respect to the distance variable x_i. And when you combined the all the rules namely the free subscript rules, the dummy subscript rules the rule that allows the subscript to come in both numerator and denominator, and the comma indicate in the differentiation you combine all these things then fairly complex expression can be written in a very succinct form using subscript notation.

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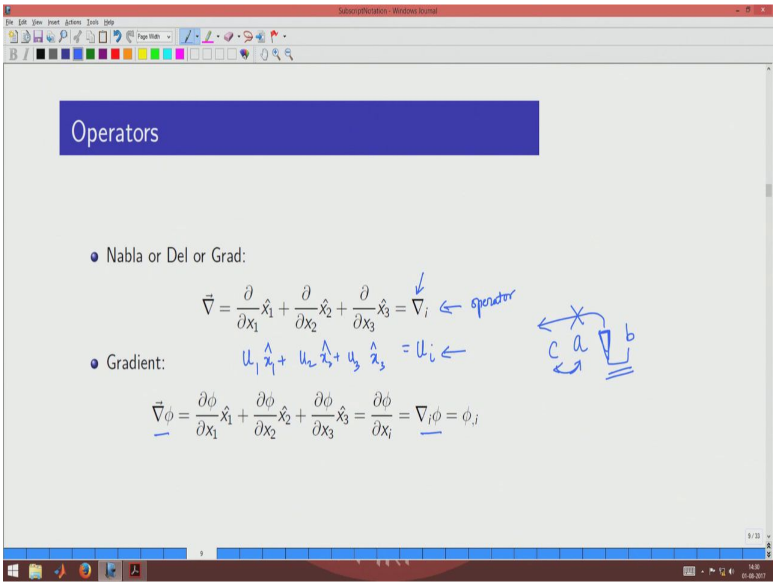

So we can see how we are using this same subscript notation even for operators. So, here if you see the where it is written; it is written somewhat like it say vector you know you could see that i am just writing below a $\vec{u}$ and then I am writing it the way we have seen $u_1\hat{x}_1 + u_2\hat{x}_2 + u_3\hat{x}_3$. So, could see that as if ∇ has a component $\frac{\partial}{\partial x_1}$ along the x_1 direction $\frac{\partial}{\partial x_2}$ along the x_2 direction and $\frac{\partial}{\partial x_3}$ along the x_3 direction, we are writing it with a subscript i, but one difference is that this is vector and this is an operator. So, while vector can come anywhere in expression because it is just a bunch of numbers and then the way we multiply numbers there is commutativity operators have be moved around a little bit carefully because they are going to operate on what is going to on the right hand side of their thing.

So, for example, whenever we have an expression like this, when you have something and an operator and then something else this is acting on this. So, this if you have something else you could switch them of back and forth, but you should not move this this is wrong. Because ∇ as an operator works on b and it should not be moved on the left hand side it can be operating only on what is on the right hand side. So, there are certain cares that must take whenever you using subscript notation, along with the operators though we write as if the operators are like any other quantity will be must take care of that.

But keeping that in mind we can then use this concept to write large expressions even much simpler, you could see what are we writing here. We are writing the gradient ϕ using the operator and then we have put the symbol i there pretending that the operator is also indicated with subscript and treating it on par with a vector ok.

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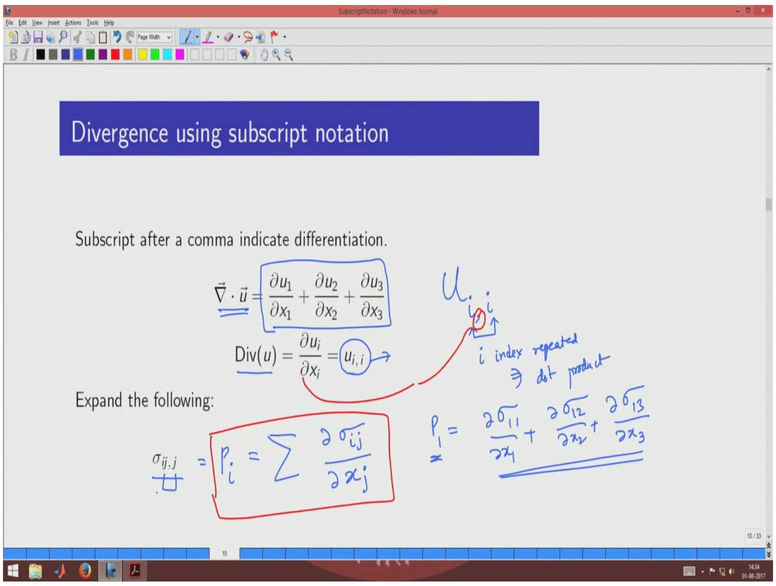

And if you then combine these and we could write it expression that it give some more meaning. For example, this quantity $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$ use divergences as we have just mentioned ahead in this class, and divergence of a vector u is written by this symbol $\nabla.\vec{u}$ and in subscript notation it is written in a very very succinct form like $u_{i,i}$. So, this involves only 3 symbols, here you have got an operator a vector and then a dot so, in that it is a dot product, etcetera, all of that actually implied. So, the way it is implied as follows; when you write something like that $u_{i,i}$ the fact that there are repeated indices. So, i index being repeated implies that we are talking about a dot product here. So, it is a dot coming in there the dot is indicated by this because we are going to sum up over the index i and then this this comma implies that we have got a differentiation that we are going to indicate here. So, very very simple you know the repetition of the index and comma between them indicates the sense that is conveyed by the divergence here. So, if this is what we have understood by applying this for a $\vec{u}$, then what is it mean by an expression like this here below?

So, let us see what this mean. So, you see here this expression the first expression σ_{ij} is not a vector it is a tensor of order 2 it is a bunch of 9 numbers, and then we have got a 2 subscripts that are repeated j is repeated. So, the free subscript is i. So, let us say that this P_i. So, P_i is nothing, but this expression we are talking about; what is this the comma should indicate that there is a differentiation. So, what we mean by is this now what subscript should I use for the x? The subscript that is to be used is what is repeated and after the comma. So, this is what we meant $p_i = \sum \frac{\partial \sigma_{ij}}{\partial x_j}$, and how to get the elements (Refer Time: 30:04) elements of P we can just write element by element we could write P_1 and P_2 and P_3 as follows.

So, you could write it this way. So, $\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$. So, there is a sense of divergence that is coming in, but there is an additional dimension because we have got these as just elements of a $\vec{p}$. So, you could think of this as divergence applied to higher order quantities namely the tensors. So, subscript notation allows you to take the definitions that we know on vectors to tensors without much algebra and the subscript notation also allows you to then interpret these in terms of the complete forms the complete forms are quite tedious whereas, the subscript notation is quite succinct.

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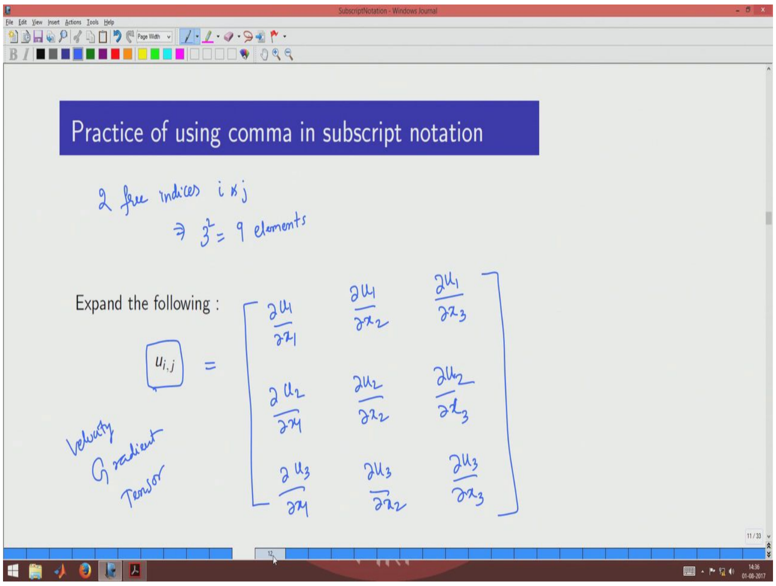

So, let us practice that and write what this mean. So, what is this quantity? $u_{i,j}$. So, we have got here 2 free indices both i and j are free. So, this should imply that we have got 9 elements or 9 components of this quantity and what are those. So, what we mean by that can be written in this fashion. So, the first element 11. So, $\frac{\partial u_1}{\partial x_1}$ second elements is 12. So, $\frac{\partial u_1}{\partial x_2}$, third is 1 3 $\frac{\partial u_1}{\partial x_3}$ similarly $\frac{\partial u_2}{\partial x_1}$, $\frac{\partial u_2}{\partial x_2}$, $\frac{\partial u_2}{\partial x_3}$, ∂u_3 by . So, this is a matrix that we implied. So, let me just correct this,ok this is what we meant.

So, such a large expression with numerators and denominators and ∂ sitting on both sides, this kind of a large expression is just indicated by a very simple expression here and this has a meaning also as we will discover later on in this course, this is nothing, but the gradient velocity gradient tensor. So, the subscript notation allows you to write this expression in a such a simple fashion. So, here what are we doing actually? We are just simply seeing that whenever there are free indices we vary them over 1 2 3 and then list out all the elements and whenever the free indices are not there whenever there are dummy indices, we just sum it up. So, we are just following the subscript notation rules diligently element by element and index by index ok.

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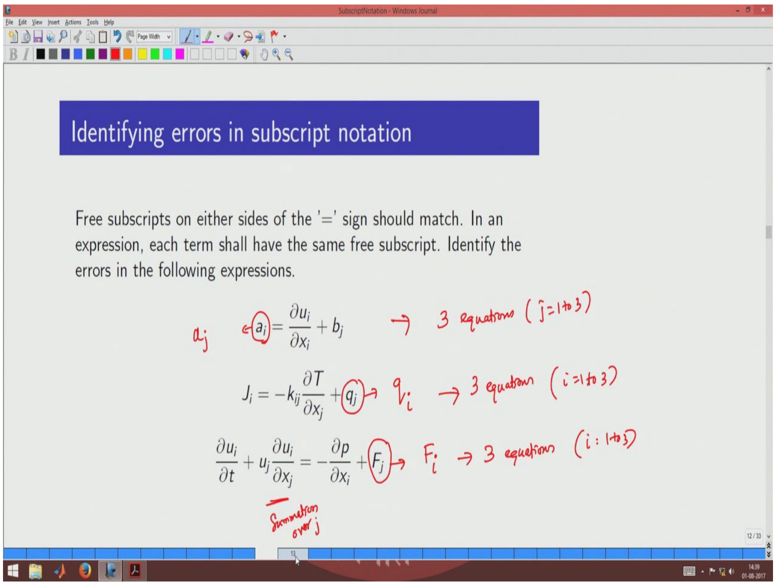

So, here we are going to now see whether we can identify any errors in this subscript notations, now that these expressions are using all the rules that we have practiced till now, we should be able to identify some errors here. So, let us see if we can identify the left hand side has free index i here you have got free index i. The first term you can see that the dummy index is there i and that the third term is free index is j the free index on the left hand is i the free index on the right hand is j. So, there is a mistake there. So, how do we correct?

The way to correct is this quantity should then be changed to a_j, then it will be fine. **So,** $a_j = \frac{\partial u_i}{\partial x_i} + b_j$ and that would be valid there is a mistake that we are correcting this now here. The second expression first index is i which is a free index then the second term the j is matching. So, j is a dummy index i is a free index, the third term j is a free index. So, we have got free index as i in the left hand side on the right hand side the first term it is i. So, we can correct this to make it also i and then the expression will be valid.

So, you can see that in each term the free index is i; however, the first term on the right hand side there is a summation that is involved, but otherwise you are basically writing how many equations are we writing here. We are writing basically 3 equations because the i can be taking values from 1 to 3, the here also how any equations 3 equations for each of j. So, here it is. So, the last equation here we have written the first term i is the free index the second term j is a dummy index i is a free index. So, the free index is matching from both the terms on the left hand side. So, that is fine on the right hand side the first term i is the free index. So, i is the free index till now, but in the last term j is the free index.

So, which means that here is where the mistake is. So, we must correct this fellow and make that as F_i. Now you can see that in every term in this equation, we have got i as a free index only thing is that here you have got a summation happening over the index j, but otherwise we have talking about essentially 3 equations and these equations are written for i varying from 1 to 3. So, you can see that this is how expressions are being used to practice a subscript notation, which are very succinct. So, instead of writing 3 equations where the second term on the left hand side have a has a summation we have written very very very very concise form by using the subscript notation and then 3 equations are written with just one equation, where the index to be kept to be free there in this case it is i is a free index. So, this is the; I would say the power of subscript notation, in being able to write a lot of things in a very succinct form ok.

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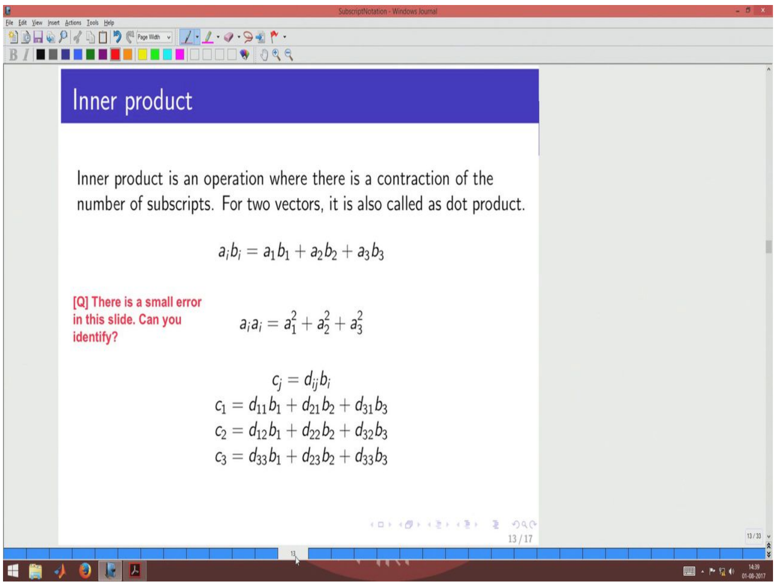

So, the subscript notation can then also be used to express things that we know and to even do some derivations of vector identities etcetera.

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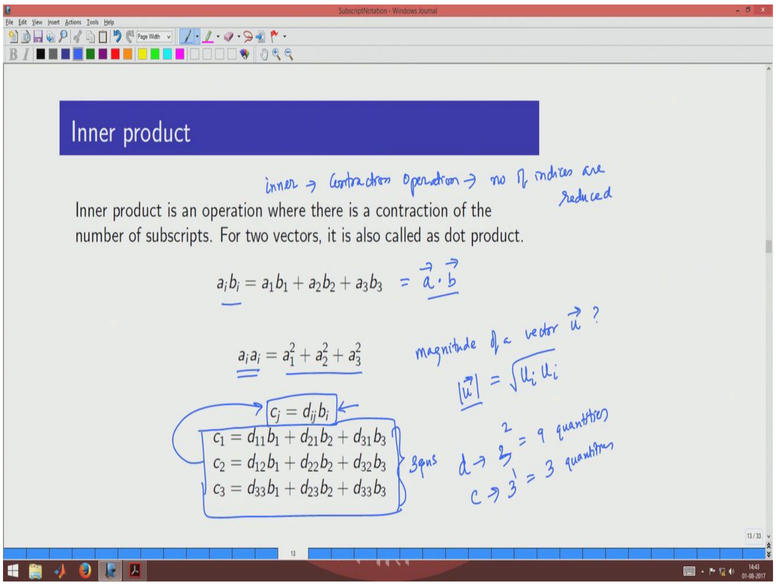

So, here we see some terminology that is coming in. So, I want to make it familiar to you. So, whenever we saw that this subscript is coming in one term with repetition we already mentioned that this like a dot product. Now the sense of dot product is usually of a vectors and when you go to tensors, we must use a term that a little bit more generic. So, we use the term which is inner product. So, inner product is basically contraction operation. So, what we mean by that inner means it is a contraction operation, what does it mean? It means that the number of indices are reduced in the expression. So, you could see that the expression here i and i there are 2 indices, but they are reduced to zero because it is repeated.

So, this expression uses the contraction operation over the index i and therefore, it is basically an inner product of 2 quantities which are vectors. So, this is nothing, but what we already know from engineering mathematics it is nothing, but the dot product of 2 vectors a and b. So, the dot product is written as 2 repeated indices and it is written in a very simple form here. You could generalize this to any order tensor also, but for vectors we already know what it is and you could expand this meaning to higher order quantities as well.

When the quantity b is same as a then you can see that the second expression is coming in and we can see that this is nothing, but the magnitude squared. So, what would be the magnitude of a $\vec{u}$ in subscript notation? So, we could see that it is nothing, but $\sqrt{u_i u_i}$ that is nothing, but the magnitude of the $\vec{u}$. So, you could see because u_i and u_i coming together would mean that it is basically a quantity like this a dot product of a vector with itself and that would then it give you a square of magnitude. So, taking the square root you get the magnitude of the $\vec{u}$. So, familiar quantity is like a dot product a magnitude can be expressed within the subscript notation in the following manner. You could also use it the idea of contraction operation can be used over quantity that are off higher order also.

So, what are we doing in this expression the index i is reduced and only the j is remaining. So, you could see that on the right hand side d is a bunch of 2 3^2 that is basically 9 quantities or 9 components and on the left hand side is C is a vector. So, there are 3 quantities. So, a quantity that has 9 components is being used to generate a quantity that has 3 components using this expression and in this process what is happening is one index is lost and that has been contracted. So, you could see that $d_{ij}b_i$ has undergone a contraction operation over the index i and then we are summing it up over index i and that summation is what is given in these equations and you can see that there are 3 equations.

The reason is that c is a vector with 3 components. So, one equation are component. So, 3 equations are written in this form. So, you could then compare how much of algebra is written here, and all this is implied by just this expression. So, that is exactly the idea of subscript notation or the summation convention giving you very brief way of writing long expressions.

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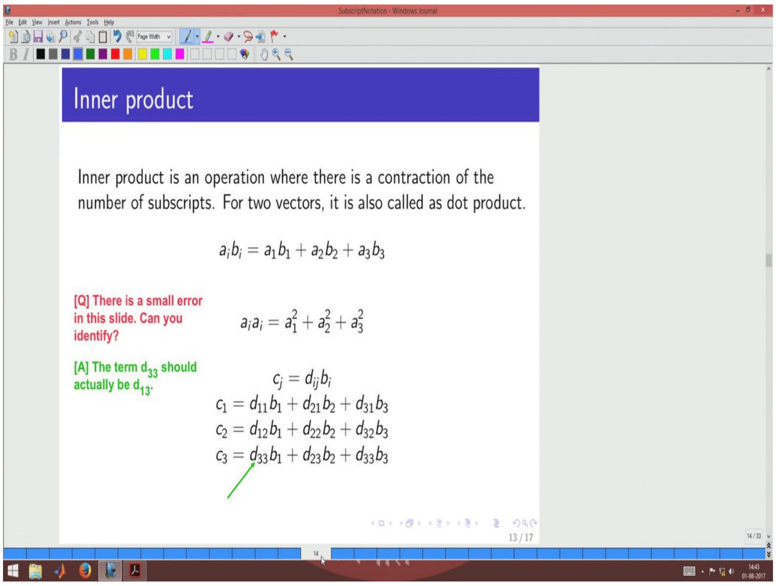

So, just like there as any inner product which means it is for the dot product a counter part of it is outer product.

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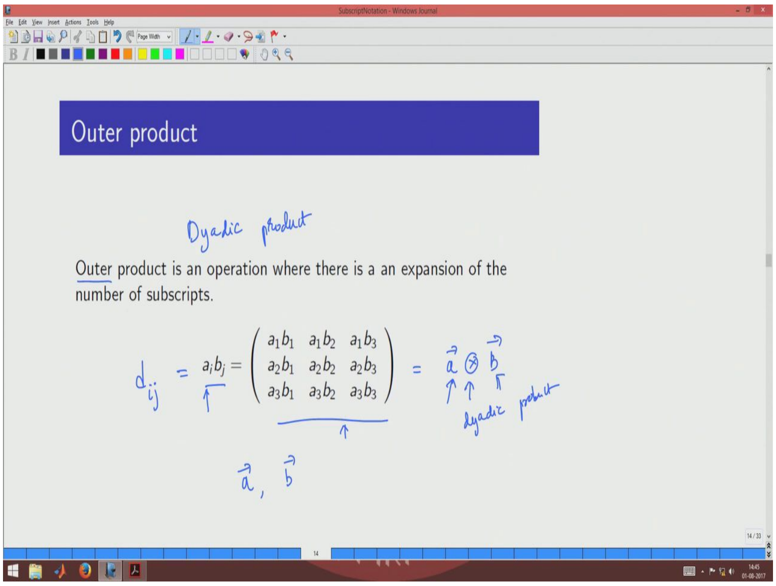

So, outer product is basically a dyadic product. So, what we mean by that is as follows. So, when we were discussing about vectors, we knew dot product we knew cross product, but when we generalize this concept then we have what are called inner products and outer products and inner product is matching with idea of dot product in some way, but the outer product we have not used that in vector algebra, but here we can see how it could be used.

So, here we have seen that a set of 2 vectors $\vec{a}$ and $\vec{b}$; a is a vector and b is a vector, but then what we have written here is a matrix of 9 elements made by using the elements of a and b. So, its a dyadic product of a and b. So, sometimes people do write this in this following (Refer Time: 41:33). So, this is a operator it is not a dot product or a cross product it is a dyadic product. So, which means that it is special way of multiplying the numbers, such a way that from 3 elements of a and 3 elements of b we get 9 elements of a quantity which if you want we can say for d_{ij}. So, it is a matrix of 9 elements and each of those are related in this following manner. So, a_i b_j is a dyadic product of 2 quantities a and b.

Now this can be generalized to create higher order quantities from lower order quantities, what did we do in inner product we have taken vectors and made them into a scalar, which means that we have taken higher order quantities and reduce them to lower order quantities by order we mean the number of elements we require to represent it. So, here in the outer product, we have done the opposite we have taken lower order quantities in every vectors and then created higher order quantities out of them namely the tensor in this case.

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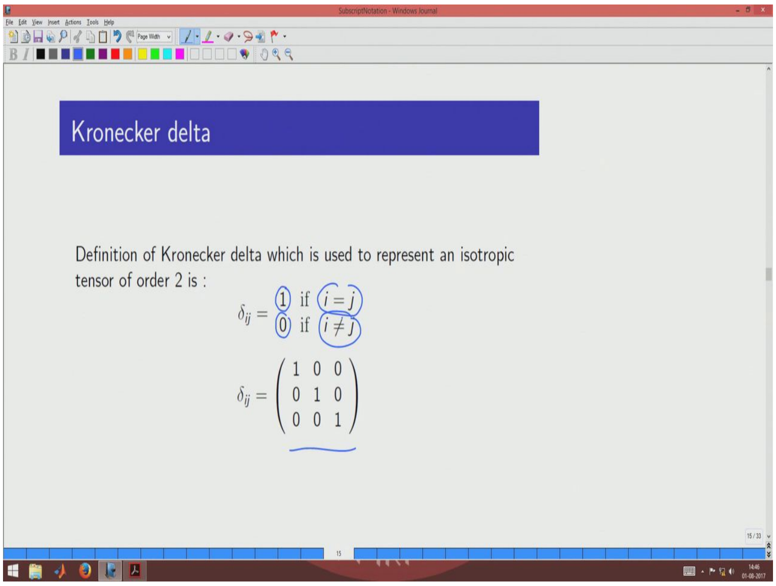

So there is a special quantity which has 9 numbers and has a very interesting behavior when we use in multiplication and that is called the δ. So, δ is represented here with 9 numbers 3 of them as unities and 6 of them as zeros. So, δ_{ij} is a very special second order quantity and this can be used with special properties because it allows us to do multiplications, because of its properties that it will take a value of 1 when i = j and value of 0 when $i \neq j$. So, this particular property makes it possible to use in multiple ways, let us see in what ways it will help us.

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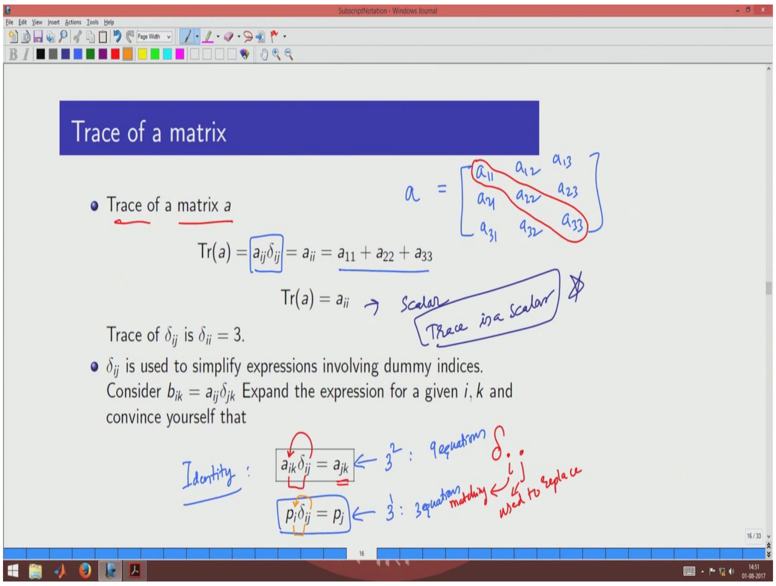

So, let us look at the expression like this matrix a is given by this a_{11}, a_{12} a_{13}. So, this expression is nothing but a inner product of a with δ and contraction operation is over the indices i and j both. So, it must be giving as just one number and we can see that this when we expand we only get 3 numbers because δ has only 3 unity rest of them are zeros. So, only when i is equal to j we have got unity there. So, only those numbers will survive. So, what is the quantity that is written here? It is nothing but a summation of these which is nothing, but trace of a matrix a. So, this is the idea.

So, whenever you take a second order tensor or a quantity that has 9 elements a matrix that has 9 elements and if you multiply with δ such that both the indices are being contracted, then the outcome is nothing, but just the trace. And because there are no subscripts this quantity both the indices i and j are dummy indices. So, this actually is a scalar which is something that is new for us to know that is trace is a scalar. So, the meaning of this we will come to it in a later lesson, because it shows us what are called as invariant quantities of the expression and here we are encountering that is something for the first time ok.

So, it is also used the δ is also used to simplify expressions so that the dummy indices can be modified. So, the way it is done is as follows, because we know that δ is taking a value of one only when the 2 indices are matching. So, it implies that whenever it is multiplied with a quantity, the one that matches is going to be then replaced. So, here we can see that i is matching. So, look at the way this quantity is done i is matching. So, the remaining index is going to be used here. So, the j is going to be then replacing that index and then we have got a j k. The way δ is used is to replace the indices of expressions. So, what we do is basically look at the indices, if δ has 2 indices in this case i and j see which of them is matching with the quantity and then put the other quantity there.

So, this is matching. So, then j is used to replace. So, we are not touching the second index of a, that is k and that k we are not touching because the index k is not appearing δ. So, we are not going to touch that the first quantity a has 2 indices i and k, k is not appearing anyway. So, we will not touch that the first index i is repeated with the index of δ. So, we see that the δ has i and j. So, the i is matching. So, the put that j index in its place. So, we get the quantity a_{jk}.

Now, this is a an identity, what we mean by that is this can be proven. So, you can prove it and do element by element and verify that it can be verified by visual inspection, but if you are not convinced you could actually do that element by element. How can equation did we write here. Basically we have written 9 equations, because for each value of j and k we have got one equation and on the left hand side we have got a summation because there is an i is that is matching index. So, we use a summation of i and only when of course, i matches the index j, then you have got some number otherwise not. So, we have written 9 equations out there.

So, look at this expression how may equations is this. So, these are basically 3 equations and what did we do the transformation, we have done the transformation of indices same way the i index is matching with that of δ. So, the j index is then going to be replacing there that is what p_j has come. So, we could see that whenever we have a necessity to change the indices of some quantities and if those quantities have δ, then we are lucky because the indices of δ can be then use to swap the indices of quantities with which we multiply and that will be of great use later on when we are trying to apply the subscript notation to prove vector identities ok.

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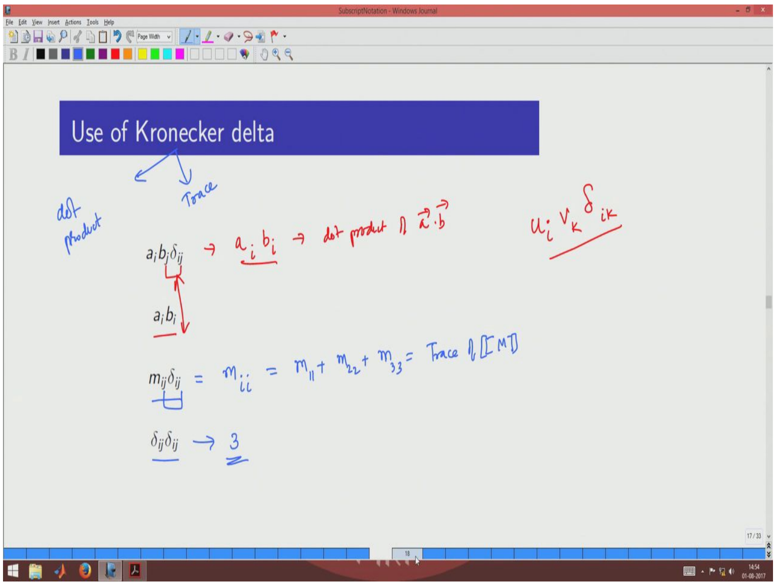

So, this is how you can then apply the δ and write the expressions simpler forms. So, the expression has more terms that we can reduce them. So, what would be this expression first expression? So, you could pick one of the 2 quantities; a_i b_j and both of them are in δ. So, we can take a pick let us take the index of b. So, the j is matching here. So, I will put the quantity there a_i and then b_i. So, you could see that this is nothing but $\vec{a}.\vec{b}$. So, which mean that a very generic way of writing dot product is nothing, but writing the 2 vectors in the respective indices, and then multiplying that with a δ with same indices, that are matching.

So, if you want to write the u_i . v_k and then you would put a δ_{ik}. So, this becomes the dot product of $\vec{u}.\vec{v}$. So, δ is that way has a special property to be used in dot products. So, what would be this quantity it is a same quantity that we have written here. So, you could see that form here to here; what are we done, we essentially replace the index of b from j to i. So, δ has the property to replace indices that has been observed here.

What about the quantity here m_{ij} δ_{ij}. You could see that δ actually can be used to transform any of them. So, you pick the index I want to pick the index j. So, then I would write it as m_{ii}. Now this is something that we already know this is nothing, but $m_{11}+m_{22}+m_{33}$ this is nothing, but trace of a matrix m. So, you could see that the property of δ is coming multifold, one way is in the dot product, another way is in getting the trace. So, for the dot product we have seen how it can be used for the trace you can see how it is used here and what would be $\delta_{ij}\delta_{ij}$. So, this would be 3 because then you can see that this nothing, but the trace of the matrix. So, here we have seen that when the δ is coming we are getting the trace of m, but the m is nothing, but here the δ itself δ trace delta is nothing, but the diagonal element summed up and that would be 3. So, you could see that these expressions are going to be helping us in reducing the number of terms. So, whenever we have got multiple deltas, we use these expressions to reduce the number of indices and that makes the working with the subscript notation a lot easier.

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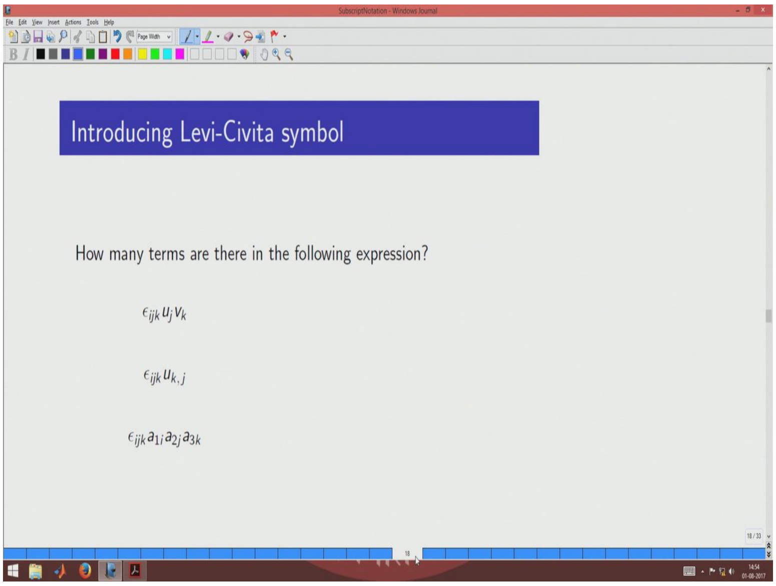

So there is another quantity that I would like to introduce, we will take it up in the next session. So, at this moment we will stop and then continue in the next session.