**Acoustics & Noise Control**
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**Module - 04**
**Lecture - 07**
**Reflection Of Plane Waves 1**

In the last time we talked about the D'Alembert solution of the wave equation. So, that was derived to you.

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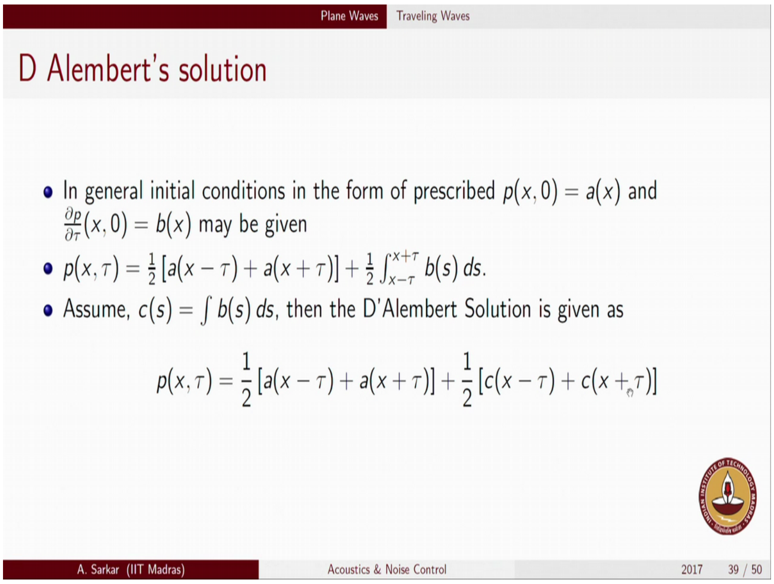


Remember, this type of solution is applicable for infinite domain problems, wherein the spatial domain extends up to infinity and for any specified time instant you have both the condition for p and del p del tau which basically stands for time derivative is known. So, using the derivation process that we went through in the last class, we are able to argue that this is the total solution that we will get.

So, we see that the solution depends not only on the initial condition on p, but also on the derivative time derivative of p, and that is obvious because you have a second order partial differential equation both in time and space. So, therefore, in this form we could also do a little bit of further simplification if the function c is define to be the indefinite integral of the function b, remember this is a definite integral where the limit have been taken from x minus tau to x plus tau, x minus tau to x plus tau is the integration limits.

So, therefore, what you have here is that c is the indefinite integral as per the substitution and what we have here is this what is incorrect is this plus sign it has to be c of x plus tau minus c of x minus tau.

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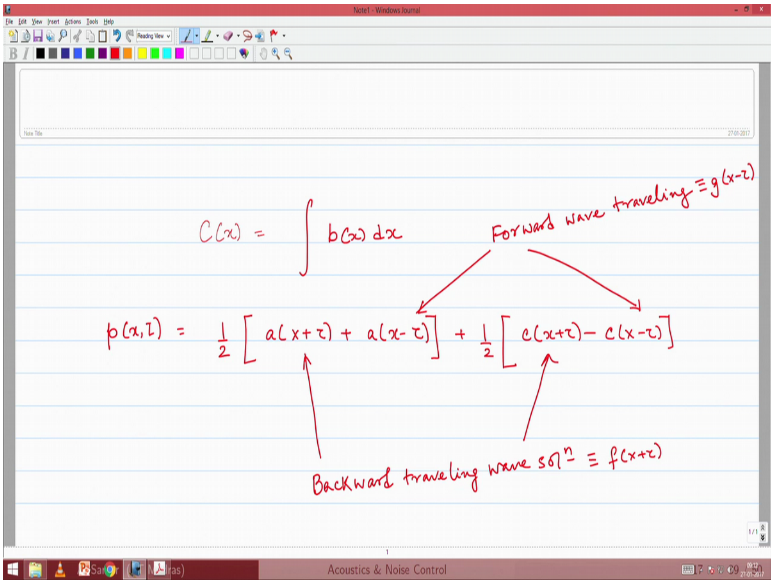


So, I will just write this once more in my notepad here. So, C of x if it is defined as the indefinite integral of b x d x, then p x comma tau the total solution which was derived as half of a x plus tau, plus a x minus tau plus half of c x plus tau, minus c x minus tau right. So, I will make that correction here in the slide this part of it is mistake, but let us try to understand what is the physical implications of this solution; please recall that functions where in the argument is x plus tau simply means that it is a backward travelling wave solution. So, these two components are backward travelling wave solutions.

We know that the general solution of the wave equation bears two components 1 which is forward traveling and the other which is backward travelling. So, we directly see the backward travelling wave component, we also see the forward travelling wave component which is this. So, this should not be any surprise that the D'Alembert solution is actually not giving you any new kind of solution, it is simply specifying that function f and g; remember what we did earlier is that we simply said that the partial differential equation has a general solution of the form f of x plus tau and g of x minus tau, what we are doing now is that we are specifying what f and g functions have to be.

So, in other words this f and g functions are to be related to the initial conditions. So, these two taken together is basically f of x plus tau. So, f of x plus tau is basically half of a x plus of tau plus half of c x plus tau, and the other part the forward travelling wave is g x minus tau the part which includes the argument x minus tau. So, we see that how from the initial conditions we can break it down, break down the response in terms of two waves the forward wave and the backward wave; let us first considered a special case.

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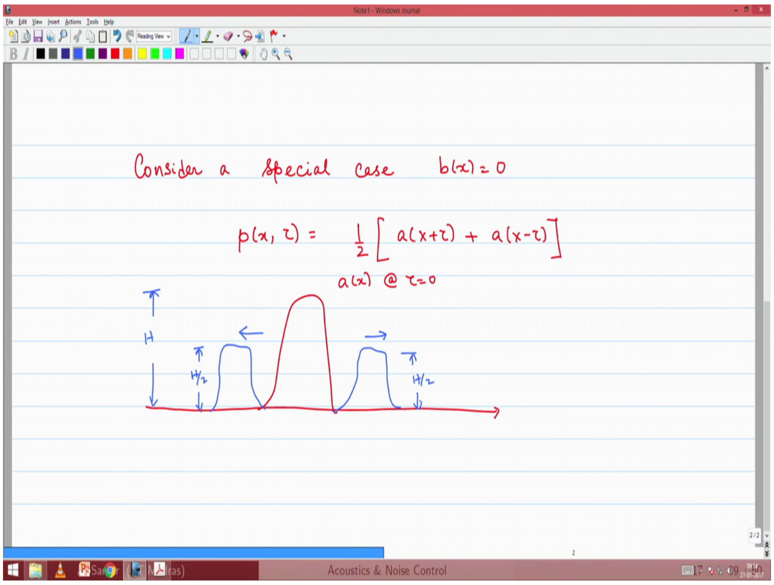


So, consider a special case with b x equals to 0, in that case what happens to the solution p x comma tau is half of a x plus tau plus a x minus tau is the total solution, which what does it mean; that means, at tau equals to 0 if we have a certain wave from. So, this is a of x at tau equals to 0 right this is the initial condition that has been specified what happens at a latest time it will bifurcates into two parts half the amplitude it will travel in the forward direction and with half the amplitude it will travel in the backward direction.

So, I will just change the colour to indicate what happens at a later time. So, at a later time you will have the same waveform, but with half of its amplitude. So, I better be careful. So, you this wave which is shown in red will now split out in two directions each with half the amplitude. So, whatever is the height here let us says H the height will be H by 2 and the height here will be H by 2. So, these make perfect sense because both the directions are asymmetry. So, whatever is the initial condition that initial condition enforces two waves going in opposite direction with half the amplitude of the initial condition, the same can be proved for the other special case where we considered a x equals to 0 and b x is non zero.

B x means non zero means c x will be non zero and again you will see two waves we will separate out right. So, therefore, for both the conditions whether it is in terms of the pressure or the derivative time derivative of pressure, you are going to say that each of them will lead to two waves which will split out; just there is the issue of negative sign which essentially means. Though for the initial pressure condition the two waves will have the same amplitude even in terms of sign whereas, for the c waves you will see the forward wave and the backward wave will flip in sign. So, what we see now is that the D'Alembert is given by the above formula which is written here.

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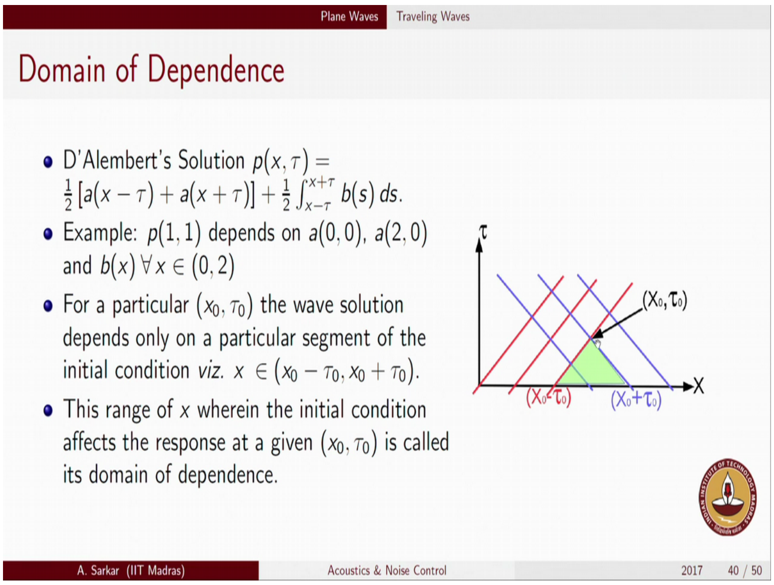


So, considering a point x 0 tau 0 we identify that these red lines are the characteristics of 1 kind, this is the characteristics for the forward travelling waves and these are the characteristics for the backward traveling waves drawn is blue; what we had earlier in our discussion of just wave solution we have identify that what if it is a forward wave whatever happens at the x 0 tau 0 is same as whatever happens at this point which is on the same red line. Similarly if it is a backward wave all points on this blue line will have identical response, but that was under the consideration that you only have a pressure based condition you do not have a derivative of pressure with condition.

So, if you look back now at the complete general solution as given by D'Alembert solution you figure out that what happens at this point x 0 tau 0 depends on let us say this point which now can be identified as x 0 minus tau 0, because this is a 45 degree line and similarly this is also a 45 degree line just that the orientation is opposite between the red and the blue. So, what happens at let us say 1 comma 1 point will be dictated by the initial condition of a at 1 minus 1 which is 0 and 1 plus 1 which is 2.

So, 1 minus 1 happen to be this point and 1 plus 1 would happen to be that point. So, we are tracing the response at a given point in or at any point in space time, to some other point at time t equals to 0 or tau equals to 0. So, what happens at this x, x 0 comma 0 is exactly the same as what happened at the initial time tau equals to 0, but at x value corresponding to x 0 minus tau 0 and at x value corresponding to x 0 plus tau 0. So, this is exactly the same as what we had figured out in our analysis of characteristic solution solutions born out from the characteristics curves, when we analyzed forward wave and backward wave in isolation.

But now we are just combining the two solutions of forward wave and backward wave we still see that our previous analysis holds for this part which is in terms of a, but look at that new part that we have got which is dependent upon the initial conditions in terms of the derivative; b is obviously, coming from the fact that at the initial condition there is crucial condition given in terms of specification of time derivative of p that is b, but the solution at x comma tau does not depend on b over the entire values of x, it depends upon the values of b only within these two endpoints.

So, whatever are the endpoints for a, within these endpoints if you perform the integration that integration region over b will obviously, have some dependency over the solution at p at x comma tau right. So, therefore, of at x comma tau will depend. Firstly, on the boundary points of this triangle, but it will also depend upon this entire interval within this region. So, wherever the two characteristic curves crossover the x axis, the a part of the solution will depend only on the boundary points, but the b part of the solution since it is an integral, but the integral is a definite integral it is integration within this interval where the red curve and the blue curve crosses the x axis.

So, accordingly this interval is identified as the domain of dependence or it is the initial condition in this interval which happens to influence the solution at x comma tau or x 0 comma tau 0. You whatever happens at this region or whatever happens at this region it is not going to affect what is the response at the specified space time point x 0 tau 0. So, that is the concept of dependence; just as an example p 1 comma 1 depends on a 0 I should have I should have said a 0 and a 2 because a depends upon only 1 argument.

So, the solution at 1 comma 1 depends on a 0 I will correct this to a 0 and a 2, that is these two extreme points and it also depends upon the values of b x over the entire interval 0 to 2, it does not depend only on the boundary points, but it also depends entirely on this interval and as such this interval will be called as the domain of dependence of the solution.

So, the domain of dependence comes from the fact that the solution at this point depends on whatever is the initial condition specified in this interval right. So, therefore, for any particular x 0 tau 0, we realized the waves solution depends only on a particular segment of the initial condition that is x lie between x 0 minus tau 0 and x 0 plus tau 0 it does not depend upon anything else. So, therefore, this segment is identified as in domain of dependence.

So, the domain of dependence corresponding to tau equals to 0 is or the initial time instant is identified to be this interval; similarly at any other positive tau the domain of dependence will progressively be the region in between the red and the blue curve. As a result the totality of all extra points which will influence the solution at x 0 tau 0 is given by this triangular patch and this is called domain of dependence. There is an equivalent concept which I would just like to touch upon which is domain of influence.

So, in a similar sense whatever happens here at this point will be influencing the solution in between this triangular region the upper triangular region. So, we will say that the domain of influence let us say of this point x 0 tau 0 will be this upper triangular region the vertex of which will happen to be x 0 tau 0 it is just the other concept. The solution at x 0 tau 0 depends upon everything which happens on this green triangular patch, similarly whatever happens at x 0 tau 0 will influence everything that is lying in this upper triangular patch that is called the domain of influence. So, this concept of domain of independence and domain of influence is a very helpful concept in analysis of wave solution that was about domain of dependence and domain of influence.

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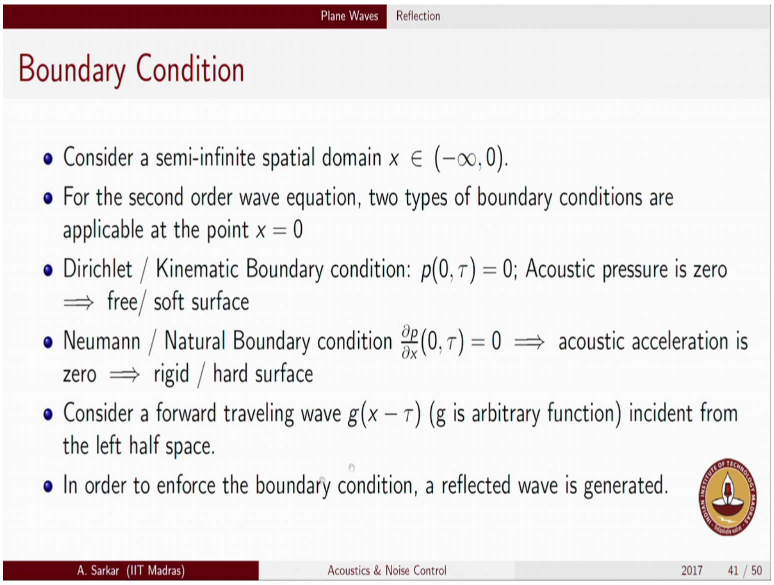


We will now start talking about boundary conditions till now we were talking about the wave solution over an infinite domain, and as such we circumvented the question of boundary condition. So, to have the question of boundary condition into our analysis we will firstly need to construct a bounded domain. So, accordingly before we move to a bounded domain let us consider a semi infinite domain. So, this time the domain is not extending from minus infinity to plus infinity, rather it is extending from minus infinity to 0. So, it is what is called semi infinite domain; and the occurrence of semi infinite domain is actually pretty natural in the case of acoustics.

If you think carefully when we are worried about acoustic wave propagation in the open space, it is basically a semi infinite domain although in three dimensions because one surface is basically the ground surface which forms the boundary, and if you assume that the ground surface is fairly rigid that is a good boundary condition and, but other than this bounded surface which happens to be the ground on the upper side it is completely open. So, there is you are basically dealing with a semi infinite condition where in on the in all surfaces excluding the boundary is basically at infinite right. So, you are basically considering a domain which is like a hemisphere, where in the diametral plan in the ground, but the spherical surface corresponds to a surface at infinite a spherical surface is basically extended to infinity.

So, therefore, semi infinite domain at actually naturally occur in the case of acoustic wave propagation, we are just going through the semi infinite domain in the case of one dimensional propagation, but we will come to the semi infinite domain for 3 dimensional wave propagation also in due course. So, we will need to understand the issues associated with semi infinite domain pretty carefully as we go along. So, this is the case where we take the bull by its horns and we do the analysis for the semi infinite special domain x equals to minus infinity to 0.

So, basically 0 forms a boundary of this domain, because 0 is where the special domain has terminated at 0 you must have a certain boundary condition. Infinity obviously, is far away and you do not need to worry what happens, but that will also come in due course that what is the boundary condition at infinite, but at present we are treating infinite is basically a part which is far away and therefore, there is no question of any boundary condition at plus or minus infinite, but where it is getting terminated is the point 0.

So, the point 0 must have a boundary condition in case we wish to solve this partial differential equation. For second order wave equation we will have two types of boundary conditions which will be applicable at the point x equals to 0; because the point x equals to 0 is a boundary. So, therefore, there will have to be a boundary condition and there can be two types of boundary condition, one is the Dirichlet or kinematic boundary condition where you specify the pressure value to be equal to 0 right this happens for example, in case of a free surface.

Suppose you are thinking about acoustic wave propagation through water right let us say there is a submarine and submarine is making some noise right the question is how much of this noise escapes from the water let us say to the air right, but at the free surface between the air and the water the atmospheric pressure is 0. So, atmospheric pressure being zero is exactly this boundary condition right same thing will happen with what we call an acoustically soft surface where the acoustic pressure is taken to be 0.

So, there will be absorbers which of different kind and some of these absorbers can be at least approximated to be a soft surface right where the pressure basically turns 0, this is called the Dirichlet or kinematic boundary condition, the Neumann or the natural condition is going to be the boundary condition in terms of the derivative; this time the derivative is with respect to space not respect to time. When you had initial conditions the initial conditions could be in terms of a specification of p or the time derivative of p when you have boundary condition it has to be a specification of the p or the space derivative of p evaluated at the boundary. So, del p by del x evaluated at x equal to 0 for all time tau has to be specified and it can be 0 or sometimes it can have a non zero value also, but let us take this case of 0 spatial derivative.

But then if you realize what is the physical implication of this boundary condition. If you look back at the Euler equation, Euler equation says that the acceleration is equals to the gradient of the pressure. In one dimension the gradient of the pressure is basically dell p del x. So, del p del x being equals to 0 essentially means the acceleration at z equals to 0 as got to be 0 for all time, if the acceleration is 0 for all times velocity is 0 for all times and I will not remember we are dealing with the situation where the articles if at all it has motion it has to be oscillate.

So, if acceleration is 0 there are two situations either the velocity is constant or it is dead 0 actually velocity being 0 is a special case of velocity being constant the constant being equal to 0, but the constant cannot have a non zero value, because if the velocity as a constant value for the particle that essentially means the particle is traveling in some kind of rectilinear motion.

As we said in acoustics we are going to deal with only situation where practices can oscillate about its mean position it cannot have a bulk motion. So, velocity equals to constant basically for acoustics boils down to the fact that velocity has to be equal to 0 because if it is anything but 0, it imply a bulk motion which is ruled out from the acoustics constant. So, therefore, when we have pressure derivative the special derivative or pressure to be equal to 0, this in turn implies the acceleration to be equal to 0 and the acceleration equals to 0 means velocity equal to 0, and which in turns means displacement equals to 0.

So, therefore, this boundary is a rigid boundary or a hard surface boundary. It is just complementary to the other boundary condition where we talk that this is a soft surface boundary whereas, the Neumann boundary condition is applicable for a rigid or a hard surface right and this is exactly what we will encounter in very very practical applications as I said in most cases when you have a single sound source mounted on floor and the floor can be assumed to be floor or the ground can be assumed to be fairly large and it can also be assumed to be a fairly hard surface; that means, the ground does not start vibrating because of the sound that is being emitted. So, that is the boundary condition corresponding to the Neumann boundary condition.

So, now the question is what is the complication that is introduced by this boundary condition? As I will show you because these two boundary conditions have got to be maintained f and g now have got to be related. Remember f is the backward traveling wave solution and g is the forward travelling wave solutions, in our previous analysis we say f and g can be arbitrary any f and any g can is possible, but now we will soon find out that if f and g are left as arbitrary then this boundary conditions cannot be satisfied. So, this f and g have got to be interrelated and the interrelation comes through this boundary condition. So, therefore, what we are saying is that if there is a forward wave in this condition where there is a boundary condition specified at x equals to 0 then necessarily for the enforcement of this boundary condition a backward wave must start right and that is what I wish to elaborate to you in this talk.

So, what we will do we will take a very simple example we will first consider a forward travelling wave g of x minus tau and it is incident from the left half space; that is from minus infinite a forward travelling wave is coming. A forward travelling wave as we now can be expressed in the form of f or x minus tau, g being just an arbitrary function it could be virtually anything. So, as I said what I wish to the objective is to actually convince you that in order to enforce this boundary condition whether Dirichlet or Neumann a reflected wave must be generated, there is no way that this boundary condition can be enforced if you keep sticking to only one wave the other wave will have to get induced and there is also a special form that has to be enforced in the backward wave, and this backward wave that will be induced by this forward wave will be the reflected wave.

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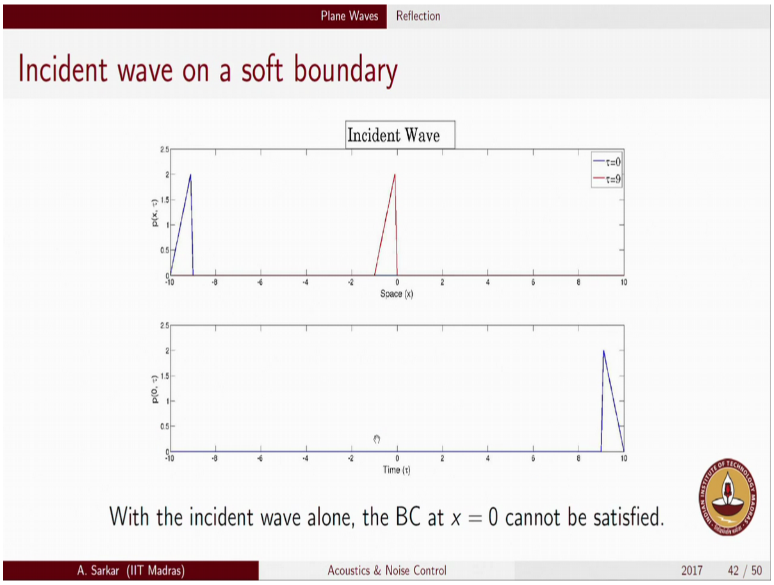


So, let us look at it in a step by step fashion. So, what I have shown here in blue is the incident wave as a function of space. So, this is a snapshot in time. So, at the time tau equals to 0 you have this nice triangular wave which is what we have been dealing with in the different examples. So, at tau equals to 0 you have this condition, remember our domain actually terminates at x equals to 0, but mathematically there is no problem in contemplating that the domain I means the for the purpose of depiction in this graph I am still insisting that it is going to plus infinity, but the physical part of the domain is restricted from minus infinite to 0, that is physical beyond x equal to 0 that we wish to consider, but none the same there is absolutely no problem in plotting and accordingly this is just a 0 plot. So, you should not get perplexed that why is it that you are having special domain which is in the positive side also though we said that we are interested only in the negative half of the real line.

So, at x equals to 0 as I said that you have sorry a time equals to 0 tau equals to 0, you have this initial wave which is from where the simulation start. As you see this wave will reach the point x equals to 0 after how much time? After 9 units of time in tau because this is minus 10 and this minus 9 and this is 0. So, the distance from here to here is actually minus 9. So, exactly after and a lapse of 9 units of time this initial profile will advance in the positive direction to this profile which is shown in red; so, so far so good absolutely no problem.

So, a tau equals to 9 you have the condition where the pulse or the wave has just reached the boundary right after then the story of reflection starts; but before we go to the reflected wave lets us quickly derive the time history corresponding to this pulse. If we know that at time tau equals to 0 you have this triangular wave, then at x equals to 0 for this forward wave alone we could determine the time history as we know this is just the flipped version of this blue triangle which is going to be this.

So, corresponding to the forward wave at x equals to 0 if you take a sensor and record the time history of the pressure at x equals to 0 which is what has been done here p 0 tau then you will see that at tau equals to 9, suddenly the sensor show the response with decays. So, this is exactly the flipped version of the other triangle.

So, now you can realize that there is going to be some trouble; because at x equals to 0 between 9 to 10 seconds you get to see between tau equals to 9 to tau equals to 10 I should not say seconds because this is our rescale time between tau equals to 9 to tau equals to 10 the pressure is not 0. We were interested to firstly, solve for the soft boundary condition or the Dirichlet condition where the pressure is supposed to be 0; if you have only a forward wave or nothing else then you readily see that the forward wave brings out a non zero condition as x equals to 0.

So, it is not possible that the forward wave alone is there in the case where you have a certain boundary condition to be specified at x equals to 0. So, what is the solution that nature will take course to? The nature will induce another wave solution and the wave solution will be such that it will exactly negate this time is history at x equal to 0.

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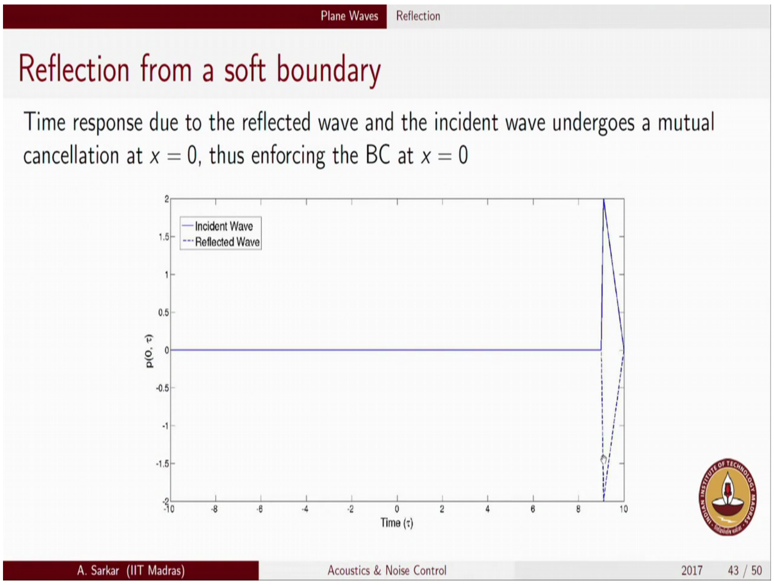


So, let us look at that. So, the incident wave has created a time history which is as shown in this solid light right. So, what nature would like to do now in order to enforce the boundary condition at x equal to 0, at x equal to 0 the condition pressure equals to 0 can be maintain only if there is a another wave that is created and that wave must have a corresponding time history which is exactly the opposite of the time history that is created by the incident wave. So, if the incident wave has created this time history, the reflected wave must create the exactly the opposite time history at the boundary point x equal to 0. So, the response due to the reflected wave and the incident wave must undergo a mutual cancellation at x equal to 0, if the boundary condition has to be enforced.

So, the boundary condition at x equal to 0 can only be enforced if there is a backward wave or if there is a reflected wave I should say which will have a corresponding time history which is exactly the negative of the time history generated by the incident wave right. And this reflected wave in other words if it gives time history equals to this dotted line at x equal to 0. Now we can construct that for this time history to occur what is the snapshot, what is wave profile over space right because we know that the reflected wave has to be a backward travelling wave right there is an incident wave which is forward travelling. Therefore, a backward travelling wave has to get generated; we know the time history at x equal to 0 corresponding to that wave. So, what is left to determine is at time tau equals to 0 what is the profile of this pressure corresponding to different spatial points x.

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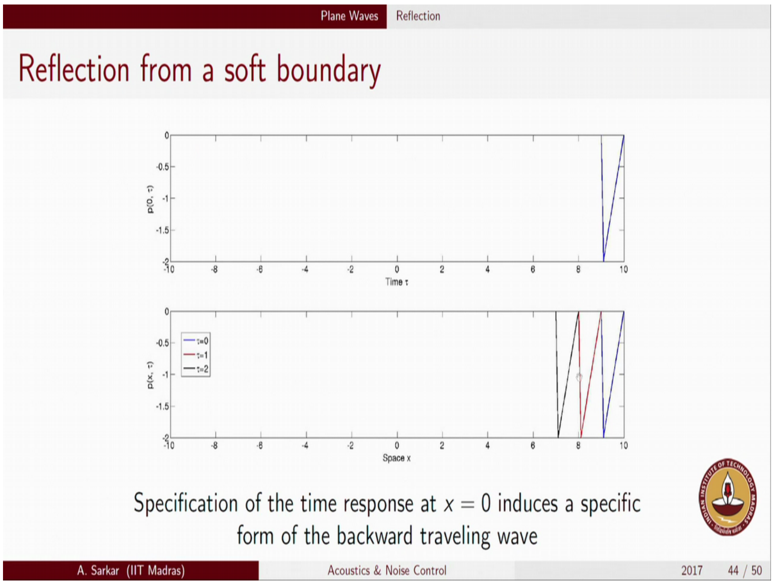


So, here is the plot. So, we have already found that what we are trying to do get at is we are trying to get at a wave such that p 0 comma tau has this profile, if p 0 comma tau has this profile and if we are sure that it has got to be backward travelling, then p 0 comma tau will have the same form as p x comma tau at tau equals to 0 or p x comma 0 and p 0 comma tau has got to be the same feature, because a backward travelling wave as we understood is given by f of x plus tau right. So, the time history plot and the snapshot in terms of space will look identical.

So, the blue time history plot that you see on the upper graph as got to be exactly same as the blue snapshot which is the pressure plotted against x, but at time tau equals to 0. So, this is what this plot it, but please note this analysis, therefore reveals that this profile is actually there in the positive part, but we have said that our special domain is only in the negative really axis there is nothing in the positive real axis. So, that is why I introduce the fictitious domain. So, fictitiously at tau equals to 0 we now contemplative that there is a reflected wave which is sitting beyond the physical extent of the medium right; it is sitting here which actually cannot be seen therefore, it is not seen because the physical extent of the medium as stopped at x equal to 0; you can only see what happens in this domain you cannot see what happens in this domain.

So, the reflected wave is there sitting at tau equals to at tau equals to 0 it is just that the reflected wave is like mathematically extending outside the physical extent of your medium and therefore, it is not visible to you physical. But then what happens progressively at different times this wave will start traveling backward. So, at each consecutive instance of time it will move ahead by a unit distance because here I have shown unit intervals of time. At tau equals to 0 if the profile is shown in blue then at consecutive tau instance of tau the profiles are shown in red and black. So, at each of these instances you have this wave advancing in the backward direction, and what will happen after a certain time? After a certain time because a progressive backward advancement it will actually cross over this wave.

So, the wave will actually be physically visible to you after it is crossed over and it has fallen in the physical extent that you are looking at, but this is a nice mathematical artifact in the sense that you are believing that the wave was always there it is just that you have it was there outside the physical extent of the body that is why you are not being able to see it right.

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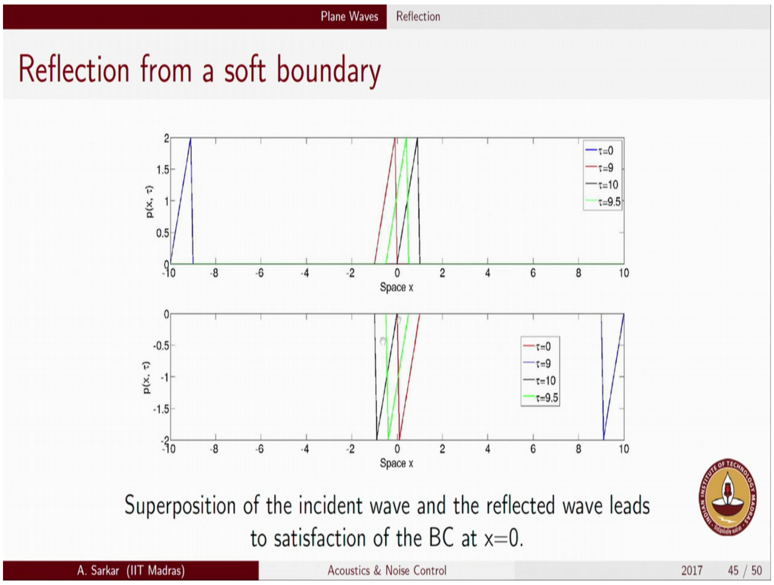


And carrying down this analogy further what we have therefore, seen is that the this is the backward wave and this is the forward wave. So, the forward wave or the incident wave as I told you that a tau equals to 0 it is here at tau equals to 9 it is somewhere here at 9.5 it is indicated by green, at 10 it is indicated by black.

Now, the backward wave at tau equals to 0 is exactly the flick version of this of this incident wave which is what is shown in blue here. Now what happens at 9 this backward wave has just reached the point x equal to 0. So, the backward wave at tau equals to sorry the red colour is corresponding 9 the blue colour is corresponding to tau equals to 0 the legend is not correct the legend has to read as the red colour corresponds to tau equals t 9 and the blue colour corresponds to tau equals to 0. But none the same at this point please realize that at tau equals to 0 you have the blue coloured wave, at tau equals to 9 you have this red profile and exactly at 9 you will see that the incident and the reflected wave have started meeting each other.

At 9.5 you have the green wave from the reflected side; you have the green wave from the incident side shown here in. So, actually there is a overlap between the incident and the reflected wave at an instant let us say 9.5, but please note during this entire course of event what is happening is whatever is the response that is generated at x equals to 0, due to the incident wave is getting actually negative by the reflected wave and thereby you are satisfying the boundary condition at x equals to 0 right and then progressively what happens for example, at tau equals to 10 is that, you gets this black white coloured incident wave which has now actually surpassed the physical extent of the medium and now the backward wave as fallen completely inside the physical in extent of the media.

So, after the instant 9 what will happen is progressively the backward wave will be seen and progressively the forward wave will vanish into (Refer Time: 36:58), but the crucial fact that you should remember is that reflection has been induced because the boundary condition had to be enforced, that is why when we travel when we analyzed the traveling wave in isolation we took this viewpoint that out medium is infinitely extended there is no boundary condition and as a result we did not have to deal two waves simultaneously.

But once you have a boundary and associated with the boundary you have a boundary condition then this argument proves that you cannot deal with an incident wave in isolation, there has to be both an incident and the reflected wave because the incident wave in isolation will not be able to satisfy the boundary condition, the reflected wave will be induced such that it exactly nullifies the effect of the incident wave, and in totality the incident wave and the reflected wave will create a condition where by the boundary condition will be satisfied.

So, we will take it from here in a next class we will meet.