Acoustics & Noise Control
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Module - 03
Lecture - 05
Plane Wave 1

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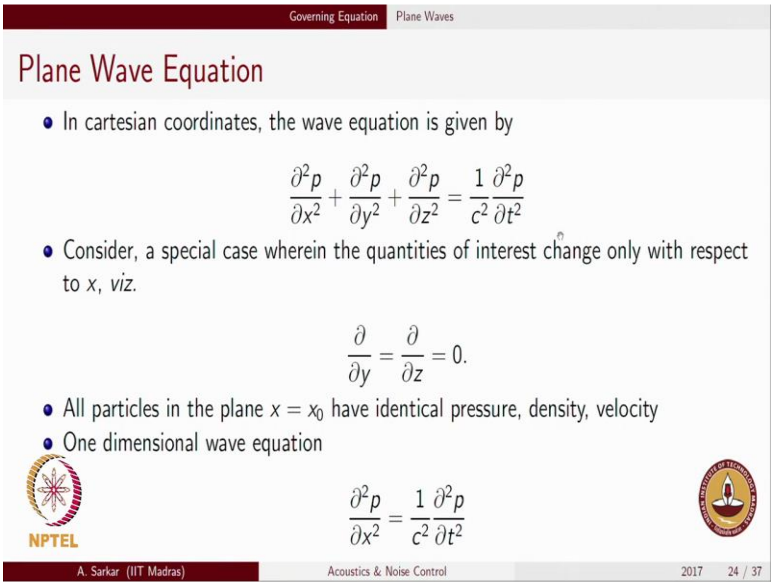

In the last lecture, we derived firstly, what is the full blown equation in Cartesian coordinates in 3 dimension for the acoustic wave equation, this was the equation for the expression Cartesian coordinates for the acoustic wave, then we specialized this to the special case and thus we arrive at this special case of what is known as the plane wave condition. So, the equation for the plane wave as we derived last time is given in this fashion, the 2 independent variables in this partial differential equation being x and t the variable of interest which we want to solve for x t is scalar variable. Please note that as per the hypothesis and the derivation process of the plane wave equation all particles in any plane with coordinate x equals to z 0 will have identical pressure density and velocity.

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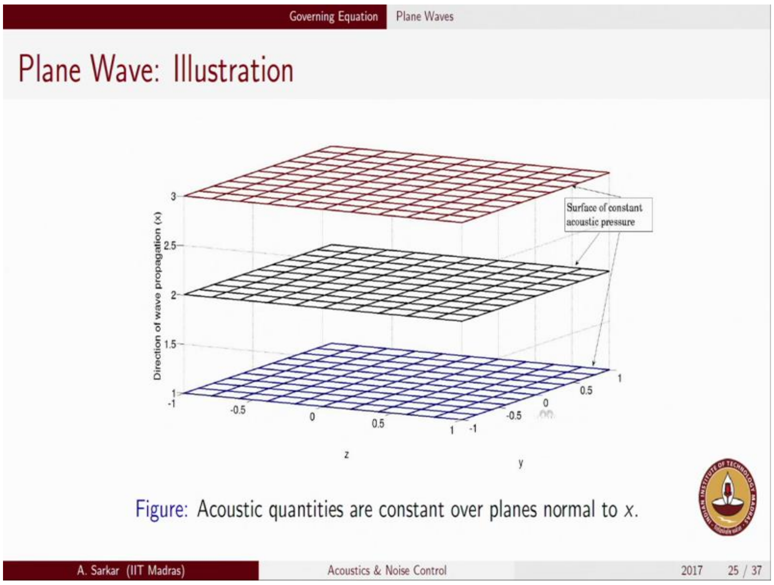

Just to give you schematic illustration of what I mean by that is that all particles lying in each of these planes which are denoted by this makes sort of a schematic here in all these 3 planes you will find all the quantities of interest are identical the quantities of interest will change between the planes, but on the plane all the quantities of interest will remain same. So, this is the illustration of a plane wave. So, whenever we are we mean clean way we essentially mean that particles on a particular plane have the same variable have the same value of all the variables of interest the variable essentially been pressure from pressure density can be derived and also we will see velocity is can also be derive.

Now, coming to the solution process of this plane wave condition, so the first thing that we will now do is that without loss of generality we will introduce a change of variable and we will redefine our time variable.

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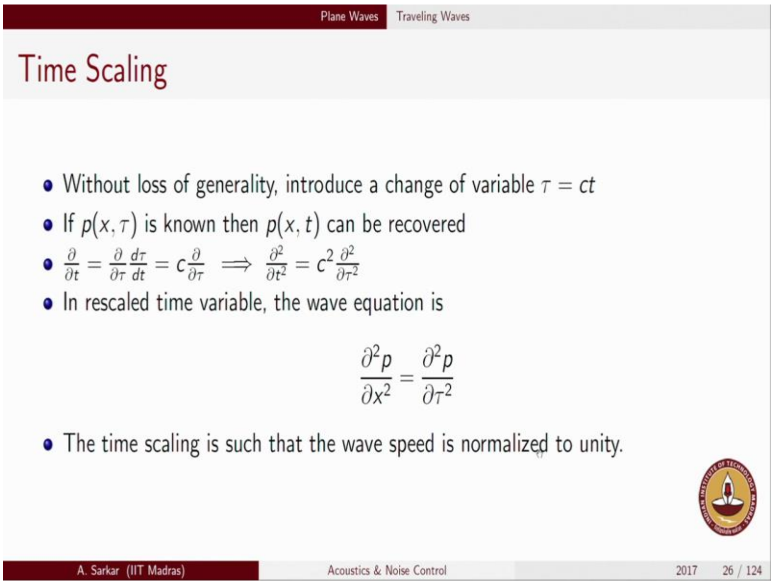

And new time variable tau which will be mathematically defined as c times t remember c is the constant which appears in the differential equation both in the 3 dimensional form as well as in the one dimensional form eventually we are going to show that this c is nothing, but the wave speed, but that is what we are we are going forward towards at present we take this seem to be just a material parameter because it depends upon gamma p 0 by rho 0.

So, way in which we are going to trace the solution of this one dimensional plane wave equation is that first we are going to introduce a change of variable tau equals to c t mathematically all this is fine, but I will try and illustrate what is actually main physically in terms of scaling this new time variable. So, the point is we are; obviously, interested in p x comma t, t is the real time, but then if p x comma tau is known then it is a big deal to recover p x comma t from p x comma tau, because the relation between tau and t is expressed in this fashion.

So, having understood how to recover p x comma t from p x comma tau then the next step would be to do some mathematical simplification as follows del del t that is the partial derivative with respect to ordinary time would be the partial derivative with respect to the tau variable multiplied by d tau d t. This just follows from chain rule of calculus that is del del t is del del tau multiplied by del tau del t, but we have already said tau is equals to c t which means del tau del t is c in other words the result that we have is the following that is partial derivative with respect to ordinary time is equals to c times the partial derivative with respect to the re scale time tau.

Now, if you do this process twice for to get the successive differentiation you will be landing up with a c square term let me just in a straight to you.

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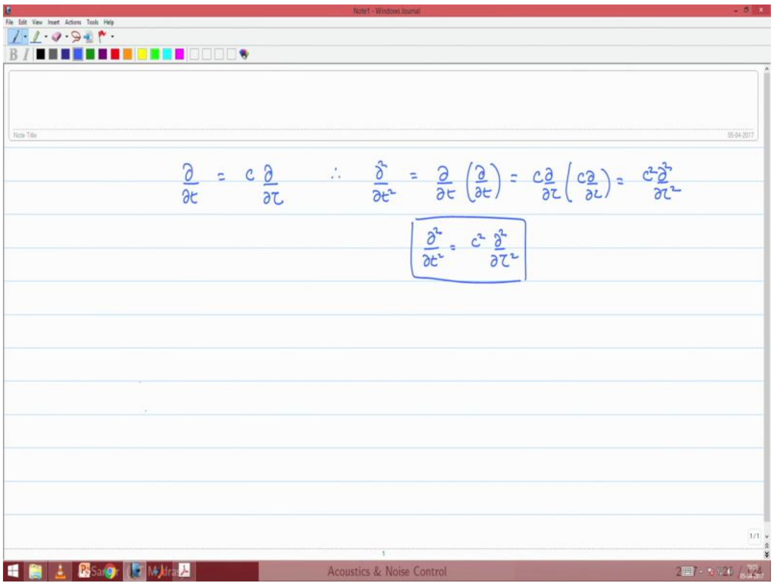

So, what we have derived in the following is del del t is equals to c times del del tau. So, therefore, we will have del 2 del t 2 its equals to del del t of del del t and each of these derivatives with respect to t can be substituted as c times derivative with respect to tau. So, that is what I am going to do and that simply suggest that it will b c square del t del 2 del tau. So, the second derivative of these 2 time quantities will be related through a multiplicative factor of c square; this is precisely what I have written in the slide. So, this is the relation. Now therefore, if we go back to the wave equation the wave equation as was shown to you was given as 1 by c square del 2 p del t 2 on the right hand side.

The left hand side is as it is we are not going to tamper with that left hand side the right hand side was reading as 1 by c squared del 2 p del t 2, but we have already is now seen that the second derivative with respect to ordinary time variable is going to be c square times the second derivative with respect to the tau variable. So, when you do that substitution this factor 1 by c square is now going to vanish and in rescale time the wave equation is going to read in the above fashion in the fashion as it is shown here that is del 2 p del x 2 is going to be equal to del 2 p del tau 2.

So, in essence what we have done is that in this rescaled variable tau we have killed the quantity see as will be shown letter c denotes the wave speed. So, what we have done is that in the rescaled time we have essentially done rescaling of time. So, as to normalize the wave speed to unity this will be also illustrated later on, but now we take it out just as a pure mathematical exercise whereby we have reduced the wave equation in x t variable and I am show that to you once again this was the wave equation the physical time and physical space with the rescaling of variable tau equals to c t, we have managed to get rid of that 1 by c square term on the right hand side and now this is what it the wave equation is from here on and we will see that it is much easier to mathematically solve this equation.

So, the time scaling is such that the wave speed which is that factor c is now getting normalized to unity.

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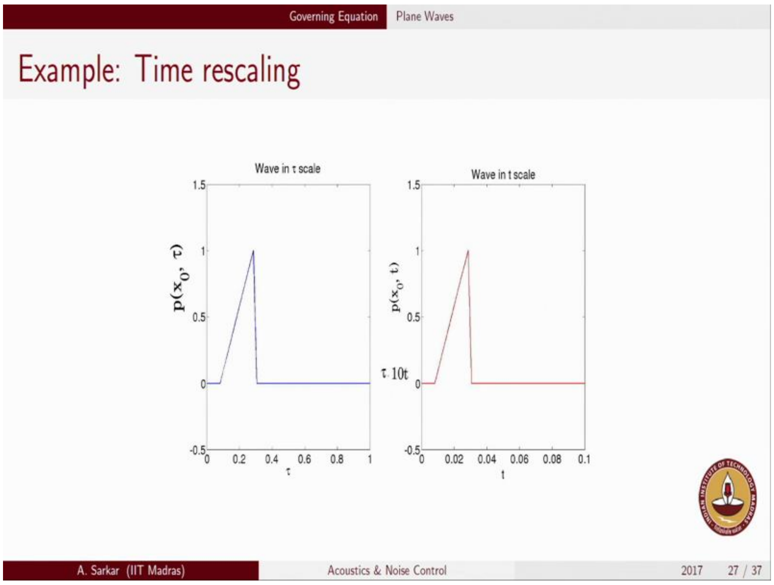

So, here is the example of time rescaling. So, here I have just taken as an example tau is equals to 10 t. So, the tau variable if it goes from 0 to 1; the t variable actually goes from 0 to 0.1. So, the p variable is as it is the fu the function p define on the tau variable if it looks like this the function looks identical in the t variable just that the x axis of this graph is rescale instead of tau going from 0 to 1, it is t going from 0 to 0.1 alternatively if you take t going to 0 from 0 to 1 then tau will go from 0 to 10.

So, the moral of the story is this that you have 2 scales one is the t scale and the other is the tau scale.

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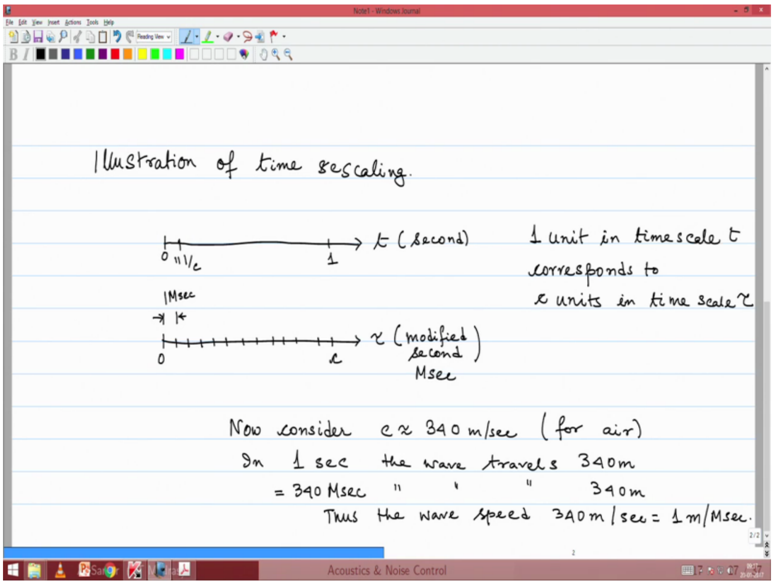

So, both of them have their 0 points at the same instant, the 0 is same because tau is equals to c t means that when tau is 0 t is 0 when t is 0 tau is also 0. So, if I take t is equals to 1. So, 1 unit in t scale corresponds to c units in the tau scale which essentially means if I considered this t to be in our usual seconds. So, if there is a single tick of my seconds hand in my watch then there are 340 ticks in that new time scale tau because c is equals to 340, so illustration of time rescaling.

So, 1 unit in time scale t corresponds to c units in time scale tau. So, coming back to the analogy that if the unit defined in t the usual time scale is seconds the unit defined in the tau time scale will be such that one second in the t time scale corresponds to 340 units in the tau time scale and if you define this tau time scale let say as a modified second unit. So, we are inventing this unit as and we are abbreviating this as m sec right. So, c units in m sec is corresponding to one unit in the ordinary time units which is the seconds unit, so now, considering c to have a numerical value of about 340 meter per second for air.

So, what we mean is that there are 340 of these small takes happening in the tau scale as there is one take happening in the t scale there is 340 ticks happening in the tau scale which essentially mean each unit is basically 1 by 340 of. So, this one m sec this is this is one m sec and that distance if you map it on the t scale we will come as 1 by c and 1 by c is 1 by 340. So, there therefore, there are such 3. So, what we have done is that in 1 second we know the wave travels 340 meter, now 1 second is also equals to 340 m sec right. So, in 340 m sec the wave would travel 340 meter.

So, therefore, in this new redefine time scale tau the wave speed is 1 meter per m sec; m secs stands for modified second which is our invented units. So, does the wave speed can be said to be 340 meter per second which is also equal to 1 meter per m sec. So, this is what the time rescaling has achieved and we are saying that even in the equation we are saying that the factor c or 1 by c square rather associated with the second time derivative has no longer persistent that has vanished which essentially means that we do not need to deal with this c factor anymore.

Initially if you are feeling a little uncomfortable with this time scaling rescaling be with me as you do some assignments this should be pretty clear how it works, but the reason why I wish to introduce this is that with this time rescaling as you will see things will become pretty easy the mathematics will become very easy to do and there is less chance of confusion with the tau scale as opposed to the t scale. So, we will get on with the mathematics now right just remember this time rescaling is not going to affect the picture along the y axis as is shown here just the x axis we will get recalibrated what has been point one units in the t scale. Now will become one units in the tau scale for the special case of tau equals to 10 t that is here I have just taken a fictitious example where c is 10 just to illustrate my point, but if it c is 340 you could work that out as well.

So, without loss that is why I say that this rescaling is without loss of generality it is not that we are considering a fictitious fluid where in the wave speed has got to be unity whatever is my the fluid whether it is air water or anything else whatever is the wave speed and I know my real time units are in second I can always rescale or reinvent a new time units such that in that rescale time variable I have a unit speed I have a unit value of c that is exactly what I have done now; now we will try and solve for p x comma tau rather than p x comma t, but the generality being if we know what is p x comma tau we can as well get p x comma t by as simple change of variables in the last step. So, last we will we have transformed the problem instead of finding looking for a solution for p x comma t we will now look for a solution for p x comma tau.

Hopefully, we will find it and in the last step we will again transform from the tau variables to the t variables to bring it back to our real time solution or the time in seconds rather than this modified seconds which we understand is our invention.

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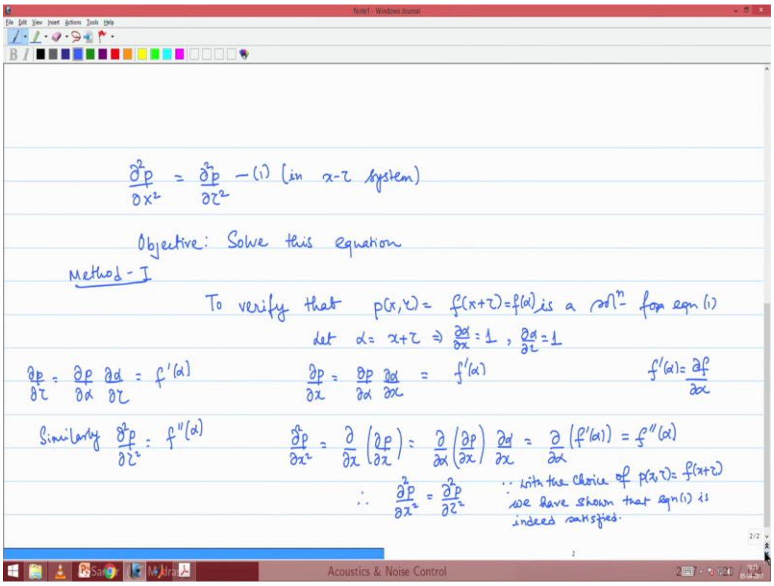

So, the wave equation as we understand is going to read as del 2 p del x 2 plus del 2 p del tau 2 in the x tau system now the objective would be to solve this equation and we are going to follow 2 methods: the first method is almost like a verification. So, it can be verified and that is precisely what we will do that the solution of this wave equation is of the form f of x plus tau plus g of x minus tau. Remember we are dealing with p d s.

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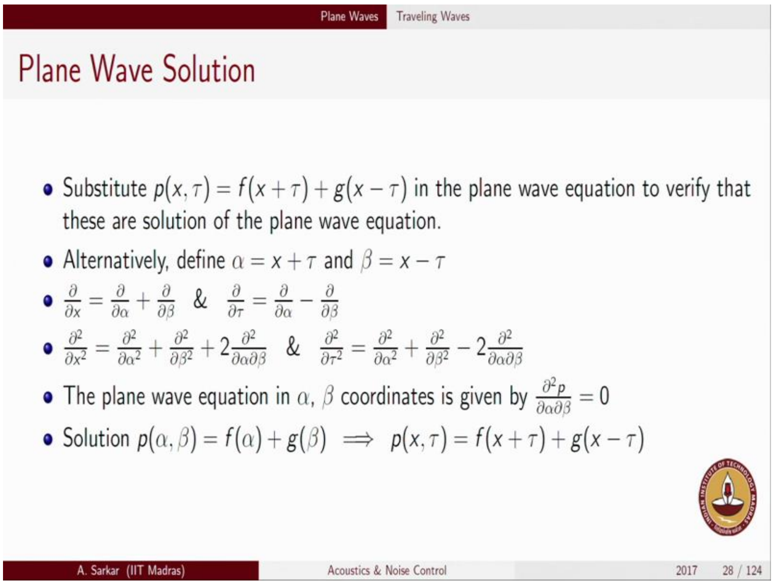

So, p d s in there general form will lead to arbitrary function just like in case of o d s you get a general solution with which involve arbitrary constants in the solution of p d s we are going to get the general solution of p d s will involve arbitrary functions.

So, these arbitrary functions are f and g, but what is important to node is that these arbitrary functions are argument of either x plus tau or x minus tau there cannot be any other combination of x and tau which will work as the solution. So, in the first objective we are going to verify that the solution of p x comma t is indeed of the form f of x plus tau plus g of x minus tau. So, let us do that in the nodes here in the tablet. So, we will verify that p x comma tau equals to f x plus tau is a solution for equation one and equation one is as marked here which is the wave equation. So, before proceeding we will just introduce a new symbol. So, let alpha be equals to x plus tau just a new variable that we are introducing.

So, now del p del x can be written as del p del alpha into del alpha del x that is no big deal and p x comma t we have already assumed as f of x comma plus tau and x plus tau be equal to alpha we could as well write this as if of alpha. So, del p del alpha is basically f prime alpha. So, f prime alpha is del f del alpha because f is a function only of alpha f is not a function of any other variable it is a function only of alpha. So, del p del alpha p is f alpha. So, del p del alpha is f prime alpha where prime denotes differentiation with respect to its argument and del alpha del x is going to be 1 from this very equation del alpha del x is going to be 1.

So, then we have del p del x as f prime alpha now we have to do the second derivative which is del del x of del p del x again del del x can be written as del del alpha of this quantity del p del x into del alpha del x, but del p del x has already been noted to be f prime alpha. Therefore, del del alpha of f prime alpha into del alpha del x which again is just one. So, we did not write that part and del del alpha of f prime alpha is going to be the second derivative which is of second derivative of f which is denoted by a double prime. So, therefore, del 2 p del x 2 is going to read as f double prime alpha. Now let us see what happens with the tau variables del p del tau could again be substituted exactly in the same form that is del p del tau is del p del alpha into del alpha del tau which again would read as f prime alpha right because del alpha by del tau is also one now big deal.

Similarly, following the same steps you will get the second derivative of p with respect to the tau variable to be f double prime alpha. So, that is to say that what we have proved is indeed that with the choice of p x comma tau is equals to f of x plus tau or f alpha we indeed have proved that we are getting to a situation where we are satisfying the wave equation both of them are actually reading out as f double prime alpha. Therefore, with the choice of p x comma tau as f of x plus tau we have shown that equation one is indeed satisfied in other words we have verified that f I f of x plus tau is indeed a solution for this equation.

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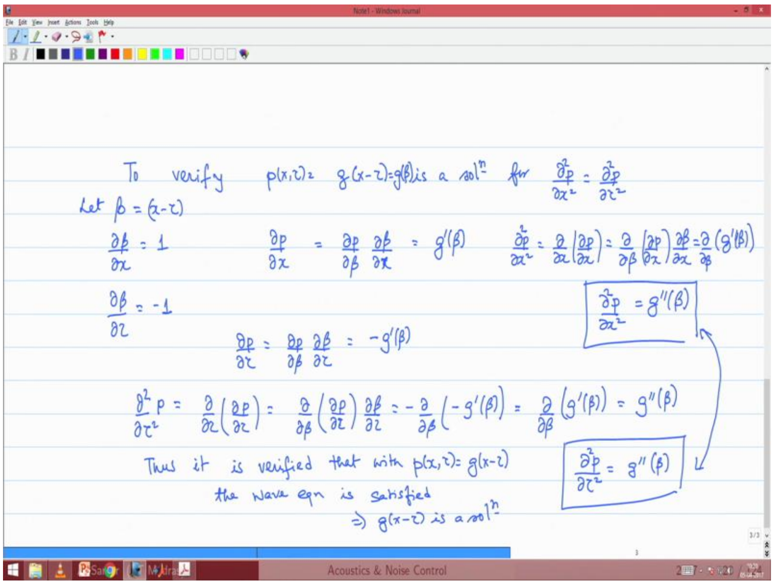

Now, let us do the second part also the second part would be to verify that p x comma tau which reads as g of x minus tau as it is shown here. So, we will now verify g of x minus tau is also a solution we will verify g x minus tau is solution for the p d which is del 2 p del x 2 equals to del 2 p del tau 2. So, therefore, again will proceed in exactly the same manner del p del x will be will do a change of variables and this time the variable that we will introduce is beta. So, we will take beta to be x minus tau and therefore, what g of x minus tau could be read as g of beta.

So, del p del x would be del p del beta into del beta del x this is by chain rule of calculus and del beta del x from this expression is clear that it is equals to 1 and del p del beta because p is g beta is going to read as g prime beta that part should be just as we had done in the previous case also now if you have to do a second f round of differentiation del 2 p del x 2 will read as del del x of del 2 del p del x and that can be written as del del beta of del p del x in to del beta del x, but del beta del x is one. So, now, point in canning it further and del p del x has already been shown to be equal to g prime beta where prime denotes differentiation with respect to its argument g is a function for which the argument is beta right and by the very definition you are now taking differentiation with respect to of beta of g prime.

So, this will read as g double prime beta. So, the left hand side of my wave equation reads as del 2 p del x 2 is g double prime beta now let us see what turns off for the right hand side for the right hand side we will have to again find del p del tau which is del p del beta into del beta del tau is node this time that del beta del tau from the expression of beta equals to x minus tau we are going to get minus 1 instead of a plus 1. So, therefore, what we will have is del p del beta which is g prime beta into minus 1 right. So, del p del tau will be minus g prime beta. Now we will have to do a successive differentiation. So, del 2 del tau 2 of p is del del tau of del p del tau and del del tau is del del beta of del p del tau into del beta del tau.

Now, we have already seen del p del tau is minus g prime beta. So, we can as well substitute that here minus g prime beta and del b del beta is also going to give me another minus sign which I am putting outside the one on the left side of this expression. So, there are 2 minus and therefore, these 2 minus is with their product we will get me plus 1 and therefore, we will have at the end del del beta of g prime beta. So, that would read as g double prime beta. So, again therefore, the right hand side of the wave equation is going to give us del 2 p del tau 2 is equals to g double prime beta. So, from these 2 expressions again we verify that the left hand side and the right hand side of the wave equation turns out to be identical and does this substitution that p x comma tau is g x minus tau is a legitimate substitution and therefore, this is a legitimate solution for the wave equation.

Does it is verified that with p x comma tau as g x minus tau the wave equation is satisfied and this implies g x minus tau is a solution. So, what we have done is that we have verified f x plus tau is a solution we have also verified g x minus tau is a solution which means the general solution because we are dealing with a linear partial differential equation the general solution is of the form f of x plus tau plus g of x minus tau. So, then let us proceed to another method by which we can actually derive this solution remember what we have done in this first step in this method one as I would say is that we have simply sort of guest this solution and then we have rigorously verified that this guess is. In fact, correct.

But we could do better and actually try to find from first principles the solution which is what is described here. So, here what we will do is that fundamentally we will define 2 variables in the first step itself we will define a variable alpha is equals to x plus tau and beta is equals to x minus tau and then using chain rules of calculus we will get del del x to be del alpha plus del beta del del alpha plus del del beta let me illustrate this procedure in the nodes here.

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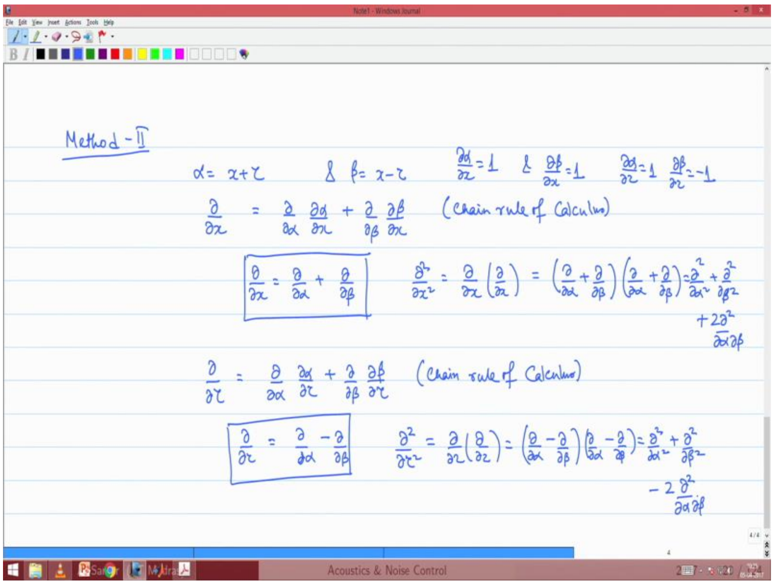

So, we are going to now, get the solution using a different method where in we are going to say alpha is equals to x plus tau and beta is equals to x minus tau and then we are going to transform all our derivative terms let say del del x we are going to say as exactly what is given here del del x we are going to transform in terms of del alpha and del beta.

So, that would read as del del alpha into del alpha del x plus del del beta into del beta del x this is again by chain rule of calculus right you are transforming between the x tau variables to the alpha beta variables. So, the first derivative del del x will be transformed in the above fashion, but then using these expressions it is clear that del alpha del x is going to be one and also del beta del x is going to be 1. So, therefore, we will have del del x to be equal to del del alpha plus del del beta. So, that is the first part of the story the second part is about the transforming the variables associated with the time derivatives or the rescale time derivatives to be more particular.

So, again the chain rule of calculus will tell us that this is del del alpha into del alpha del tau plus del del beta into del beta del tau again by chain rule of calculus, but now we can observed that while del alpha del tau is 1 from this expression del beta del tau is. In fact, minus 1, so we will have to account for this correctly. So, del del tau is going to read as del del alpha minus del del beta.

Now coming to second derivatives: so second derivative del 2 del x 2 is nothing, but del del x of del del x, but we have already seen that the del del x operator is basically balling down to del del alpha plus del del beta. So, that is that operator will have to act on itself which is what this will give us. So, this gives us del 2 al second derivative with respect to alpha plus second derivative with respect to beta plus 2 times del 2 del alpha del beta.

Similarly, the second derivative with respect to that tau variable is going to read as del del tau of del del tau and if you substitute the expression for del del tau this is what we are going to get. And finally, if we take this product its again going to read as del 2 del alpha 2 plus second derivative with respect to beta and the only difference that you are going to get here is that there is to be a minus sign in the cross derivative to del to alpha del beta that is it. So, that is exactly what I have noted down in the slides for you. So, del to del x 2 is going to read as this and del 2 del tau 2 is going to read as this. So, now, if we make this substitution in the wave equation remember the wave equation would read as del 2 p del x 2 is equals to del 2 p del tau 2.

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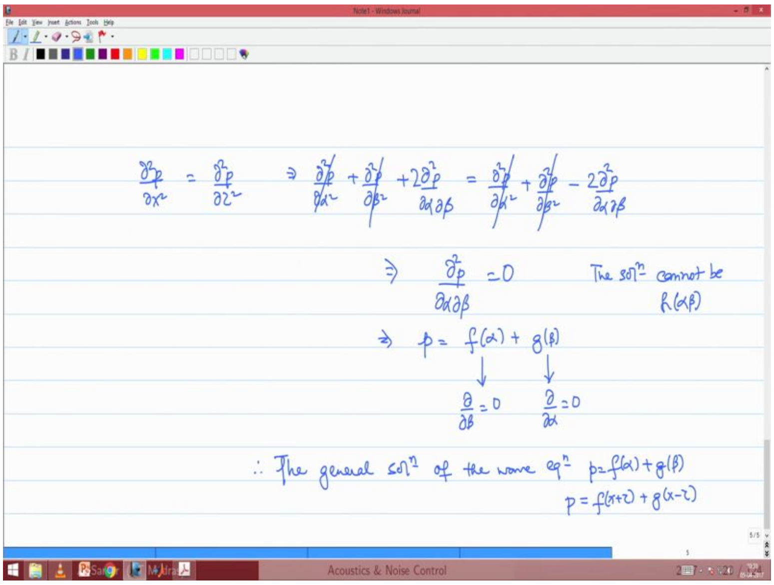

Now, all that you have to do is you have to replace this second derivatives with respect to x and tau in terms of second derivatives with respect to alpha and beta. So, these expressions have already been derived here. So, if you now use these expressions where in the second derivatives of x and tau are changed over to from x and tau to alpha and beta the this is what you are going to get you are going to get del 2 p del alpha 2 plus del

2 p del beta 2 plus 2 times del 2 p del alpha del beta should be equals to del 2 p del alpha 2 plus del 2 p del beta 2 minus 2 times del 2 p del alpha del beta is observed that there is a change in sign of the cross derivatives in the 2 derivatives with respect to x and with respect to tau.

So, this is what we get for wave equation in terms of alpha and beta, but then we observe that 2 of the terms is identical so; that means, these 2 terms can get cancel. So, effectively the wave equation in alpha beta coordinates reads as del 2 p del alpha del beta is equals to 0. So, the cross derivatives of p with respect to alpha and beta both are supposed to be 0 which means that p has to be either a function of f or a function of sorry either a function of alpha or a function of beta or a superposition of 2 such functions it cannot be a function of both alpha and beta simultaneously contain.

So, in other words this is will imply that the solution has to be of the form f alpha plus g beta it cannot be for example, the solution cannot be f or h alpha beta because that will lead to a non-zero cross derivative that will remain and therefore, that is not possible f alpha will have derivative with respect to alpha, but the derivative with respect to beta del del beta of this term will go to 0 similarly the derivative with respect to alpha of the second term will go to 0 and as a result if you take either f alpha or g beta your sure to get the con in to the conclusion that the cross derivatives of both f alpha and g beta will get vanished. Therefore, f alpha and g beta are both valid solutions and therefore, the general solution should be a superposition of both these individual solution which is f alpha plus g beta.

Remember the general solution. So, with this argument we are getting that the general solution of the wave equation is going to be p equals to f alpha plus g beta and now if we switch over again from alpha beta variables to x tau variables we are going to get as f of x plus tau plus g of x minus tau as p. So, this is exactly the same result that we have obtained using method one, but method one as I said is more of a verification that yes indeed this solution does hold, but here we have almost derived it from very first principles.

So, let with that let us again go back to the slides and see what we have to offer here. So, what we say we have just derived is this that once you get to this stage that the cross derivatives has got to read to 0 therefore, you can say that the solution is f alpha plus g beta which means the solution in terms of x tau variables is going to got to be f of x plus tau plus g of x minus tau.

Remember f and g are completely arbitrary functions they will be getting fix from certain initial conditions of your problem, but other than that if I compare they are completely arbitrary they demand certain smoothness requirements, but we will assume that the smoothness requirements are satisfied by the physical process other than that f and g are completely arbitrary functions and the reason why these arbitrary functions appear is because, we are now talking about solution of p d s remember in the solution of o d s arbitrary constants appear and those constants are actually evaluated through initial conditions in p d you go one step up and their hear you have arbitrary functions appearing in the general solution and the exact form of these arbitrary functions can only be fixed by further information associated with initial condition.

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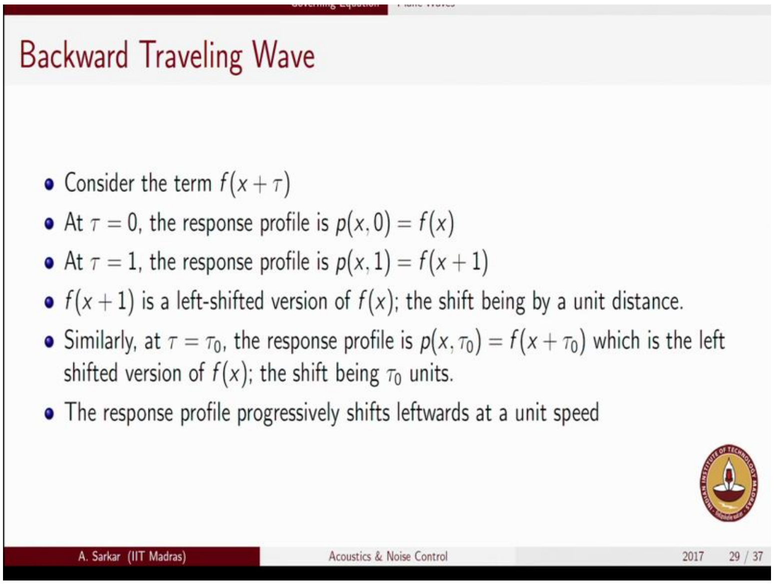

So, now you will considered the term f of x plus tau the function which comes with an argument x plus tau at t equals to 0 the response profile is p of x comma 0 now we are ignoring the g part; we are only taking the f part. So, the p of f comma tau is basically f of x plus tau because then g part is just suppressed in this discussion. So, because we wish to understand only the f part of the solution we will come back to the g part eventually. So, at tau equals to the 0 therefore, p x comma 0 is just f of x at tau equals to 1 p of x comma one will be f of x plus 1, but how are these 2 solutions related if you want if u recall that if f of x is a certain function or the graph of it looks in a certain fashion f of x plus 1 the graph of this function will be left shifted or backward shifted.

So, that is why we call this as a backward shifted wave or a backward travelling wave. So, f of x plus 1 is a left shifted version of f of x the shift being unit distance right because again by elementary ideas that you have got in plotting graphs you can well understand that if f of x has a certain graph f of x plus 1 will just be a left shifted version for it I will illustrate that with an example in the next slide, but what I wish to drive home at this point is that f of x plus 1 is not an another arbitrary graph of the function it has you know quite a distinct resemblance with f of x the only point of disparity being that it is just left shifted in comparison to f of x. So, in other words the response profile the acoustic pressure profile at time t equals to 0 and the acoustic pressure profile at time t equals to 1 are completely resembling each other except for the fact that one is the left shifted version of the other.

So, this is what we will take into our being now generalizing this further. So, if we consider any arbitrary time tau 0, the response profile at that arbitrary point tau 0 point of time I mean is going to be f of x plus tau 0. So, this f of x plus tau 0 also bears the same resemblance with f of x the point of disparity again being that it is just the left shifted version of f of x the shift this time being equal to tau 0 units. So, the moral of the story is this that if you know what is happening at the initial time or anytime instant of your interest for all progressive time instants the response profile is going to be identical except for left shift which is happening and because we have done rescaling in the time variable.

Now, you see what benefits we get because of this rescale time variable because of the rescale time variable for each unit progress in time this graph proceeds by a unit distance in space. So, basically time and space have been made equal numerically right the for each advance of the time scale tau by a unit modified seconds if you may wish to call it the response profile shifts by a unit distant and because the response profile shifts by a unit distance we are saying that the wave speed is unity each progressive time instant by unit second leads to a picture which is shifted by unit distance left words I will again explain to you with an illustration in the next slide.

So, the important take away is that the response profile progressively shifts leftwards, but the speed is unit because for each unit increment in time the response profile is shifting by a unit distance left words. So, it is a backward travelling wave or a left word travelling wave because right is generally taken as positive x left is taken as negative x. So, it is a backward travelling wave or a leftward travelling wave.

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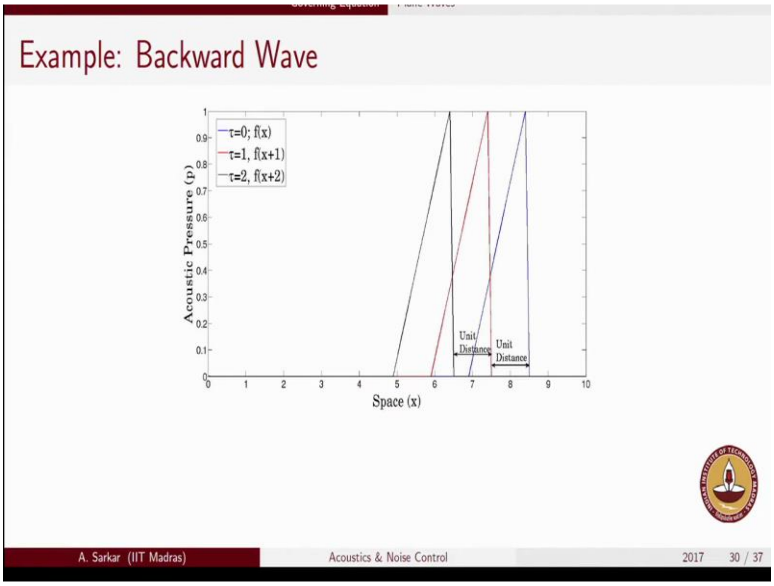

So, let me illustrate to you now with this with this plots for your assignment you will be required to draw these kinds of to generate these kinds of plot and animate them also in this p d f file it was not possible for me to embed the an animations, but you will possibly do the animations as a part of your assignment.

So, again this illustration is for sort of simplistic wave profile. In fact, If you see carefully I have taken a triangular wave profile which actually define defies certain continuity requirements it has a sharp edge for example, so, I am sort of scheming that below the carpet we are not really bothered. So, much about the continuity or smoothness requirements at this stage we understand that whatever is the continuity requirement will be taken care of which essentially means that the sharp edge will actually be rounded of in reality, but the more important point to drive home is what is meant by these concept of backward wave.

So, if you see this blue color plot it essentially denotes the acoustic pressure profile at time t equals to tau equals to 0 which essentially means that it is f of x at tau at time tau equals to 1 we know that the acoustic pressure profile will now be f of x plus 1 this is plotted in red if you note carefully f of x plus 1 and f of x is just looking identical to each other except for a left shift which is by a unit distance. Similarly if I plot what happens at time tau equals to 2 at time t tau equals to 2 p of x comma tau is given by f of x plus 2? So, f of x plus 2 is shifted version of f of x the shift being 2 unit distances are 2 units.

So, for every incremental progression in time by unit steps we see that this initial response profile which was plotted in blue keeps shifting by a unit distance right. So, this is essentially why we are calling this a wave this is what is a wave that is the response profile remains identical in time except for a shift and the wave speed is precisely this how much does the response profile shift by every unit time right. So, it turns out because of our rescaling of variables we will get to see always unit wave speed in the rescale time not in the actual time, but we can recover what happens in the actual time as per the discussion we have just had.

Therefore, we have this notion now that this is a backward wave next lecture we will see; what is the forward wave, but this is something that you should have very good hang one and a very group good grip on that what is the wave and you should be able to close your eyes and visualize waves. So, as I said there will be some assignments given to this effect which you could work it out in MATLAB and that will possibly facilitate your visualization of what is the wave we will end the lecture here.

Thank you for your attention.