Acoustics & Noise Control

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Module – 33
Lecture - 38

Kirchoff Helmholtz Integral Equation

Welcome friends to this talk on acoustics and noise control. So, till now we had looked at.

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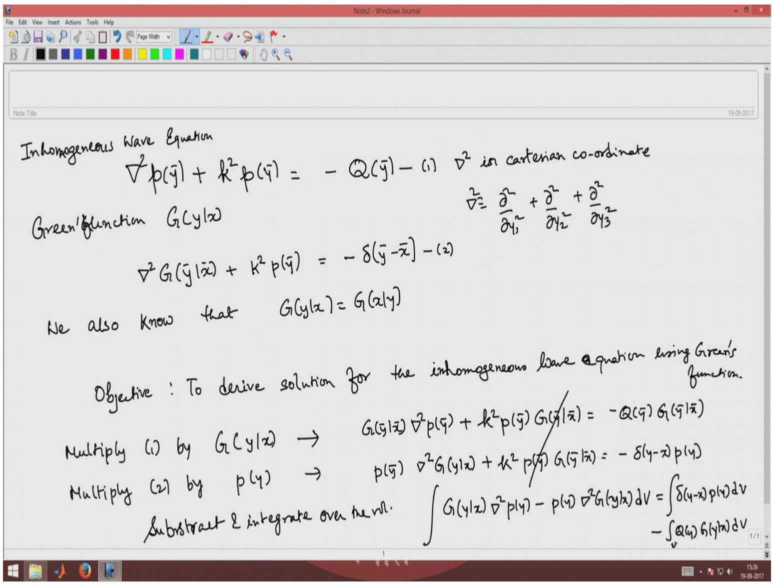

So, the inhomogeneous wave equation is del square p plus k square p is equals to minus Q. Till now possibly we had looked at situations where the space variable would have been denoted by the vector x, the position vector x, but here I am changing over my notation that is for a specific reason which you will be able to appreciate, instead of the vector x I will call this the vector y. So, just I am interchanging my notation between x and y. So, pressure at any point y is given as a solution of this inhomogeneous equation.

So, here you must note that the del square in Cartesian coordinate will be given by del 2 del y 1 square, plus del 2 del y 2 square plus del 2 del y 3 square. So, this is the Laplacian operator. So, this is the inhomogeneous wave equation which we have studied in some detail, though the solution has not yet been made available to you, and this is precisely the objective of this lecture. We have also looked at a greens function, and the greens function depends upon 2 arguments y and x. So, the greens function is the pressure profile at y, due to a delta type of inhomogeneity which is acting at x. So, that is why we changed over to the notation where in we used y as the field point in x as the source point. So, minus delta y minus x that is the definition of greens function.

We also know that by reciprocity theorem greens function has this reciprocal property G of y comma x is y by y with respect to x is same as G of x with respect to y. That is if you interchange the source and receiver location in any domain whether unbounded or bounded, you are going not going to affect anything the formula remains just the same. So, today what we will do is that we will actually try to derive the solution of this inhomogeneous equation one using the greens function.

So, the objective will be to derive solution for one, solution for the inhomogeneous wave equation using greens function. And this is done by the following steps. So, if we multiply equation one by the greens function itself G y comma x, what we get is the following. G y comma x into del square p y plus K square p at point y into G y x is equals to minus Q y into G y x, and we will multiply the second equation with p y. So, multiply 2 by p y. So, what we will get here is the following, p y multiplied by Laplacian of the greens function plus K square p y multiplied by the greens function is equals to minus delta y minus x into the into p y.

Next we will substract these 2 equations, substract and integrate over the volume. So, when we do that this term these 2 terms are going to get canceled, and we will be left with the following G y bar x del square p y, minus p y del square G y bar x integrate over the volume and that will be delta y minus x, p y within the volume plus or minus Q y G y bar x d V. So, this is just an elementary calculation, but we will make some further simplification which is why I will copy and paste this derivation this final form in the next page.

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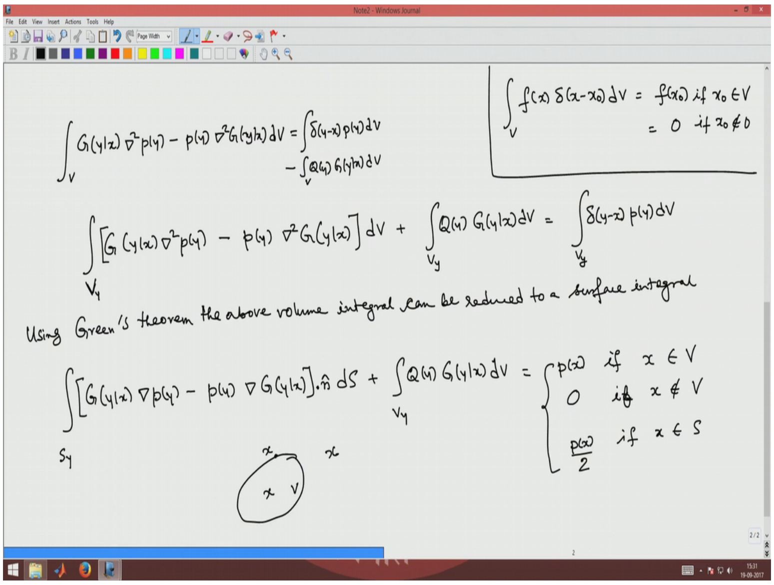

So, here we have the relation which we have derived.

So, we will have G y bar x del square p y, minus p y del square G y bar x, the entire thing is a volume integral we will also bring the other volume integral on Q on the left hand side with a positive sign. So, therefore, this will be Q y G y bar x d V over the volume and on the left hand side what remains is delta y minus x p y d V. Remember here we are doing the calculation wherein the variable is essentially y it is not x it is essentially y. So, each of these variables of integration is basically on y. So, just to remind ourselves we can put that y in the subscript here. So, that we do not we are able to appreciate this fact.

Now we can also reduce this volume integral, this is this can be reduced to a surface integral using the greens function using the greens theorem, and the greens theorem was discussed elaborately in one of the previous details I am not repeating here. So, using greens theorem this above volume integral can be reduced to a surface integral which is given by S y G y bar x. So, the Laplacian will now become the gradient, minus p y again the Laplacian becomes the gradient and finally, you have to take a dot product with the outward normal of the surface, and the other terms remains as it is.

Greens function d V over v, and on the right hand side we have to appreciate that if x belongs to the volume then the right hand side is going to be p of x. So, this is going to be p of x. If x belongs to this volume, but if x is outside the volume then this quantity is going to be 0, because just like we had delta functions in one dimension the essential property of delta function if you recall is f x delta x minus x naught integrated over the volume is going to be f of x naught, if x naught is within this volume of integration. If it is not if x naught let me write it cleanly if x naught is within the volume then this is what the integration should be, and if x naught is not in the volume then this integral should actually will 0 that is the definition of the delta function.

So, similarly we have in three dimensions now. So, this is this being the digression of what a delta function property should be. So, if x is chosen to be within the volume interior to the volume, then this should read as p x. That is no big deal and it should read as 0 if it is true that x does not belong to the volume, but there is a crucial transition between these 2 cases, what if x belongs to the boundary. So, you can have x within the volume. So, let us say this is the volume. So, x can be within the volume x can be outside the volume or x can be on the boundary. So, it is neither within the volume nor outside the volume. It so turns out that which may be actually intuitively obvious, but there is a very rigorous mathematical proof, but for this result, but the essential result is if x belongs to the boundary of the domain of interest, then this integral on the right hand side is going to read as p x by 2.

So, this is essentially going to be the right hand side depending upon where do we choose x. So, p x can be either if x is on the volume, the right hand side will read as p x if x is on the boundary, it will be p x by 2 and if its x is on outside the volume exterior to it then it will be 0. So, therefore, this is the equation which is the grand equation in acoustic it is called as the Kirchoff Helmholtz equation in acoustics.

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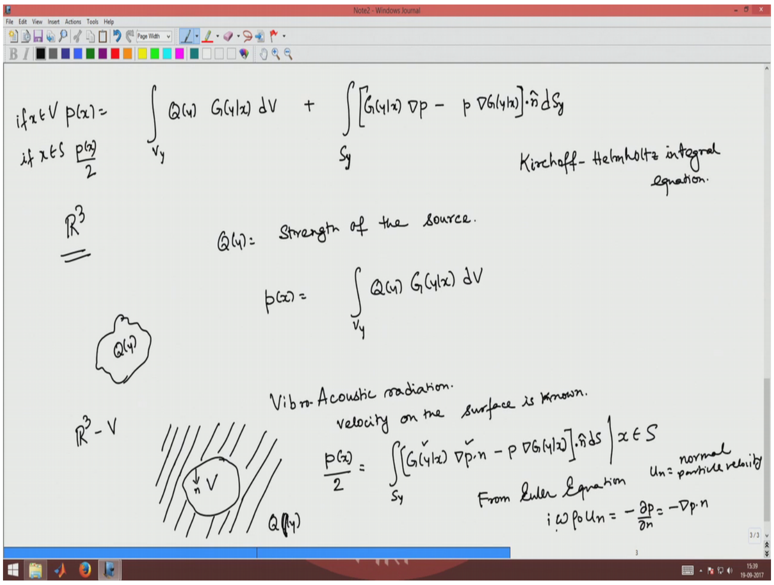

So, this is p x is given by integral Q y G y bar x d V over the volume, will keep putting that subscript to remind ourselves that the field of integration is essentially the y variable and in the surface integral we have the following G y comma x, gradient of p minus p gradient of G y bar x dot with the outward normal n d S again we will put that subscript y to remind ourselves that this is what it should be for one of the integration variable is essentially the y variable.

Now, let us try. So, this this equation as I said is called the Kirchoff Helmholtz integral equation. So, let us see what is the applicability of this equation. If you choose to analyze the entire three dimensional domain, you do not wish to analyze any bounded domain or I mean there can be 2 many many cases, lets first consider that you are interested in the entire three dimensional space, and within the three dimensional space; obviously, there is some region which is the source of sound and this region could be an aeroacoustic source or it could be a vibratory source. In this approach what you are going to do is that you are going to demarcate a certain region, and find this quantity which is basically the in homogeneity associated with the Helmholtz equation.

So, Q y is basically the strength of the source and since in this approach there is no boundary. So, therefore, the boundary integral terms will vanish. So, in this approach when we are taking the entire three dimensional domain into consideration, p x will be simply given by integral of Q y G y bar x d V. Provided you know what is the distribution this is a very useful way in which you can determine the acoustic pressure at any point. This is usually done for aero acoustic applications where if you know the flow field and if you know the turbulence, and you have a CFD simulation which accurately predicts, the properties associated with the turbulent flow field, then it is possible to determine this source distribution the strength of the source Q y tub, it can be determined. Once it is determined the only calculation that is left is to integrate this out together with the greens function. I have not given you the expression of greens function, but I will give that to you, it is a very simple expression and once that is done it will the pressure at any point can be determined.

Next consider to be r to be your region of interest to be in the in the zone which is R 3 minus some quantity V. So, what I mean is that there is a space which is V and you are interested to know what happens outside to of this space. So, this is a typical problem of vibro acoustics, you may have a vibrating body and you may be interested to look what happens exterior to the vibrating body. So, this is exactly the acoustic radiation problem which appears in the case of vibro acoustics. So, this is called vibro acoustic radiation.

So, in here the velocity on the surface is known right. So, the first job is to know the surface acoustic pressure and towards that end we appeal to the Kirchoff Helmholtz integral equation, this integral equation is if x belongs to the volume, but if x belongs to the boundary, then it will be replaced by p x by 2 as I argued this out. So, p x will be replaced by p x by 2 if I wish to determine the surface acoustic pressure. So, once I determine the surface acoustic pressure, I will show you that I can determine the pressure everywhere else.

So, p x by 2 is equals to Now, what happens is that? There is no other source of interest except this vibro acoustic source. So, therefore, the Q y term associated with this volume the shaded volume in particular not the V volume, the volume exterior to it the shaded volume and in this case the normal will point inwards because the volume of interest is the shaded volume. So, when you have this condition Q y will be 0 because there are no other sources in the shaded volume which is the volume of interest. So, because Q y is 0 therefore, p x by 2 is replaced by is going to be evaluated by just this surface integral equation, which is del p dot n minus P on the surface gradient of G y bar x dot n d s.

Now, note that from Euler equation we know that I omega rho 0 U n where the part U n is the particle velocity, U n is the normal particle velocity, i omega rho 0 U n must be equal to minus del p del n which is also minus gradient of p dot n. So, typically the vibra caustic problem the vibration analysis is first done and you know what is the vibration velocity on the surface. So, from the knowledge of the vibration velocity on the surface U n is known and therefore, del p del n is known. So, del p del n is known the greens function is known I have not give you the given you the expression, but I will give it to you before the conclusion of this class let us say the greens function is known. So, therefore, the only unknown here both in the left hand side and the right hand side is pressure on the surface.

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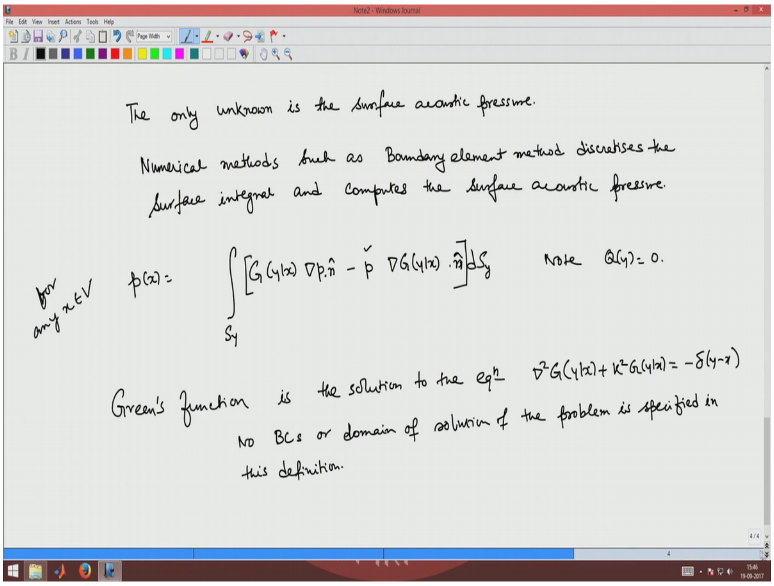

So, in this integral equation the only unknown is the only unknown is the pressure is the surface acoustic pressure. So, at least you have a well posed problem in the sense that you have only one unknown to be determined; obviously, you cannot expect that this can this problem can be solved in pen and paper, rather it has to be solved numerically, but you must understand that in this first step we are we are determining the surface acoustic pressure numerically, in this equation there being only one unknown which is the surface acoustic pressure, you can adopt a discretization procedure, and the discretization procedure that we choose in the numerical process which is called boundary element method is go to discretize this pressure variable exactly the same way, in which the finite element method does for structural simulations and as a result the only unknown that will be left over is the pressure variables, and that pressure variables can be solved for and at the end of the day you will have a pressure at the boundary.

Please note also from this integral equation what is evident is that you need to know only the velocity conditions at the boundary, you do not need to know what happens interior to the boundary interior to the volume, you because this is a boundary integral equation. So, accordingly when the discretization takes place the discretization has to take place only on the boundary, you need not discretize the volume. So, despite being a three dimensional problem this shaded region is the three dimensional region excluding this volume v. So, this is definitely a three dimensional region, but because of the form of the Kirchhoff Helmholtz integral equation, which we have seen and because of this condition that Q y that there are no sources within the region of our interest therefore, Q y has gone 0. So, therefore, there are no volume integrals that are left with, and you are left with a greens function a surface integral of the greens function.

So, therefore, what we have is that we have a well posed problem which can be solved numerically, and the numerical procedure essentially is a methodology is a methodology by which you discretize this boundary, and once this boundary is discretized you are able you should be able to get the solution of the pressure at discrete points on the boundary. So, essentially numerical methods such as b e m such as boundary element method discretize is the surface integral and computes the surface acoustic pressure. Once the surface acoustic pressure is available pressure at any point can be calculated very easily using the Kirchhoff Helmholtz integral equation, again we will appeal to the same equation essentially, but without this factor of half. So, p x will be given by exactly the same quantity G y bar x p gradient of p minus p into gradient of G y bar x dot product of with the outward normal, and this is a surface integral. Note that Qy is going to be 0 because we are not considering any source which is outside this vibrating body.

So, therefore, as a result of this please note del p dot n is known. So, I should have taken the dot n inside. So, del p dot n which is related to the surface velocities is known. So, we need to know the del p dot in on the surface that is already known because the vibration velocity on the surface is known through a vibration analysis. We also know the p at this point which is the which was the first step. So, p at the surface is also known and therefore, what we have is the entire right hand side is known. We could in principle integrate this out and determine pressure at x, for any x in the region of our interest. This is exactly what boundary element method does; it first finds the surface acoustic pressure which is actually the critical step. Once the surface acoustic pressure is known numerically it uses again the same surface acous the Kirchhoff Helmholtz integral equation puts this back where only the surface acoustic pressure and the surface velocities are required surface velocities are anyway given to us as a matter of boundary condition, surface acoustic pressure is what has been determined in the previous state. So, this being the essential step of boundary element method, this is where the numerics takes over and this is where this is why I said that any problem in acoustics will be boiling down to just a game of greens function, and just a game of monopole sand dipoles.

I will close this talk with by giving you the expression of at least one form of greens function. As I said greens function is the solution to the equation del square G y comma x minus plus k square, G y comma x equals to minus delta y minus x right. So, x is the location of the source which is taken as an impulse. So, greens function is a solution of that. Please note that at this in this definition of greens function we are actually not specified the domain of the greens function, it could be anything we have not specified any boundary condition or the domain of application of this equation. So, no boundary conditions or domain of solution of the problem is specified in this deformation in this definition. Accordingly for different boundary conditions and different domains you will have different greens function.

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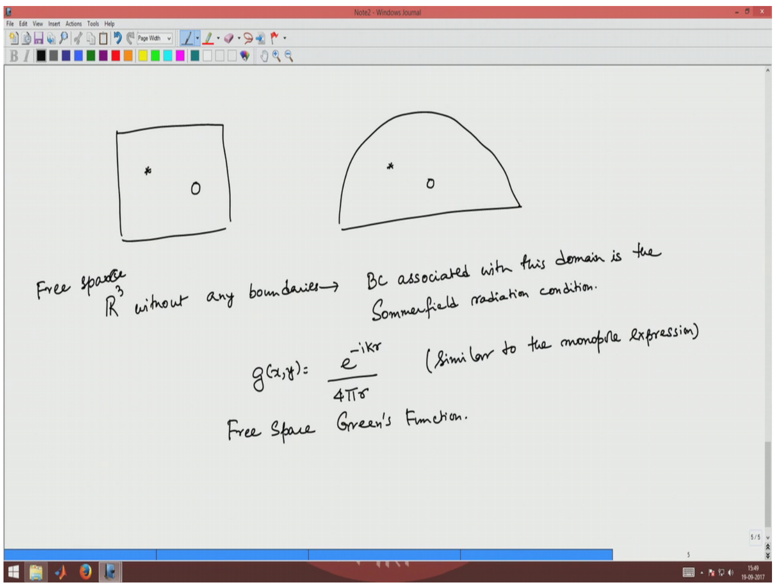

For example if you have a room and in a room you are interested to know that what if you have a delta source at a certain point, what is the response acoustic response at some other point. This greens function will not be the same as if the room is let us say a anechoic chamber. If it is a semi anechoic chamber and you may put the source and the receiver exactly at the same location, but just the bound the domain under consideration is different. So, the greens function associated with the domains will be different and it will capture the idea that what is the room under which you are doing the acoustic analysis. In particular one domain which is of high interest is the unbounded domain. So, if you take the free space or R 3 without any boundaries. So, this is like saying that we are interested in the entire three dimensional spaces which does not have any boundary.

So, it is like a completely anechoic chamber it is not having any boundaries and as such as no extra flexions are expected. So, associated with this condition you can expect that the boundary condition associated with this domain is the Summerfield radiation condition. Which basically states that you are going to have only outwardly propagating wave inward waves are not allowed, and we have talked in great details about the Summerfield radiation condition in one of the earlier talks. So, I am not repeating it again, but the point is this that the greens function at least for the free space turns out to be a very easy expression, and that expression is denoted by g x comma y given as e to the power minus i k r by 4 pi r and you will recall this is very similar to the monopole expression, which we also derived in one of the previous classes.

So, the free space greens function is given by this form and unfortunately we in the current course we do not have time and opportunity left for the derivation of this idea, it is not a very difficult idea, but none the same it could not be derived within the time and opportunity permitted within this course. So, I request you to take this derivation derivation as an exercise you will find it in advanced books in a caustic, that this is called the free space greens function that is greens function in a completely unbounded three dimensional domain. So, what this is one choice of greens function by the way there can be many greens function depending upon which domain which boundary condition you are choosing. If you happen to choose an infinite three dimensional domain in the absence of any boundaries, then you will end up with the free space greens function. But please note that the Kirchhoff Helmholtz integral equation that we have talked here the derivation which we have given, is applicable for any of such greens function.

So, the greens function whether it is a free space green function or it is a greens function in a particular room Kirchhoff Helmholtz integral equation is always applicable. And therefore, in particular in the Kirchhoff Helmholtz integral equation if we choose the greens function to be the free space greens function it is a perfectly legitimate choice and that is why I said that greens function is known and I will give the expression for greens function, which is what I have done here. So, returning back to the Kirchhoff Helmholtz integral equation you will now appreciate that. In fact, all these greens function expressions can be replaced by the free space greens function which is just a monopole like term, and this also sort of again reinforces the idea that when you have this sort of an integration with respect to greens function, it basically suggests that it is a superposition of monopoles. Similarly you should be able to verify that the gray gradient of greens functions or rather the normal derivative of the greens function is actually a dipole like term right.

So, therefore, in one of the earlier classes I made a remark that every vibro acoustics source can be decomposed into only monopoles and dipoles, and the Kirchhoff Helmholtz integral equation is to is the answer to that question. So, Kirchhoff Helmholtz integral equation all resources have been reduced to integration over either the greens function or the gradient the normal derivative of the greens function. It turns out that the green one choice of greens function is the free space greens function which has the expression exactly like that of a monopole and the derivation of it is available in the literature, and similarly the normal derivative of the greens function is exactly having the expression like that of a dipole. And therefore, all vibro acoustic source turns out to be super position of monopoles and dipoles.

From here onwards it will be a computational analysis procedure as I said this equation cannot be solved by pen and paper and by analytical calculation, it needs to be solved using computational procedure and what we have done is we have been able to formulate the equation which has to be numerically implemented through a boundary element method type of calculation, and that will lead us to the solution or at least the computational solution of the acoustic response due to any vibrating object. And with that comment I think you should be able to appreciate that at this point at least all the equations of acoustics are known to you, and from here on it is, but a numerical implementation it is, but the numerics that will play through that you have to implement on this equation to get the solution procedure, fortunately there are lots of commercial softwares which are available which already implemented this idea you should be able to use such commercial softwares and get meaningful result.

So, with that I would like to conclude this lecture as well as this course, I hope you enjoyed the course we from our side we have really enjoyed preparing this course material, we also enjoyed interacting with you during the course and the course forum. So, we hope that this course will set you up for your career in acoustics and if you wish to know anything further, you definite this is not definitely the end of the course in the sense that this is not all that is there to acoustics this subject is vast, but this is just an introduction for you such that from here on you should be able to pick up more advanced literature, and pursue those read possibly read and understand them by yourself.

And please remember the stress on this course has been the formulation related issues, the objective was to be able to appreciate how the equations are formulated computational procedure as I said is anyway available these days. So, the compute the exact calculation procedure using a computer; obviously, depends upon the exact procedures available in each softwares. So, that was not a part of this course. So, with that I would like to take your leave I hope this course is beneficial to your life.

Thank you.