Acoustics & Noise Control

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Module – 32
Lecture – 37
Green's Function

Today to start with we look at some elementary results in vector calculus the first one is gauss divergence theorem.

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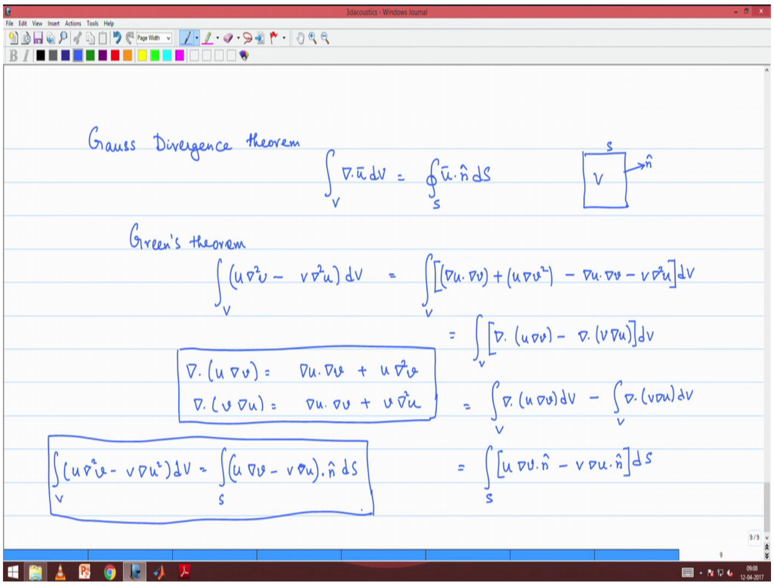

So, this states that a volume integral of the form del dot u d V can be written as a surface integral u dot n d S, it is like it is a close surface integral. So, if you have a certain volume v and the boundary of that volume is s, the outward normal is n, then this is what gauss divergence theorem states we are not going to prove it we will just going to take it on fat, but the proof can be found in any book on vector calculus or some advanced engineering mathematics book.

So, this theorem has got usage in let us say fluid mechanics and so many other places, we are going to use this theorem to know proof what is known as greens theorem in vector calculus. So, greens theorem talks about the reduction of integrals of this form u del square v, minus v del square u and del square is the Laplacian. So, we are going to see how we can reduce this volume integral into a surface integral. So, towards that end what we do is the following; we add del u dot del v and also substract the same in this expression.

So, will add and subtract del u dot del v, that should be perfectly fine, but then this first integral would look like gradient of some quantity. So, if you look at del divergence of u del v is del u dot del v plus u del square v. Similarly divergence of v del u is, del u dot del v plus v del square u this is u del square v. So, this is again just elementary rules of vector calculus I am not proving it. So, using this identity we can now reduce that volume integral into 2 divergences del dot u del v, minus del dot v del u d V. And now we know that each of these divergences can be put in the form of a surface integral using gauss divergence theorem.

So, these two volume integrals which are divergence of some vector field can be written in terms of two volume integrals and the volume integrals sorry each of these two divergences of volume integrals can be written as two surface integrals. The surface integrals therefore, will be u del v dot n minus v del u dot n over the bounding surface where n is the exterior normal. So, the moral of the story for greens theorem is this volume integral of u del square v minus v del square u d V has got to be surface integral of u del v minus v del u dot product it with the unit normal vector. So, this is greens theorem and we will use that in our study of greens function.

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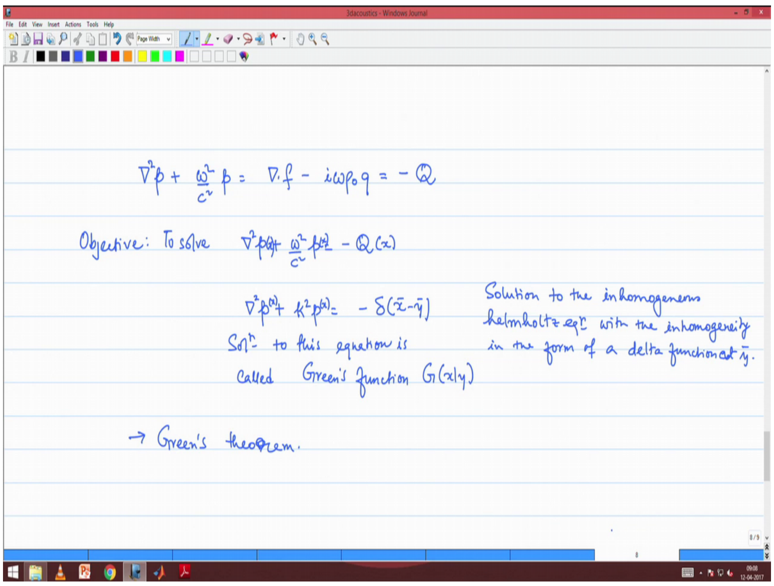

In the last class we introduce the notion of greens functions, greens function it was said is the solution to this differential equation hear. That is it is the inhomogeneous Helmholtz equation, the in homogeneity being in the form of a delta functions centered at a specific point y right. So, the solution of this equation was called the greens function.

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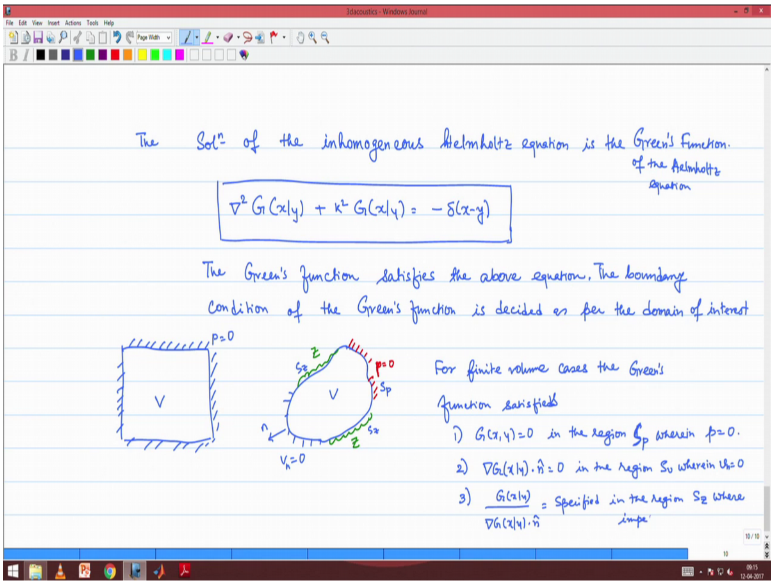

So, we will take it up from there now. So, the solution of the inhomogeneous Helmholtz equation is the greens function. Greens function is actually a very generic concept for any partial differential equation with inhomogeneity you can talk about it is greens function, but in the present course we are only dealing with the Helmholtz equation. So, this is the greens function more particularly of the Helmholtz equation ok.

So, in particular we will have this condition the Helmholtz function is designated as G x comma y or bar y, plus k square G x bar y is equals to minus delta x minus y. So, this is the equation which the greens function is supposed to satisfy right. Remember at this stage we are not talking about the boundary condition, but depending upon the domain of interest the boundary condition could be virtually anything. So, the greens function satisfies the above equation the boundary condition of the greens function is decided as per the domain of interest.

What do I mean by that suppose I am interested in a domain which looks like this. So, if this is my V and if I know for sure that the pressure at the boundary is equals to 0, then what I will say that a greens function will also satisfy this boundary condition. This is a finite volume problem. So, if for a certain finite problem we are sure that the volume that we are looking at terminates with a rigid boundary, then the corresponding greens function will also satisfy the very same boundary condition right. Similarly so, sorry this is this is P equals to 0 this is not velocity equals to 0, because the greens function is after all a pressure like quantity it is not a velocity like quantity right.

So, if similarly if you have any other arbitrary shaped quantity arbitrary shaped volume, and there in you have a portion of the boundary where in P equals to 0, and some other portion where V equals to 0. So, here we have let us say V equals to 0 and on the red hashed surface we have v equals to 0 sorry P equals to 0 and then on some other portion somewhere here we have a certain impedance Z that is specified. Then the corresponding greens function will also have to necessarily have to satisfy these boundary condition that the greens function itself must be 0 over the region where P is 0. Because greens function is pressure like quantity, similarly if the normal velocity as 0 this I should say as normal velocity it is not velocity at every direction, but normal velocity.

So, if the normal velocity is 0 then the gradient of the greens function along the direction of the normal should be 0, because del p dot n is the particle velocity along that normal direction. So, the moral of the story is this that for finite volume cases, the greens function satisfies either G x comma y equals to 0 in the region gamma P or S p let us call this S p wherein p equals to 0 right. So, this red region is S p where pressure has been specified it will also satisfy gradient of G x comma y dot n equals to 0 in the region S v where in V n equals to 0. This is the region where V n equals to 0 and in this region you must have the gradient of the greens function dotted with the outward normal which is something like this that also should be 0.

And in some other regions where it is only the impedance which is specified neither the pressure nor the velocity, but it is the pressure by velocity ratio which gets specified due to the effect of an absorbing boundary conditions let us say. So, therein again you will have this ratio which is G x y, gradient G x y dot n could be specified in the region S z. So, this region I will call as S z where the impedance is as specified in the region where S z where impedance is specified. So, that is the story as far as finite volume situations arise the greens function can be chosen to satisfy the exactly the same boundary conditions as is dictated by the domain of interest but what happens for a infinite volume case?

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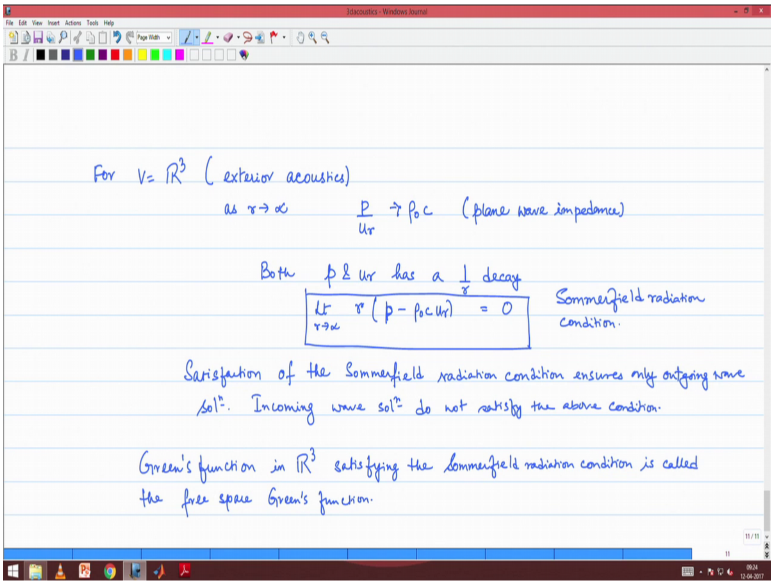

For an infinite volume case as I said that in acoustics exterior acoustic specifically, we would be interested in the volume which is the entire three dimensional space. So, for V which is equals to entire three dimensional space and this type of problem generally occurs in exterior acoustics. Where we are interested in the complete domain there is no boundary that we wish to consider. Let us say we were talking about the acoustic radiation due to a submarine which is at the middle of the Pacific Ocean. So, therefore, it can it is virtually in an infinite media similarly we are talking about an aero plane which is media. So, we are talking again about an infinite media.

So, in that case how do we specify? The boundary condition associated with this domain and the corresponding greens function. So, all that we will now call upon is the fact that what we have seen in our study of pall setting fears and monopoles, is that everything must reduce to plane wave situation. And plane wave situation demands that the pressure by velocity of the impedance should finally, v row c that is only thing which is there in our hand. So, somehow we want this quantity pressure by velocity along the radial direction should go to row naught C which is the plane wave impedance, but this should happen only as at large distances as r tends to infinity.

So, at large distances the boundary conditions that is demanded is for a complete infinite domain problem is that the pressure by velocity is should go to row naught C, but then you will realize there is also another condition that both pressure as well as the velocity will have a 1 by r decay. We have seen the 1 by r decay effect in the pressure you could work out the 1 by r decay for the radial velocity also by appealing to the momentum equation. So, both P and u r has a 1 by r decay when we are talking about three dimensional acoustics we have proved this from the point of view of monopole.

Now, therefore, what we will say is that P by u r should approach row naught C as r tends to 0, it should not only be equal rather it should take the form as defined here. As r tends to infinity P minus row naught C U r should definitely be 0, but it is something more demanding than this u it will be 0 even if I multiply r why is that? Because both P and u r has a 1 by r dependence, that 1 by r dependence. So, this equality will be maintained at a rate which is faster than the linear then the limiting condition will be attain at a rate which is commensurate with this 1 by r decay law. So, basically what I am saying is that this P and U r both fall as 1 by r.

But the constant part associated with the P and U r they will have a relationship and scaling factor which is exactly row naught c. So, this is the boundary condition associated with exterior acoustic and this is called the Somerfield radiation condition. So, the Summerfield radiation condition is given mathematically by this, but this is just an intuitive proof that what is a summerfield radiation condition. Basically it is a condition which implies that far away from the source you should have the plane wave impedance getting satisfied and the mathematical form of it also accommodates for the fact that both the pressure and the radial velocity has a 1 by r decay. So, that brings us to the Summerfield radiation condition

Another way of interpreting Summerfield radiation condition is this, the solution for the pressure which in fact, satisfy the Summerfield radiation condition is. In fact, the outgoing solution the incoming solution will not be able to satisfy this condition because the incoming solution will demand a change of sign here. For the incoming situation you will have P plus row naught C U r, but if you if you put it as P minus row naught C U r; that means, the radial velocity is directed outwards right which means it is an outgoing wave. So, satisfaction of the Summerfield radiation condition ensures only outgoing wave solutions.

Incoming wave solutions do not satisfy the above condition. So, the greens function which is in V equals to R 3 in the entire infinite domain, but satisfies the Summerfields radiation condition as mentioned above is supposed to be called as the free space greens function. Free space means there are absolutely no boundaries, the space that is of interest is complete three dimensional space nothing is excluded in that, but then the only exclusion principle that we apply is the principle that we are only going to look for outgoing waves we are not going to look for incoming waves, and that seems physically obvious also because incoming waves cannot arise in an infinite domain problem incoming waves only arise if there is a boundary.

But in an infinite domain problem there is no boundaries, there are no incoming waves. So, the greens function in R3 satisfying the Summerfield radiation condition is called the free space greens function. So, please understand that there are many greens function associated with every domain you will get it is greens function, associated with every domain and every boundary associated with that domain you will get one incarnation of a greens function associated with that specific domain and it is boundary.

Associated with the free space when you are dealing with the domain which is as big as the complete three dimensional real space, that greens function is called the free space greens function provided it additionally satisfies the Summerfield radiation condition; which essentially means you are looking for only the outgoing wave solution the incoming wave solution is sort of filtered out. So, we are going to our next milestone would be to actually derive the form for this free space greens function, and we will see that deriving this free space greens function will be adequate for our purposes

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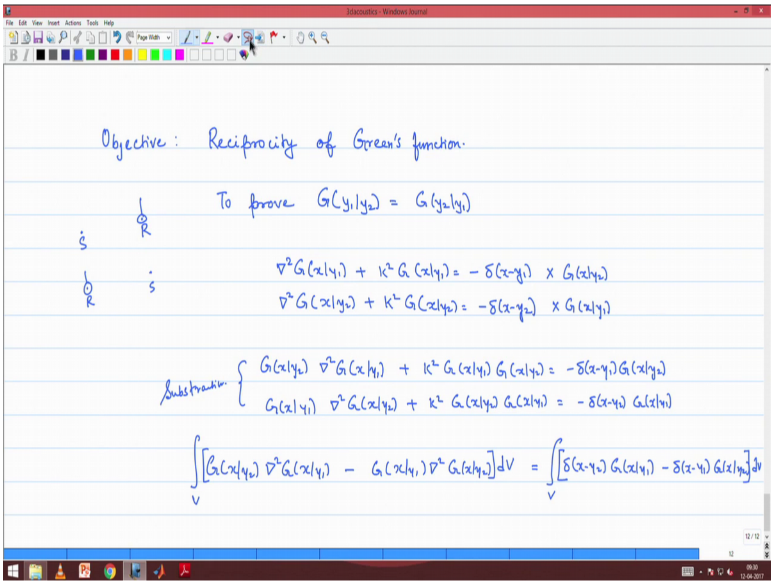

So, towards that end we will make a few steps but the first objective before we derive the form of that free space greens function will be to study this very important effect of reciprocity. This is again something very intuitive, but can be rigorously proved mathematically also. So, what it says is that considered there is a certain source and there is a certain receiver. Greens function is after all the effect of the source on the receiver, that is if you have a certain source at a certain point then what is the reception at the point R. Now the point is if you flip the positions of the source and receiver; that means; now you have receiver here and source there.

The point is will the by receiver I essentially mean a microphone or your human ear. So, if the position of the source and the receiver is interchanged, will there be any change in the signal that is received by your microphone or by a human ear; obviously, not it only depends upon the distance, it does not depend upon the orientation or the location right it only depends upon the relative distance. If me and you were to interchange our position my sound would be appearing exactly the same, it will not change it will not change provided I mean it will not change whether I am in this recording studio or in the open area or in the playground does not matter. In any domain if source and the receiver interchange their position then the acoustic quantity or even any other signal quantity of interest will not change, that is essentially the principle of reciprocity.

With that intuitive understanding let us now look for a little more formal understanding. So, what we are looking for is that, the greens function of y 1 at y 1 when the source is located at y 2 must be the same as the greens function at y 2 and the source is located at y

1 this is what we are required to prove we will recall that the greens function has this property that is del square G x y 1 plus k square G x y 1 is minus delta x minus y 1. This is exactly what we had for the greens function rule here, we have just replaced y with y 1. Similarly we will have we also have del square G x y 2 to be satisfying this equation right.

So, now what we will do is that will multiply this equation with G x bar y 2 and the other equation with G x bar y 1, as a result what we will get? From the first equation we will get G x bar y 2, del square G x bar 1 plus K square G x bar y 1, G x bar y 2 equals to minus delta x minus y 1, G x bar y 2 right and from the second equation we will get G x bar y 1 del square G x bar y 2 plus k square G x bar y 2, G x bar y 2 equals to minus delta x minus y 2 G x bar y 1.

And then what we will do is we will subtract these two between these two equation we will do a subtraction operation and as a result the second term will cancel out because they are identical. So, what will have is the following G x bar y 2 del square G x bar y 1 minus G x bar y 1 del square G x bar y 2 is equals to delta x minus y 2, G x y 1 minus delta x minus y 1 G x bar y 2 right simple enough and then finally, we will do a volume integral of these quantities. So, we will integrate both sides over the volume of interest which is actually the entire three dimensional space sorry in three dimensional space in the case that we are looking for an exterior acoustic problem it could also be a finite dimensional space in which in case we are looking for a finite acoustics domain right it could be both.

So, the reciprocity definitely holds for both exterior acoustic problems as well as interior acoustics problem. So, the volume could be anything at this stage. So, now, what we will do in the next step is that we will use to very important results, before stating that let me just copy and paste this in the next page.

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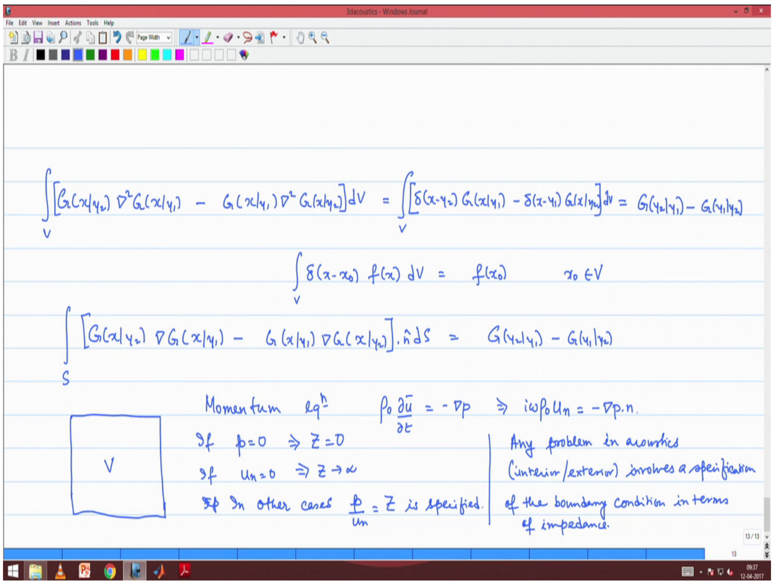

So, having obtained this relationship, we will use to very important results again in mathematical in mathematics we have this very important result, which is the integration of a delta function with any function. So, integral of delta x minus x 0 f of x over any volume, is supposed to be f of x 0 provided x 0 is within the volume of interest right.

So, this using this result what we have for the right hand side of this equation, that we simply need to substitute in place of x the value of this shifted the shift in the delta function. So, that would read as G y 2 comma y 1 minus G y 1 comma or bar y 2. So, that is what is there in the right hand side of the equation. If you look at the left hand side of this equation this is exactly in the form that is required by the greens theorem. So, we will appealed to greens theorem to reduce the volume integral into a surface integral which is G x bar y 2 gradient of G x bar y 1, minus G x bar y 1 into gradient of G x bar y 2 dot n d S and that must equal to G y 2 by y 1 minus G y 1 by y 2 ok.

Now, the point is as we said the principle of reciprocity demands that in fact, the left hand side surface integral should be 0 right because G y 2 comma y 1 must be equal to G y 1 comma y 2 or bar y 2. So, to get to that effect that y is the surface integral 0 will recall that from momentum equation, we have row 0 del u del t u is the particle velocity is minus grad P, which means i omega row 0, u in the normal direction which is having a unit vector n outward normal is having a unit vector n. So, U n will be minus grad p dot n right. So, this understood now also for any volume of interest whether finite or infinite u have an impedance condition if. So, this is the volume of interest. So, if a certain portion of the boundary has pressure equals to 0 that essentially means impedance is 0.

So, if p equals to 0; that means, impedance is equals to 0 if we have U n to be equals to 0 that will imply that impedance is infinite, and in other cases like in the case of three d acoustic exterior acoustics problem you will only have a specification of an impedance condition p b p by un equals to z is specified. So, the different boundary conditions that can arise in any acoustics problem for them matter interior or exterior, boils down to a specification of the impedance condition. Both the (Refer Time: 31:28) and normal boundary condition which are basically pressure conditions or the normal velocity conditions both of which are captured by the impedance boundary conditions.

Not only that there are some mix boundary condition problems, which requires a specification of pressure by velocity. That ratio pressure by velocity again specified and therefore, we claim that any problem in acoustics interior or exterior involves a specification of the boundary condition in terms of impedance. In particular I would like you to note that for the exterior acoustics problem it is only the boundary conditions the exterior acoustics problem finally, specifies the Summerfield radiation condition as is boundary condition for away from the source right. And the Summerfield radiation condition essentially is a specification of the impedance.

The impedance at large distances away from the source basically boils down to the plane wave impedance which is row naught c that is the moral of the story. So, therefore, the point is for any volume of interest for a well post acoustics problem you must have on the boundary the value of the impedance specify right.

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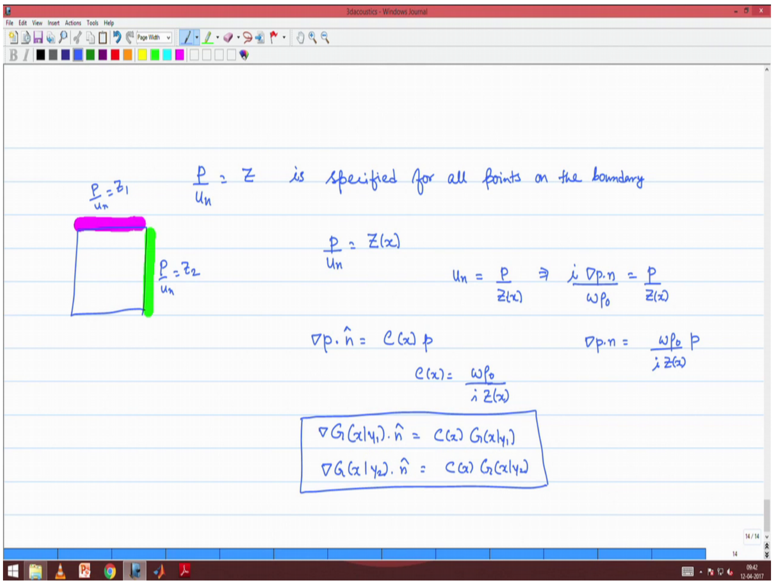

Which means P by U n equals to Z is specified for all points on the boundary. So, it is not necessary that on the boundary every point must have the same value of P by U n it may be possible that some portions of the boundary will have certain. So, it may be possible that this portion of the boundary maybe having a value of P by U n is equals to Z 1.

And it some other portions of the boundary it may be possible that you may have P by U n to be some other quantity Z2. Depending upon what is the absorbing layer that has been put in the different regions or in the worst case it could a one of them could be completely rigid one of these walls is it is completely rigid then the associated impedance for that wall will go to infinite right, but if some other walls is made of some corporate cardboard sheets as you are seeing in this wall then it will have it is own impedance. So, therefore, the impedance need not be a constant value throughout it is throughout the boundary, but the important point is there has to be a specification of p by U n which could possibly depend upon the locations of the boundary, but none the same it has to be specified right.

So, the point is U n therefore, will be P by Z x, Z as a function of x and you n intern we have seen is given by. So, U n here we have seen is given by gradient of P dot n divided by i omega row 0. So, i, here what I can do is, I can put the i on the top i times gradient of P dot n divided by omega y row 0 minus 1 by i is plus i on the numerator.

So, this is exactly what I will do. So, i times gradient of P dot n divided by Z x is basically P. In other words gradient of p dot n is Z x sorry I missed out the i omega row 0 omega by row 0. So, there is omega by row 0 here right U n is i delta p by n omega by row 0. So, there is an omega by row 0 here let us do this little.

So, this implies i omega i dealt p dot n divided by omega row 0 right is equals to P by Z x right which means delta P dot n is equals to omega row 0 divided by i Z x into P. In other words gradient of p dot n is some quality I called it C of x times p where C of x is omega row 0 divided by i into Z x, Z is the impedance. Now this is what is true for pressure is also true for the greens function because greens function is finally, a pressure like quantity right that is the basic premise. So, therefore, this sort of a relation will also hold for the greens function.

So, delta G x comma y 1 dot n, n is a unit normal vector is C of x, G x bar y 1 and similarly gradient of G x bar y 2 dot n is also C of x G of x bar y 2. So, this is how we related the two gradients in the I mean two direction derivative in the normal direction of the greens function, with the value of the greens function itself right. So, now, with this we turned back to this equation which again I will copy and paste. So, we were left with this equation in our analysis.

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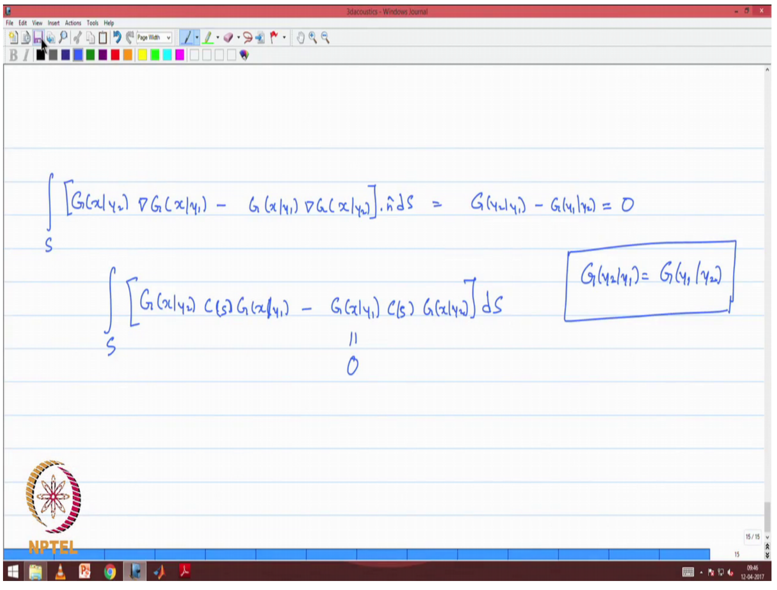

So, in our analysis we were left with this equation, now we what we are doing is that we are simply going to replace gradient of G dot n to be the value of G multiplied by some constant which does changed over the boundary, but we can keep track of that.

So, this quantity will b G x comma y 2 into c of x G x comma y 1. So, what I have done is I have said that the gradient of G x, I have basically used this formula which I had derived right. So, with that derivation I am led to the following G x comma y 1 gradient of G x y 2 dot n is also C as a function of x into G x y 2 right d S. So, this c possibly depends upon the boundary. So, I think I better notation would be instead of giving x. So, this c depends on S the location S right x is a is the point where we are receiving the acoustics signal. So, let us not use x. So, c that constant which in turn depends upon the impedance right could vary from point to point on the boundary that is the points. So, therefore, I will use a notation S.

But anyway you can understand at this point that both of these terms in the surface integral are exactly the same and therefore, this will vanish. Which leads us to the crucial finding that the greens function has the crucial reciprocity property, which I mean this basically turns out as 0 from where we get G y 1 y 2 is equals to G y 2 y 1. So, this is the reciprocity of greens function proved in a mathematically rigorous passion; however, you can have a very intuitive understanding as I just said that greens function the intuitive understanding of greens function is that if there is a unit source or if there is an impulse if there is a source which is located at just one point, and you are interested to find what is the signal that is received because of a certain sources at a certain point then that signal is denoted by the greens function right.

Obviously the signal itself that the greens function itself will depend upon the domain of interest, whether it is open playground and open space on a closed space. So, greens function does depend upon the properties of the domain that you are interested with, but no matter what be the domain if simply the position of the source and the receiver is interchanging, without affecting the strength of the source then you can understand that the quality and the quantity of signal received at the receiver location though the location has changed it has swapped with the source location the signal itself will not change right.

That is the idea of reciprocity, but none the same we have done the mathematical proof also. The reason why I did this mathematical proof is that, exactly the same ideas will go through in the next proof where in we will derive the Kirchoff Helmholtz integral equation so that will sort of round things up. But the proof of Kirchoff Helmholtz integral equation will make use of the reciprocity and it will proceed exactly along the same lines where we involve greens theorem where we involve properties of the delta function, and then once we look at Kirchoff Helmholtz integral equation we will be able to setup the problem in three dimensional acoustics which is basically the corner stone of the numerical I mean the equation itself may be too difficult to solve by pen and paper or analytical means.

So, we will set things up. So, such that it can be done in a numerical fashion from they are all, in that numerical techniques called the boundary element method which; obviously, is not within the scope of this course, but none the same the basic point is that we will lead up to the formulation of the boundary element method right. So, with that lets end the class here we will see you again in the next class.

Thank you.