Acoustics & Noise Control
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Module - 31
Lecture - 36
Inhomogeneous wave equation

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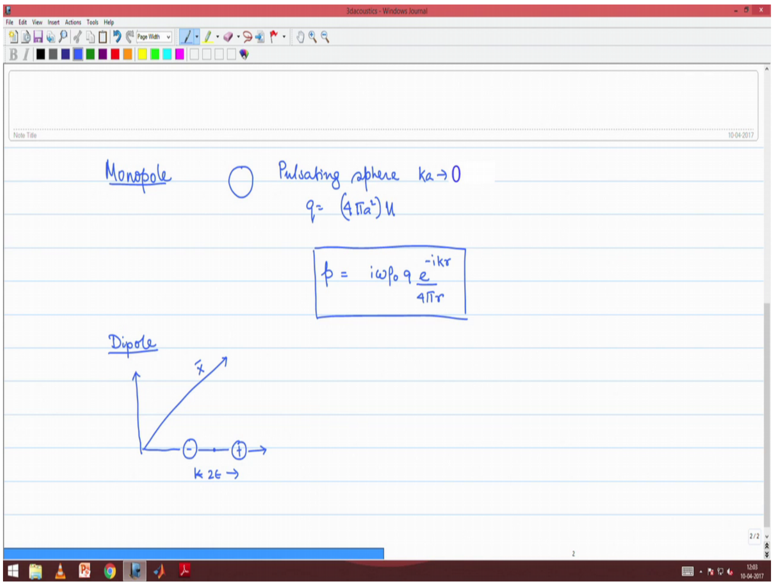

In the last class we looked at the important source which is called monopole and we approached monopole as a pulsating sphere. With K a tending to 0, but then the strength of the monopole which was defined as 4 pi a square into the velocity is held constant as this limiting process is undertaken. So, it turned out that the pressure at any point was i omega rho 0 times q e to the power minus i k r by 4 pi r.

That was the illustration that we did in the last class. Also we looked at a simple case of a dipole wherein we had two monopoles vibrating in, pulsating in opposite phase and we aligned this dipole and along the coordinate axis and we calculated the. So, this distance was 2 epsilon and this location was given as y and we found the pressure at any point x in the previous derivation. We will slightly generalize this derivation today by saying that this dipole is arbitrarily aligned to any coordinate axis, need not be along. We will no longer simplify the situation by saying that the dipole axis is align with a coordinate axis, the dipole axis could be completely arbitrary.

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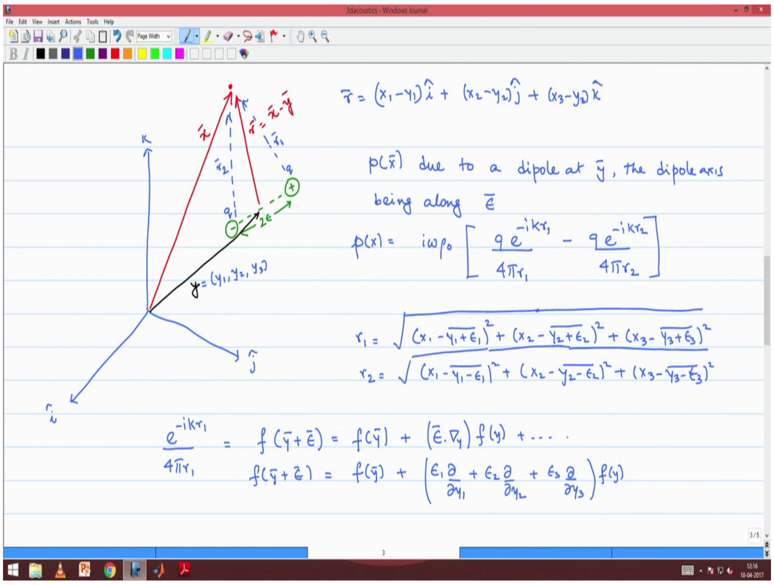

So, in other words what we will do is that, we will employ a perfectly general coordinate axis, call this i j k. We will have as usual two monopoles; one with a plus sign and the other with a minus sign, but this time the vector joining these two monopoles which is of length 2 epsilon. That is not necessarily aligned with any coordinate axis and the center of this dipole is again located at y and we would be interested to know, what is the pressure at the receiver location which is denoted by the vector x and r would be the vector from y to x. So, this is the problem at hand. So, we were interested to determine the pressure at the point x due to dipole aligned, due to a dipole at y; the dipole axis being along the vector epsilon. So, this 2 epsilon or epsilon is going to indicate the direction of the dipole axis from the negative monopole to the positive monopole.

Again exactly the same situations are true, that both of them monopoles have volume velocity source or volume velocity source of magnitude q, is just that one is coming with a minus sign. So, pressure at any point x would; obviously, be given by i omega rho 0 that part is common into q e to the power minus i k r 1 by 4 pi r 1 minus q into e to the power minus i k r 2 by 4 pi r 2. And what are these r 1 and r 2 quantities? r one is exactly from the positive monopole, you will have the vector to be denoted as r 1 and from the negative monopole the vector is denoted as r 2. So, r 1 and r 2 in this diagram are the vectors. When I indicate r 1 and r 2 without the vector sign, I mean the magnitude of those vectors.

So, in particular r 1 would denote square root of x 1 minus y 1 plus epsilon 1 plus x 2 minus y 2 plus epsilon 2 whole square plus x 3 minus y 3 plus epsilon 3 whole square. So, this is the quantity r 1 and similarly the quantity r 2 would be denoted by under root x 1 minus y 1 minus epsilon 1 this time whole square plus x 2 minus y 2 minus epsilon 2 whole square plus x 3 minus y 3 minus epsilon 3 whole square. So, these are the expressions for r 1 and r 2, both treated as perturbations to the quantity y. So, remember the y here is having three coordinates; y 1, y 2, y 3. By the subscript 1, I did not mean the components along the direction I, the subscript 2 stands for components of the vector along the direction j and by subscript 3, I imply the components of the vector along the direction k.

So, in this form we realize both r 1 and r 2 are taken to be functions of port all values of this y vector which has components y 1, y 2 and y 3 along the three coordinate axis respectively. So, what we are going to do is that this quantity e to the power minus i k r 1 by 4 pi r 1.

This will be taken as a function of y plus epsilon vector. This time it is a vector not a scalar. Last time when we did the derivation the y associated was a scalar it was in fact, a perturbation of only y 1, but now it is turning out to be a perturbation of y plus epsilon vector; both y and epsilon are vector. Now by Taylors Rule applied to a multivariable problem we will have this simplification that f of y plus epsilon could also be given as f of y just like you had it for single variable, the same thing will go through except for the fact that you will now have to deal with a multivariable calculus operation.

So, epsilon dot del y is what comes in the first order, I will tell you what that exactly means into the function evaluated at y plus higher order terms which does not worry us. I will just open out this expression a little bit more for you to be unambiguous. So, epsilon dot del y would mean epsilon 1 del del y 1 plus epsilon 2 del del y 2 plus epsilon 3 del del y 3. So, that is why I put a subscript y along with the gradient sign to remind myself that this gradient operation, in this gradient operation it is the primary variable is y not in terms of x. So, this will be the relation for f of y plus epsilon right.

Similarly, we could write the other expression which is e to the power minus i k r 2 divided by 4 pi r 2.

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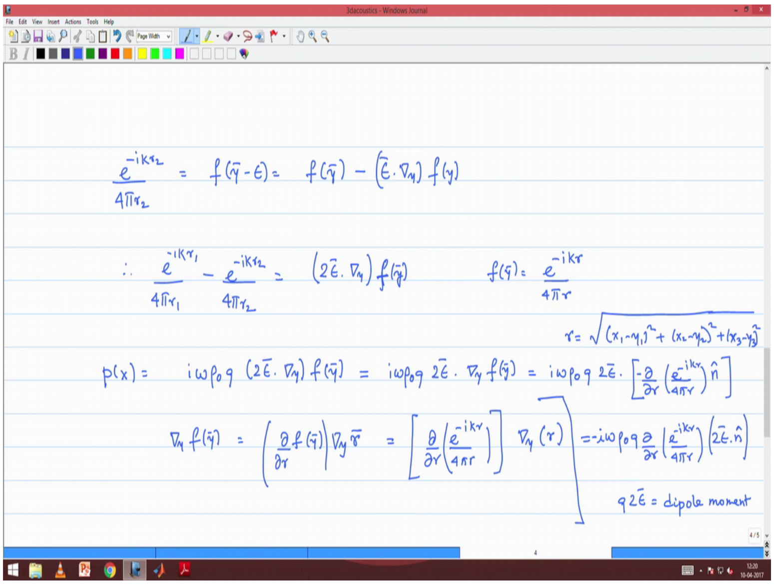

That would be written as f of y minus epsilon which will be f of y minus epsilon dot del y of f evaluated at y. So now, what we have to do is we have to subtract the two expressions out, which means if you subtract these two expressions written for e to the power minus i k r 1 by 4 pi r 1 and e to the power minus i k r 2 by 4 pi r 2. The f y term will get knocked off what remains is the second term which has the gradient effect.

So, now what we will do is therefore, e to the power i k minus i k r 1 divided by 4 pi r 1 minus e to the power minus i k r 2 divided by 4 pi r 2 will read as 2 epsilon dot gradient in respect of the y variable of the function y calculated at the vector y of the function f calculated at the vector value of y. So, this is the expression and remember f y would be e to the power minus i k r by 4 pi r, not r 1, not r 2, but at r. So, this is what we will have. So, therefore, the total pressure at the point of interest will be i omega rho 0 q into 2 epsilon dot del y into f evaluated at y, but epsilon is a vector we will keep track of that. Similarly gradient is a vector operator; we will keep track of that also.

So, we could write this expression in the following fashion; i omega rho 0 q 2 epsilon dot del y f evaluated at y, but then del y of f evaluated at y could also be written as f of y is this quantity. So, we will write this as del del r of f evaluated at y into del or into gradient with respect to y of r. This is just the chain rule of calculus, that here we are first taking the derivative with respect to y and then we are doing this chain rule. And then f y if we substitute that above expression we will get del del r into e to the power, del del r of e to the power minus i k r 4 pi r. This is the first term and the second term is going to be del y of r, the del y of r; r as a scalar sorry not vector r is the scalar quantity. And what was r? r is square root of x 1 minus y 1 whole square plus x 2 minus y 2 whole square plus x 3 minus y 3 whole square.

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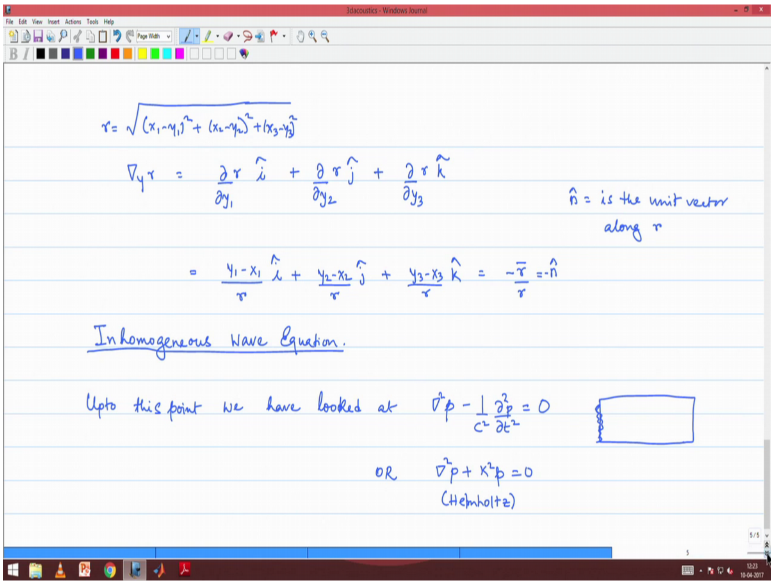

So, therefore, let us evaluate del y of r. So, del y of r where, r is this quantity needs to be evaluated and I will just open up this gradient operation once more. We will get del del y 1 of r along the first coordinate direction plus del del y 2 of r along the second coordinate direction, del del y 3 of r along the third coordinate direction. And then what is del r, del y 1? You will get x y 1 minus x 1 by r right because we have a square root x 1 minus y 1 whole square, if you take that derivative you will get 2 in the denominator and then entire thing will come out.

So, therefore, this is the expression for the differentiation I am not detailing out that step. So, this will be y 1 minus x 1 into in the i direction and similarly this second term will be y 2 minus x 2 by r in the j direction and similarly y 3 minus x 3 by r in the k direction, but then this is exactly the numerator is exactly minus r, what is r? r is a vector, if you look at this diagram r as a vector is x vector minus y vector. So, r is x 1 minus y 1 in the i direction plus x 2 minus y 2 in the j direction plus x 3 minus y 3 in the k direction; from this diagram it is evident.

So, therefore, this quantity can be written as minus r vector. So, i have to be careful about r as a vector and r as a scalar. So, minus r by r is what I have the numerator r is a vector, the denominator r is just the scalar which is the magnitude of that vector. So, this quantity can be represented as the unit vector in the direction of r. So, n is the unit vector along r which is from the source point to the receiver point. The source point is given the coordinate y or is given the position vector y and the receiver point is given the position vector x. So, therefore, you have n to be denoting that above quantity and if you make that substitution in the derivation, you get the following. So, here we will have i omega rho 0 q 2 epsilon dot of this quantity and this gradient of y f y is now computed as del del r of e to the power minus i k r by 4 pi r into del y of r. So, that quantity is minus n, n is a unit vector. So, then a minus has to be accounted for; that is all we have.

So, therefore, this expression can be just simplified to the following i omega rho 0 q del del r of e to the power minus i k r by 4 pi r into 2 epsilon dot in, but it comes with a minus sign because there is a minus sign associated with n. Please note that the final answer is a scalar, that is true for because we are looking for a pressure quantity and 2 epsilon dot n is going to give you that scalar. You would have seen the effect of Cos theta in the last derivation. So, that Cos theta effect is now replaced by this dot product and this dot product is also precisely pointing towards the same sort of directivity relation. The directivity happens because there is a direction associated between the vector pointing from the source to the receiver and the dipole axis itself.

So, in particular if you are pointing along the direction which is perpendicular to the, if the receiver is perpendicular to the dipole axis then, again you will see that this quantity will give you a 0. So, this is something that we had noted in our simplistic calculation also where we sort of align the dipole axis along one coordinate axis and obtain a Cos theta in contrast to this derivation. This derivation is slightly more general because if you have multiple dipoles then, there is no way that you can align your coordinate axis to each of those dipole axis; you rather work with one fixed coordinate axis and have expressions associated with each dipoles.

So, here this expression serves that job. It just uses multivariable calculus and little machinery from vector calculus is also used in this derivation, but the final answer is this that you have the pressure at any field point to be given by i omega rho 0 q del del r of e to the power minus i k r by 4 pi r with a negative sign popping out and there is a directivity term, which is captured through this effect which is 2 epsilon dot n. Sometimes q times 2 epsilon is called the dipole moment.

q is this scalar which is the associated velocity strength of the monopoles and that multiplied with the distance between the two monopoles taken in a vector form is going to give the dipole moment. So, with this machinery of monopoles and dipoles in place will go forward into the next topic which is about inhomogeneous wave equation. Till now what we have been looking at is, that we have looked at up to this point we were looking at the solution of the homogeneous wave equation. Either the homogeneous wave equation which is del 2, which we derived in its full entirety as del square P is equals to 1 by C square del 2 p del t 2. It is a homogeneous equation because if you transfer this term on the right hand side all quantities associated with the unknown P, if you transfer it on the left hand side the right hand side is left with 0.

So, it is like the situation of a x is equals to 0 or m x double dot plus k x equals, this is homogeneous equation. And we understood that the solution in such systems we looked at either this wave equation or we looked at the steady state form of this which is del square P plus k square P equals to 0, which is the Helmholtz Equation. In either case both of them are homogeneous equation and this solution get triggered because of a certain boundary condition. You had a certain domain and specifically with the duct problems and in the duct problems when we said that one of the boundaries is given a certain excited that over bridge oscillator piston of a flexible piston once there was a boundary condition that was non homogeneous that triggered a certain pressure waves with in this domain of interest and that was what we are solving for. We will adopt the slightly different perspective now. We will say that the region of interest now, will be the entire three dimensional space.

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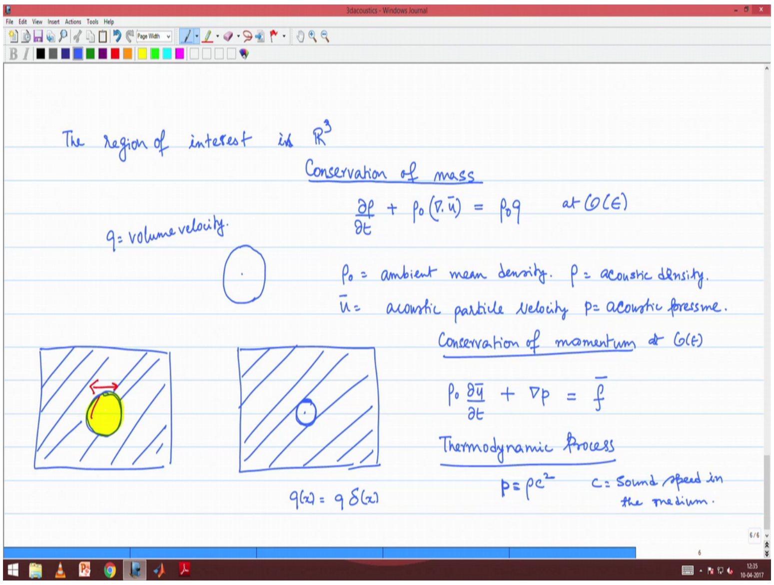

And within these three dimensional space, we will try to embed all these sources of sound we will not exclude anything, but we will try to include these sources of sound itself. What we were doing in the previous approved is that we were excluding the sources of the sound; that means, the oscillating piston was just at the boundary of our domain. It was not included in the domain right, the rigid oscillating piston or fixable piston in the case of the duct problems were at the boundary of our domain of interest. Whereas now we would like to include all such oscillating sources or fluid dynamics sources if it is of interest will be included within the region of interest which is the entire three dimensional region.

So, therefore, our fundamental approach will change will we will not try to solve a P d with a given boundary condition, but we will try to solve an inhomogeneous P d which includes the effects of the sources of sounds. So the effects of the sources of sound will be includes in the inhomogeneity associated with the wave equation. Just like you know any non homogeneous equation would imply, that there is a right hand side to that equation which is the non homogeneous stuff. So all the sources of sound would contribute to such right hand side in the equation and they will be called as in homogeneities. But physically speaking we must understand that, what is the reason of this in homogeneities? Till now if you understood the derivation of acoustic wave equation, the fundamental principle was that the conservation laws namely the conservation of mass, conservation of momentum and thermodynamics consideration together with certain asymptotic arguments and some mathematical simplifications lead as to this,

Essentially saying that mass is conserved and momentum is conserved within the volume of our interest. And obviously, we had good reasons of thinking so, but now let us say that we are worried about this entire three dimensional world, but in this three dimensional one there is a pulsating sphere, a sphere with just pulses readily inwards and outwards right. So, now, can we say that the volume is really concerned? No because of the mass in turn is not concerned because now this sphere is pumping in the mass and also sucking in the mass at alternative time cycles right.

So, we would like to include the effect of such in homogeneities and again rederive from those very first principles by including these effects of non homogeneities. So, instead of, see the way we approach the solution of monopole or pulsating sphere was to consider what happens exterior to the sphere. We were trying to match the vibration velocities, exterior to the sphere and thereby we got the outgoing wave solution, but now you are saying that instead of trying to match the exterior to these sphere we will keep this sphere included within our domain of interest, but we will just say that at this point there is an imbalance of mass or volume which ever you may call it. There is an imbalance in mass and this imbalance in mass will have to be rightly accommodated. If that can be accommodated we will get the inhomogeneous wave equation.

So, let us turn to that argument. So, we will look at conservation of mass. So, the conservation of mass gave us del rho del t plus rho 0 del dot u is equals to 0 in the case of at the order one. I am not redoing the entire analysis, but if you look sorry not at order one at the order epsilon. So, at the acoustic order this was the equation that was obtained, right rho 0 is the ambient or mean density, u is the acoustic particle velocity and rho is the acoustic density; that means, the total density is the ambient density plus of fluctuation over it which is the acoustic density. So, when we did the derivation for the wave equation that we have done at order epsilon we had this expression if you turn back to your notes that is what you are going to find.

But now this expression needs to be modified instead of getting a 0 here you are going to get a change in a mass imbalance that is going to come and the mass imbalance is going to be the volume imbalance multiplied by density. So, this is the imbalance in the mass.

So instead of 0 you are now going to get an imbalance which is precisely going to be rho 0 times q; q is the volume velocity of the source q equal to volume velocity. So, what we are pretending is that in this entire three dimensional region there could be lots of this small pulsating spheres or monopoles whatever you may call and each of these pulsating spheres are going to cause imbalance in the mass conservation. And we have to strictly account for them, the other way of doing the problem would have been to leave out these volumes and to considered the fluid which is exterior to pulsating sphere. In which case there is no imbalances in the conservation of mass right at all those. So, the two approaches I will just high light once more suppose; in one case we could consider the region of interest to be everything outside this pulsating sphere.

This is my pulsating spheres if we consider all point which exterior to this pulsating spheres then, there is no question of any imbalance in the conservation law associated with the mass. But in the second case which is what we are doing now if we wish to considered the entire volume as our region of interest. So we are considering the entire volume as our region of interest right. We have to understand that it this case there is a mass in balance which will be set up at this point right. So, the mass imbalance function will be q x equals to q into delta x; assuming this is at the origin. So, at the origin this pulsating sphere is going to create an mass imbalance. We have to account for that strictly in the conservation laws right that is exactly what we done right.

Now, consider the conservation of momentum equation also. So, conservation of momentum equation act order epsilon which is the acoustic order reads as rho 0 del u del t equals to minus of grad P right and u is the vector, I should have put a vector sign here also u is the acoustic particle velocity rho 0 is the ambient density and P is the acoustic pressure, p I should write as acoustic pressure.

So, therefore, this is a vector equation I could just turn this equation around by saying that rho 0 del u del t plus gradient of P equals to 0 at order epsilon. This is what happens when there are no imbalance is within the region, but just like we contemplated the situation of pulsating sphere; now if you considered this sphere instead of pulsating it is rather oscillating. So, considered this that if you have a strider and in a beaker of water you are just oscillating this strider then, what are you doing with your stridering effect? You are only imported the momentum to the fuelled right. So, there is, so in top view the situation would look like this. This is the region of the strider I could as well mark this with the highlighted. So, this is the region of the strider. So, this strider is supposedly oscillating up and down in this fashion right. So, around this point there is a force which is given from the external source to the fluid domain of interest.

So, there at this point precisely there is a imbalance of the momentum equation and you need to account for the right. So, the imbalance of this momentum would now cause a right hand side to appear in this case which I will denote as f. So, f is the force per unit volume because everything here is per unit volume, f is the force per unit volume which is there within the fluid domain. Because in the present picture we are not excluding the sources we are including the sources. So, the effect of the sources would be to either create an imbalance in mass or an imbalance in the momentum loss and this needs to be accommodated correctly in the equations of motion and finally, there are just these two primary equations of motion I mean continuum laws that we have to satisfy. The other very important law, that of the thermodynamic process that is exactly the same and does not need any modification.

So, the thermodynamic process basically tells us that P is equals to rho C square where, C is the sound velocity, sound speed in the medium using the arguments that it is an adiabatic process and using the Taylor Series in all of that you can, this part remains just identical. So, I am not repeating. So, what I will now do is that I will do the same essentially do the same simplifications that we did in arriving at the homogeneous form of the wave equation and we will just using the same step we will now be derive the modification to that homogeneous wave equation which we will now see will have a certain right hand side and will call those as a inhomogeneous wave equation. So, the two equations I will just copy may be I do not need it, I can just write it once more. So, the two equations I have is, I think copying is better.

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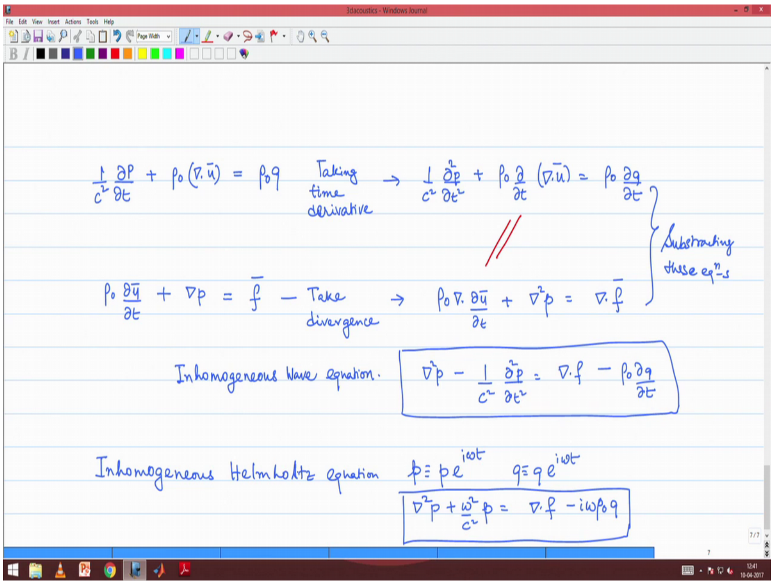

First part is that we will change over from rho to P rho the acoustic density, but we have already seen that the accounting density can be related in terms of the acoustic pressure in the following fashion right. So, that simplification is done, now we will take the as usual will take the time derivative of this equation; so taking time derivative of the first equation.

What will get is this 1 by C square del 2 P del t 2 plus rho 0 del del t del dot u plus equals to rho 0 del q del t. That is the first equation and the second equation we will take divergence. So, will take the divergence this is what we will get, rho 0 del dot del u del t and u is the vector plus del dot del P which is del square P plus equals to del dot f and f is the vector right.

So therefore, at this stage we see that these two terms are equal and therefore, if we subtract. So, I will just indicate that these two terms are equal. It does not matter whether you take divergence first and then take time derivative or you take time derivative first or divergence if the rules of calculus says that, if the function of interest is sufficiently smooth then order of the derivative processes can be interchange. And thereby you will have those two terms to be equal. So, therefore, if we subtract these two, we will get 1 by C square or maybe I will take del square P minus 1 by C square del 1 P del t 2 is equals to del dot f minus rho 0 del q del t right.

So, this is the inhomogeneous wave equation in contrast to the homogeneous wave equation which had the right hand side to be 0. Now you have the right hand side to be little more complicated, but the benefits here is that you do not have to deal with boundary conditions any longer. In the homogeneous wave equation you are dealing with the equation in a specified domain together with its boundary and it was the boundary condition which was the trigger for the solution. Here the trigger for the solution is not in the boundary, there is no boundary in fact, because everything is included. The entire volume of interest is that the entire three dimensional space which means there is absolutely no boundary of our interest.

So, therefore, the trigger to the solution is not the boundary condition. You do not have to deal with the boundary condition; that is the advantage of inhomogeneous equation, but you done the same have to deal with this right hand side in homogeneity. So, that is the two contrasting approach between the homogeneous wave equation and inhomogeneous wave equation. So, this is marked as the inhomogeneous wave equation and if the inhomogeneous wave equation has come then the inhomogeneous Helmholtz Equation cannot be any different. To arrive at the inhomogeneous Helmholtz equation we understand that the quantity of interest which is p will have to be replaced by e to the power p times e to the power i omega t. Not only p quantity of interest will be harmonic right and we have to do frequency by frequency analysis.

So therefore, any quantity will can have only harmonic time dependent. In particular p will have a harmonic time dependences, q will also have exactly the same type of harmonic dependences; which means the above equation will now read as del square p plus omega square by C square p is equals to del dot f minus i omega rho 0 times q. This is the inhomogeneous wave equation right.

And now sorry inhomogeneous Helmholtz equation and now we have to see how we can solve this equation right. It is fine to formulate the equation, but we now need to look at how to solve this equation.

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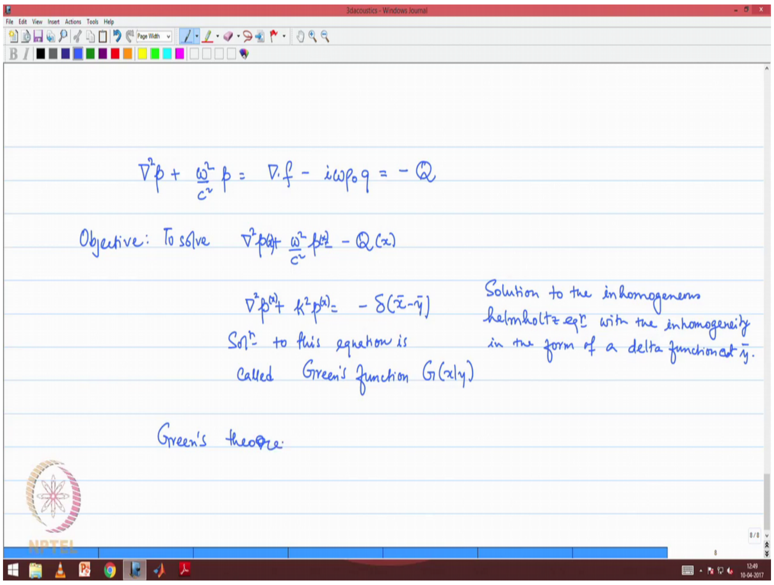

So, del square p plus omega square by C square p is equals to del dot f minus i omega rho 0 q is the equation that we are looking to solve. And just to shorten our notion will call this entire term as minus Q. So, del square p. So, the objective will be to solve del square p plus omega square by C square p is equals to minus of capital Q, which in journal in the function of the space variables also. Because the how does this x arise? Because depending upon the locations where you are oscillating spheres and your pulsating spheres are located at different positions, you will have different sources and all sources also would have different strings right. The strength of the oscillations of the source, strength of the pulsation of the source could be different at different points and as a result you will have a spatial dependence.

Now, the question is how do you solve this equation? Any ideas at this point? A PDE with a right hand side; yes you know how to solve this equation?

Student: It is like or a spring mass.

Spring mass in ODE, it is a PDE with the right hand side.

Student: Separation of variable.

Separation of variables would have work with zero left hand side, but so where as zeer or non zero does separation of variables. How does it keep track of? In the case of ODEs, how do you solve non homogeneous ODEs? Sorry.

Student: Complementary function; particularly complementary function.

All that is very good, but complementary function; obviously, does not depend upon the non homogeneity. It is the solution for homogeneous system right.

Student: particular integral.

Particular integral depends. So, you find the particular integral associates with the

Student: Right hand side.

Right hand side. How do you do for spring mass system with the particular right hand side? You found particular integrals really?

Student: (Refer Time: 43:38)

But for to go to convolution integral, before convolution integral something else is required, what is required?

Student: Impulse response.

Impulse response function. So, what is an impulse response function? You are actually for solving, for the system with an impulse right. The actual system that you are interested is M x double dot plus k x is equals f of d, but to solve that problem you water it down and you say let us first try to solve with f of t is equals to delta of t. The solution to that is impulse response and then using convolution integral you con cog the solution with for an arbitrary forcing. Because that particular integral the problem is very strange that you know we remember everything that is written in our school books, but we forget everything that we study in our universities and higher education system that or that never settles anyway. So, the point is that finding that particular integral is not possible for, except for very specific cases. Whereas convolution integral is at list numerically can be implemented for any for arbitrary function right.

So, therefore, convolution integral is the better approach. So, we will adopt exactly the same technique here. What we will do is that; we will first solve this equation for the case of a delta function and where should this delta function be located? So, instead of solving for q of x we will solve for a delta function which is located at say which is centred at some point say y. All of these are functions of x, p is the function of x, q is a function of x. Now this would denote that this is a solution to the inhomogeneous wave equation or inhomogeneous Helmholtz Equation.

I am sorry with the in homogeneity, in the form of the delta function. And I hope you know, what are the different properties of delta function? Most of those properties are going to be same whether you considered in one dimensional time or three dimensional space; does not matter. So, we just carry over at least loosely most of those properties and will deal with them as and when it is required. So, the point is that we will try to solve this equation which is a in homogeneity wave equation with inhomogeneous in the form of delta function, at a certain position which we are calling it as y. The solution to this equation, anyone knows what the solution to this equation is called with the in homogeneity? In the form of delta function? This is called Greens Function and this can make you grow go green actually literally.

An entire this string notation also; it has to include this notation of y because it depends upon both the source and the receiver. Basically what it is saying that the, it is an in homogeneity which is located at y, but it gives you the answer of what is the pressure at any point x, due to an in homogeneity located at a specific point y.

So, we have to look at how to find Greens Function and basically what we will see is that it, at least in one special case the Greens Function is exactly found out by now actually. It is the monopole source, the monopole source will turn out to be the Greens Function at least for one very nice case which is called the Free Space Greens Function and that will enable us to solve the complete problem of acoustic in terms of an integral equation. So, that is some mathematical derivation which is left in the course. This part is going to be really rigorously mathematical and you will do well to brush up your vector calculus and there is something called Greens Theorem. Please learn Greens Theorem, before you come to next class anyway.

Thanks you.