Acoustics & Noise Control
Dr. Abhijit Sarkar
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Module – 29
Lecture – 34
Spherical Waves

Welcome friends, this is the next lecture on acoustics and noise control. So, till now, we had been studying the acoustic plane waves. In particular, we have formulated the equations, we have looked at the solutions, we had also looked at a very important application of acoustic plane waves which applies in industrial application for muffler design. We had quite elaborately discussed the procedure of muffler design based on plane wave theory, but now in the remaining part of the course, we will quickly run through the 3 dimensional acoustics and in particular the spherical waves.

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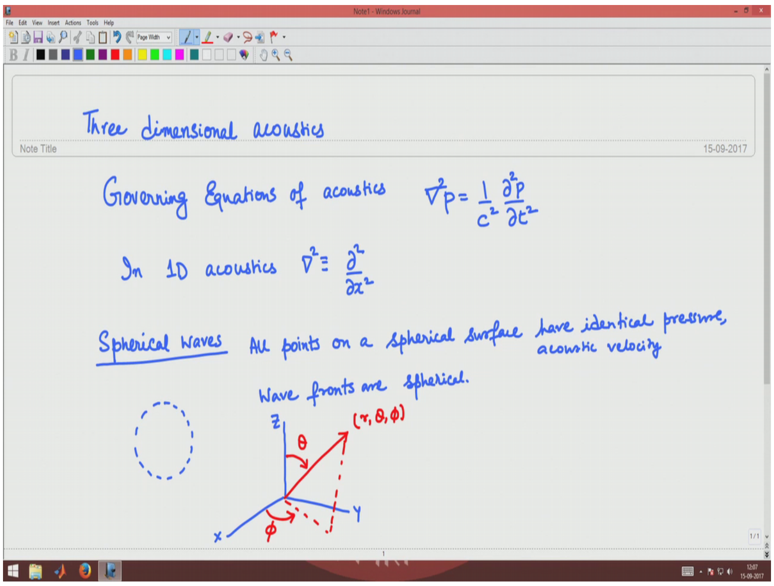

So, the topic of this last module is 3 dimensional acoustics. So, towards that end, we start again with the governing equations of acoustics. You will recall that we had derived the governing equations of acoustics in the early part of the course and that equation was derived in a general 3 dimensional settings. So, the equation if you recall would was given in this form del square P is equals to 1 by C square del 2 P del t 2.

So, will start from here and we will develop the 3 dimensional acoustics wave solution just as we did for the one dimensional case. So, in the one dimensional case, again to recall in 1 D acoustics, we simply simplified this Laplacean operator to be the second derivative with respect to our space coordinates which was defined as x. So, what we will now firstly look at in 3 dimensional acoustics is the category of spherical waves and in spherical waves as the name suggest the wave fronts are no longer planar. So, what we will expect is that all points on a spherical surface this time and since I cannot draw a sphere in my plane of the tablet, I would rather draw a circle, but you should understand this is actually a sphere which is drawn.

So, all points in this sphere will have identical pressure identical velocity identical intensity all quantities of interest. So, just like in a plane wave, we define all quantities of interest are identical in planes and these planes happened to be normal to the direction of wave propagation exactly the same wave for spherical waves all points on spherical surface have identical acoustic pressure acoustic velocity and everything related to acoustics is same at these points.

So, thus we say that the wave fronts this time are spherical so in contrast to the planar wave fronts which we studied in plane waves. Now we will study the spherical waves which bears spherical wave fronts and towards that end, we will change over now to a coordinate system which is spherical. So, will recall as spherical coordinate system is indicated by an r theta phi by an r theta phi specification where r is the radius vector theta is this angle and when you drop a perpendicular then the angle that is subtended in this plane would be marked as phi. So, r theta phi r the variables in the spherical coordinate system as opposed to X, Y, Z in the Cartesian gold net system.

So, we will start from the solution of the governing equations of acoustics in 3 D, but in a spherical coordinate system.

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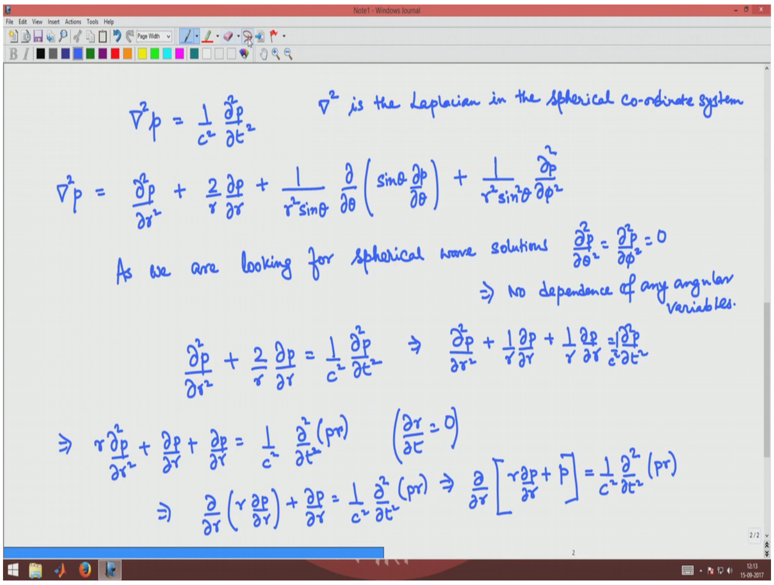

Because we are looking for a spherical wave solution in our case. So, here is what we will do. So, we will try to solve del square P is equals to 1 by C square del 2 P del t 2 and del square is the Laplacean in the spherical coordinate system and this expression for a Laplacean in a spherical coordinate is available in any book on engineering mathematics, in particular, I will refer the book by Kreyszig engineering mathematics by Kreyszig, it has the expression for the Laplacean in a spherical coordinate system. So, I do not wish to derive these expressions.

So, I will simply you the result which is del square P in the spherical coordinate system will be written as del 2 P del r 2 plus 2 by r del P del r plus 1 by r square sign square not signs square sin theta 1 by r square sin theta del del theta sin theta del del theta plus 1 by r square sin square theta del 2 del 5 2 and there is a P here; there is a P here. So, with this is the expression for the Laplacean of the variable P acoustic pressure is given in this form, once you open it up in the spherical coordinate system r theta phi and r theta phi is as indicated in this diagram.

Now, going forward, we realize that we are looking for a spherical wave solution which means there is absolutely no dependence in the angular variables, all points on this sphere will have identical acoustic pressure that is after the definition of spherical waves and therefore, if all points have go to have the same pressure distribution at a particular radius, it cannot have any dependences with the theta variable or the phi variable. So, therefore, as we are looking for spherical wave solutions we can happily set these 2 conditions to be 0 which implies that there is no dependence of any angular variables.

So, with that being the simplification, now we can look to solve this simplified equation which is del to P del r 2 plus 2 by r del P del r is equals to 1 by C square del to P del t 2 and what we can now do is we can open this term a little further to give us del 2 P del r 2 plus 1 by r del P del r plus 1 by r del P del r is equals to del 2 P del t 2 1 by C square. So, what I will do is I will multiply throughout by r and get it in the following form. So, if I multiply throughout by r, I get r del 2 P del r 2 plus del P del r plus del P del r again is equals to 1 by C square and the r multiplication here can be carried within the differentiation process. After all the space variable which is the variable r symbolized by the variable r does not lead to any problems here because del r del t is 0; that is space variable and time variables are independent. So, this step should be perfectly valid as far as the mathematics is concerned.

Next we can group these terms in the following fashion, we can write it as del del r of r del P del r because r del P del r is going to give when you differentiate it. It is going to give the first term as r del 2 P del r 2 and del del r are of r is just 1. So, this is the second term arising out of it plus del P del r again is equals to 1 by C square del 2 del t 2 of P r and this we can further simplify in the following fashion. We can collect the differentiation process together and then write r del P; del r plus P, right. This is the left hand side of the previous step and that is equals to 1 by C square del 2 del t 2 P r. So, we have to simplify this a little further. So, I will just copy paste this equation in the next page to make that simplification happen.

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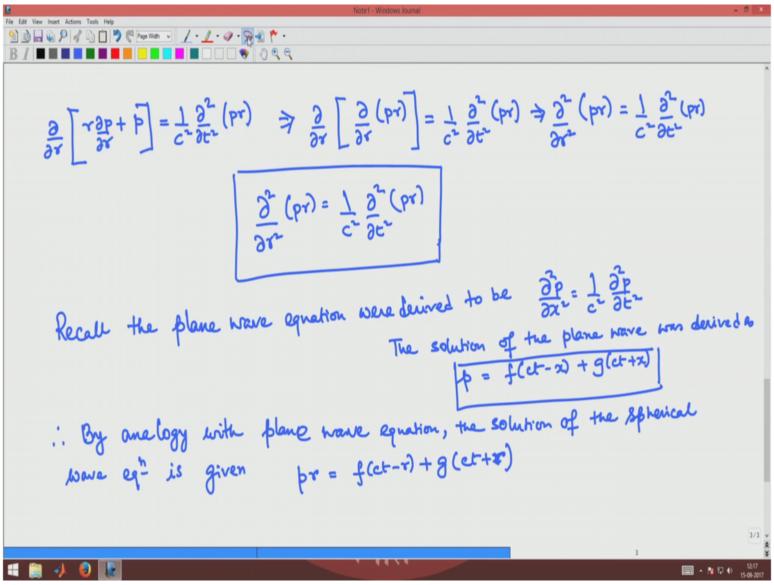

So, again I restart from this last step and let us see what the next step leads us to. So, in the next step, I realize that the quantity under the bracket is nothing, but del del r of P r. So, again this is a product of 2 quantities P and r. So, when you take the differentiation with respect to r, the first quantity you will get is r times del P del r and the second quantity you will get is P times derivative of r with respect to r which is unity which is exactly what is shown in the previous step.

So, this simplification should also be justified to you and finally, we get it the right hand side is as it is. So, this equation in turn implies that I have to actual take second derivative of the P r variable on the left hand side and that is essentially equals to 1 by C square del 2 del t 2 o f P r. So, the final equation that I am looking to solve, after all this simplification for these spherical waves looks like this; so, here you will realize this equation is identical to the plane wave equation except for the change that instead of P r; you had P in the plane wave equation. So, I will just make a recall here. So, recall the plane wave equations where derived to be del 2 P del x 2 equals to 1 by C square del 2 P del t 2.

So, the only difference between the plane wave equation and the spherical wave equation as we have derived lies in the fact is in that the space variable x has been replaced by the radius variable r and instead of solving for P. We now seemed can solve for P times r other than that everything else is just the same and we will also recall the solution of the plane wave was derived with D Alembert solution and everything was derived as P equals to f of C t minus x plus g of C t plus x.

So, this was our solution for plane wave equation, the equation itself was stated as I have given here. So, therefore, by analogy with plane wave equation, we should be able to construct the solution for the spherical waves in no without any trouble. So, that is what we are going to do. So, that by analogy with plane wave equation, the solution of the spherical wave equation is therefore, given as is given as. So, this time, it is going to be P r which is going to be my variable of interest and P r will have an identical type of solution like a plane wave just that the variable x will get replaced by the variable r and therefore, if P is our quantity of interest, the final solution for P is now pretty trivial to contemplate.

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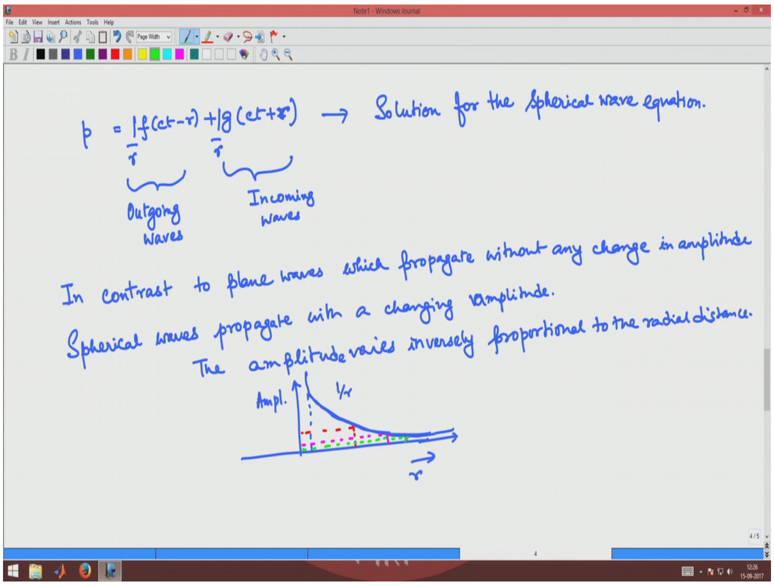

So, the solution for P can be written as follows. So, all that we have to do is to divided the r over here. On the left hand side, we instead of multiplying by r, we multiply the right hand side with 1 by r, I mean; we divide the right hand side with r. So, this is the solution for the plane wave equation. So, this is the; sorry, solution for this spherical wave equation. So, this is the solution for the spherical wave equation. So, this represents waves which are completely symmetric which are dependent only on the radially coordinates; does not depend upon the angular coordinates and the characteristic of this wave is this that all points which lies at the same radial location have got identical pressure, identical velocity, identical intensity, everything, if pressure is same velocity is after all derived from the pressure through the Euler equations.

So, therefore, if pressure is same velocity also has got to be the same the intensity is also have got to be the same. So, therefore, what I have presented to you is that the spherical wave equation is an easy fallout, if you can understand the development of the plane wave equation, spherical wave equation is a very smooth transition, if not an easy transition. So, let us carry on to understand the interpretation of these 2 solution as was talked earlier, this represents just like in the case of plane wave; in the case of plane wave, we understood that this represents an outgoing wave or a forward wave and this represents an incoming or backward wave right AFNG has this characteristics

Now, here also, the same rule applies except for the fact that as it goes out or comes in, the amplitude no longer remains constant, but it has; it varies in proportion to 1 by r. But firstly, let us make this observation that this first term is associated with outgoing waves and the next term is associated with the incoming waves and also we will make this observation that in contrast to plane waves which propagate without any change in amplitude. Spherical waves propagate with a changing amplitude or a varying amplitude the amplitude varies in proportion to I or I should say inverse proportional to the radial distance radial distance.

So, this is very useful that the amplitude by itself is going to have a fall if I just plot the amplitude it will have a L by r D K. So, the amplitude against distance; so, this is the distance which is from the origin where you are interested. So, the distance; as the distance from the origin increases then the amplitude is seem to fall. This is very interesting in terms of applications. So, in terms of plane waves, what we understood was that the amplitude being constant, we had to put in muffler such that we get some reduction of the amplitude at the point where the exhaust is opening out to the atmosphere, but in case of spherical waves it so happens that as the wave is spreading; as the wave is going to a far; going to distances which are further of then the wave itself suffers a drop in amplitude and that essential means that the noise associated with these spherical waves will drop drastically as you go a little further away from the source.

So, in terms of noise control applications, it has got huge implications one way by which you can appreciate this result is by doing; by recalling something which you very often do in your everyday experiences. When you actually put your ear phone on to the ears then you are residing the ear phone place out as sound such that the radial distance between the location of the sound and your eardrum is very minimal which is why the ear phone when inserted into the ear is actually producing a very appreciable quantity of music and sometimes it could be very loud music also, but hold the earphone just away from your ear and the sound is barely audible. So, what happens at that time is that when you are holding the ear phone just away from your ear drum, you are occupying a position as indicated by this red line. So, here you see, you have now in encountered a drastic fall in the sound levels.

So, therefore, you cannot hear the music, if you happen to hold the ear phone away from your ear, but even if it is very close, but away from your ear, there is a drastic fall in the sound level and you cannot here it any longer. What happens in a speaker on the other hand when you are hearing a sound through a speaker as possibly you are at this time in that case, we are talking about 2; if you are hearing music from a speaker, then possible you are already standing far away from the speaker and then if you go a little bit further then you possible are at let us say this position and between these 2 positions, there is hardly any difference in the amplitude of sound.

So, once you are all; your base line is already faraway then going a little further away does not create much of a perceptual difference in terms of perception of sound, but if as in the first example, if the base line or if you are referencing a sound measurement with respect to the situation where the sound is produced at almost very close to your ear drums then; obviously, it will very loud. But once you increase the separation between the ear drums and the place where it is produced, in this case, it is the ear phones then you will find a drastic fall in the sound levels and that drastic fall could in actually mean that the sound is not even audible to your ears. So, this has important ramifications as I explained.

So, what I will now do is that form the pressure expression that we have obtained, will go to the velocity expression, but before that just like as we did for plane waves, we will now no longer take an arbitrary time dependence, remember in the development of the plane waves, we first took the time variable into consideration, but when we said, we will be interested only in a harmonic time dependence because that is what leads to study state noise and in case of harmonic time dependence, we found that the solutions are much easier to handle and yet it has got very many applications.

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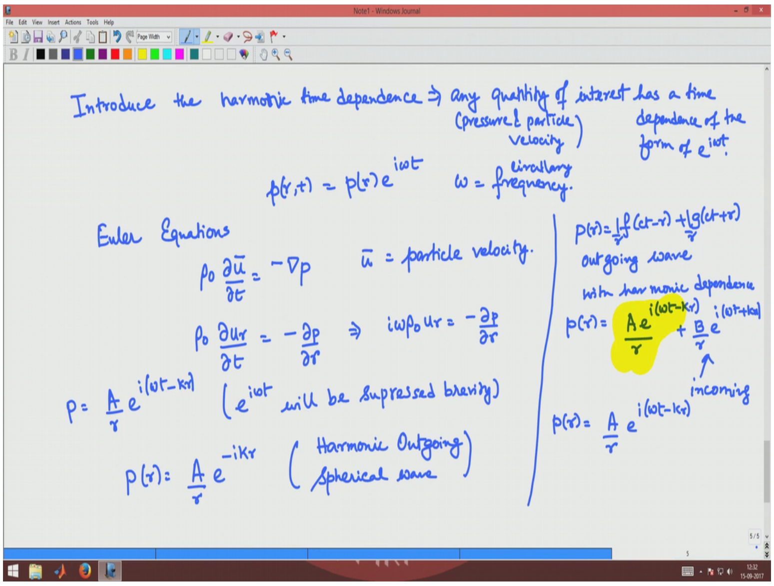

So, therefore, we will introduce from here on we will be introducing the harmonic time dependence which again to recall then this implies any quantity of interest I mean it is mostly the pressure and velocity, I would rather be specific by saying pressure and velocity or even more specifically pressure and particle velocity has a time dependence of the form of e to the power i omega t as was again done for plane wave theory. In plane wave theory, we said that all the pressure, we assume that the pressure does not have an arbitrary time dependence, but it has a time dependence which is harmonic and that essentially meant that n pressure at for plane wave P x comma t, we said is given as P x into e to the power i omega t. Same, we will do for the spherical waves also, just that we will replace the x variable with the r variable, r denoting the radial location. So, an omega is the frequency; it is a circular frequency rather than the frequency which is reciprocal of the time period. So, this is circular frequency and you know what that means.

So, with this substitution, we can appeal to the Euler equations which again is a general equation for a fluid and that stated that main density times del u del t should be equals to minus of gradient of P where u is the particle velocity. So, derived this equation in one of our very early classes, I will just recall that result for you at this stage, what we need is the radial velocity at this point because we are talking about a spherical wave when we are taking about a spherical wave, please understand; again from the figure that all particles will have to move only in the radial direction because of the symmetry because of the symmetry there cannot be any angular motion because any angular motion of the particle will break the cemetery of the problem. Everything should depend only on the radial coordinates and therefore, the radial velocity alone should be non 0 the other angular velocities should be 0 that is again coming from the symmetry arguments.

So, if we specialize this equation towards the radial velocity, this is what we will get. Radiant of P will be simplified to del P del r, but now since all variable of interest are having harmonic time dependence, we can write del del t equivalent to i times omega and rho 0 stays times u r to be equals to minus del P del r and we already know that P of r has got to be of this form f of C t minus r plus g of C t plus r. So, let us consider the outgoing wave to start with. So, will consider the outgoing wave and the outgoing wave form will be given as so, the outgoing wave with harmonic dependence just like we did for plane wave.

So, in the plane wave case, we replace this C t minus r to basically omega t minus K r where omega by K turned out to be C, exactly the same thing we are repeating now. So, with harmonic dependence P at any radial location will be given in this form e to the power i omega t minus K r by r plus b by r e to the power i omega t plus K r, but then we said that we are not going to look at incoming waves. So, this is the incoming wave, if you are interested only in the outgoing wave, you should take only the outgoing wave term which is this is the outgoing wave term the other term is the incoming wave.

So, for the present derivation we will take only the outgoing wave term which reads as P r is equals to A e to the power i omega t minus K r. So, when we make this substitution here and we calculate the del P del r. So, if P is equals to A by r e to the power i omega t K minus r and e to the power i omega t will be suppressed from here on because again this is in consonance with what we did earlier for plane waves we just suppressed this for brevity e to the power i omega t is implied we do not have to write it again and again.

So, therefore, we write this in a short form as P r e to the power minus i K r. So, this is the harmonic outgoing spherical wave. So, let us do a little bit of analysis related to this equation.

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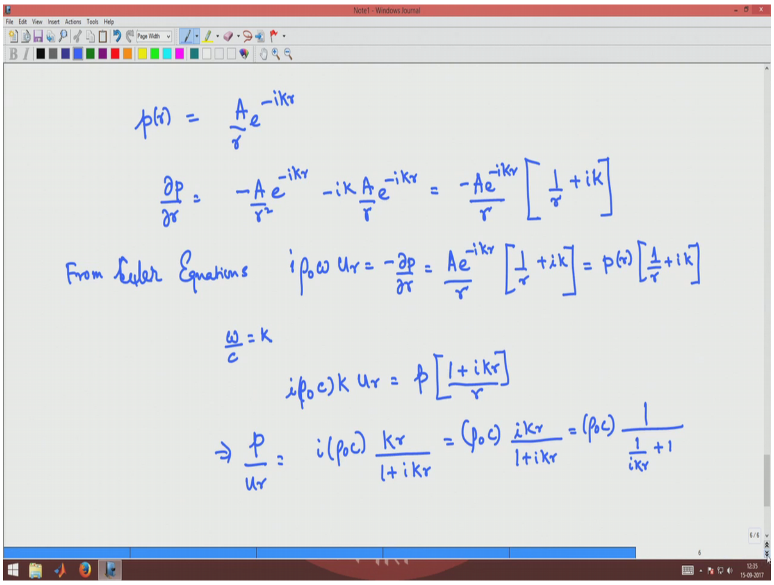

So, if P r is A by r e to the power minus i K r then del P del r is given by minus A by r square e to the power minus i K r minus i K A by r e to the power minus i K r and if we take minus A e to the power minus i K r by r as common, then what we will get is the following; 1 by r plus I K and from the Euler equations, what we need is the following. We need rho 0 i omega rho 0 into u r must be minus del P del r which now reads as A e to the power minus i K r by r into 1 by r plus i k, but then this group of term A e to the power A by r e to the power minus i K r is nothing, but the pressure at the location r.

So, that change we will make and as a result we will get to see, sorry, 1 by r plus i k. So, then in the next step what we will do is we will change A we will recall that omega by C is K and as a result i rho 0. So, instead of writing omega I will write it as C times K u r will be equals to P into 1 plus i K r by r and that too in the next step we will see as P by u r to be i rho 0 C K r by 1 plus i K r little more simplification rho 0 C times i K r divided by 1 plus i K r which could also be written as rho 0 C 1 divided by 1 by i K r plus 1.

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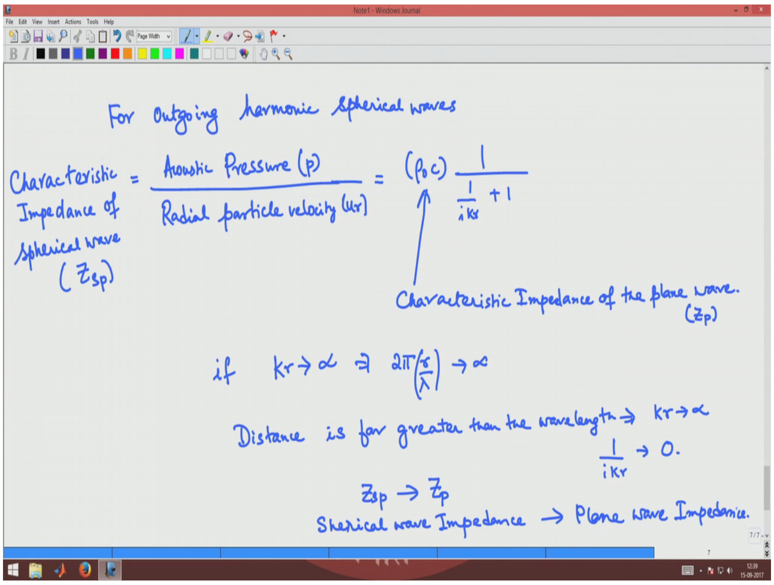

So, this is very crucial result that for outgoing harmonic plane waves A, sorry, not plane spherical waves, we have this crucial result which is the pressure denoted by P acoustic pressure denoted y P divided by radial particle velocity denoted by u r that quantity is turning out to be rho 0 C into 1 plus 1 by i K r plus 1, we identified this sort of a quantity pressure by velocity in our earlier analysis as impedance. So, we will continue doing so. So, we will call this quantity as the characteristic impedance of spherical waves and that turns to be rho 0 C multiplied by this quantity and rho 0 C; you will remember is the characteristic impedance of the plane wave.

So, the characteristic impedance of the plane wave and the characteristic impedance of this spherical wave is related and the relation is given in this equation; what we can quickly note is that if we have the situation that K r is large; K r is going large which means that this also implies that 2 pi by lambda into r is a large number which means that the distances that we are looking at is much larger than the wave length of the acoustic wave that is propagating and wavelength again has the same connotation, it basically means the distance between the 2 maxima or the 2 minima of the wave.

So, if you have this situation that the distance is far greater than the wavelength then you will land up into this situation that K r is a large quantity and if K r is a large quantity then 1 by i K r will become a small quantity and therefore, the characteristic impedance of the spherical wave which I can call as ZSP will be same as the characteristic impedance of the plane wave ZSP will tend to ZP. So, spherical wave impedance will approach the plane wave impedance.

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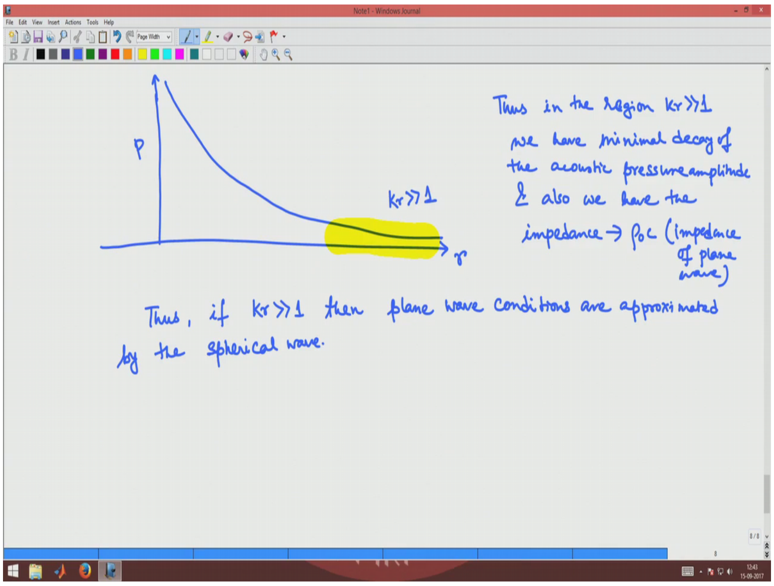

So, in other words, let me just consolidate the findings, what we have to found is that as the distance from the source increases, the pressure is going to fall off and therefore, we when you are talking about distances which are large vary at very large distances, let us say this zone where in you have this condition coming in that K r is large. So, here let us say K r has gone much much greater than 1 which is like it is going towards infinity.

So, here you see that the acoustic pressure amplitude. So, this is the plot of pressure by the way here we can say that the decay in this region of space for the acoustic pressure is minimal and also the impedance in this region we will see we will say is almost approaching the plane wave impedance because K r is large if K r is large we seen have that this spherical wave impedance has become the plane wave impedance.

So, if the impedance is same as rho naught C and the pressure is not dropping; that means, we essential have the plane wave condition that is the velocity in the direction of propagation of the wave will be given by rho not C multiplied by the pressure and that is exactly the same situation that we had in plane wave. So, thus in the region K r much greater than 1, we have minimal decay of the acoustic pressure amplitude and also we have the plane wave, we have the impedance to be approaching rho naught C which is the impedance of plane wave.

So, therefore, the particle velocity in the direction of wave propagation which was u r is basically rho naught C times the acoustic pressure and the same formula is obtained was obtained even for the plane wave. So, in this sense spherical wave, propagation at large distances away from the source is basically boiling down to the case of plane wave propagation and that is why we spend the major part of this course in the analysis of plane wave propagation because usually more often than not we are interested in the propagation of the sound which is quite far away from the source.

So, where you will usually have this condition K r must greater than 1 to be satisfied and at least in that case, you can completely rely on the plane wave analysis to give you good approximate answers. So, does if K r is large then plane wave conditions are approximated by the spherical wave. So, that is really nice that all our analysis for for plane waves carries over to this case also together with some I mean to I mean it is not exact that is understandable, but at least it is approximates.

So, we can hold on to that part of the analysis the other point which I must elaborate is about the incoming radial wave solutions.

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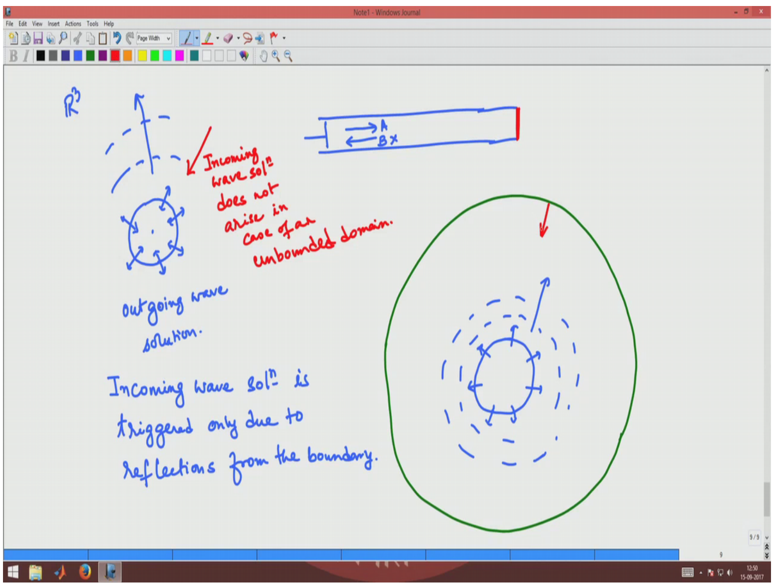

So, you can understand that if you have something like sphere which is pulsating radially in and out. So, we have this balloon like object balloon like structure if you make. So, call it and just like heart beating it is pumping in out and contemplate this situation that this balloon like object is there in the complete 3 dimensional place r 3.

So, in there is only one object which is this balloon like object which is pulsating radially inward and outward and there is absolutely no enclosures no barriers nothing. So, you expect that it will lead to outgoing radial waves, right and these are exactly the outgoing spherical waves that we have talked about and as the wave propagates there is a decay in amplitude in propagation to the distance from the center of this sphere and also as you go to distances further and further away then you get to the condition where the impedance of this spherical wave approaches that of the plane wave and thereby you get your plane wave conditions to be approximated very nicely.

But what happens on the other. So, this was the outgoing wave solution if we simply contemplate the situation of a time reversal; that means, if we have just like there can be an outgoing wave solution there can be mathematically an incoming wave solution also remember this happened even in the case of plane wave in the plane waves also when we did the problem of a duct, which is infinite and then we had only one excitation mathematics showed us that there could be a forward wave. There could be a backward wave, but then we applied our physical logic and physical intuition associated with the problem and then we said that no the incoming wave is infect not possible.

This is not possible, this will only be made possible if the duct is bounded. The duct once got bounded, we understood that there is an incoming wave which will get reflected because of the incident wave, it will get reflected and you will get to see left word or a backward travelling wave, but if the duct where to be infinite, then this b wave does not arise that we argued pretty much the same pretty much elaborately.

Relying on that logic, the same thing happens in this spherical wave propagation problem also, mathematically, we have understood that there was 2 solutions as was derived. There is an outgoing wave solution, there is an incoming wave solution, but then in this case, just like you do not have reflection in an infinite duct and thus the incoming solution does not arise. Similarly the incoming wave solution does not arise incoming wave solution does not arise in case of an unbounded domain unbounded domain that is a domain where in there is no boundary.

However if you can contemplate a slight complication of the same problem that is you have a pulsating sphere, but this time, this pulsating sphere is residing in a spherical cavity, you have a spherical room in which you are doing this experiment of having a pulsating sphere. So, the sphere is pulsating as before right it is pulsating as before and that leads to spherical wave propagation as we understood. So, in initially there will be only outgoing waves, right. So, these blue waves will travel outwards and at certain instant of time, it will meet the boundary which is denoted by this green line and once it hits the boundary the incoming waves will start, right.

So, the incoming waves can start only because of the boundary effect, it cannot start by itself, right. Just like we had it in the case of planar wave propagation in an acoustic duct, the b wave is possible only if this duct ends in certain termination, whether it is a rigid termination or whether it is a open termination or whether it is an impedance termination, any of these situations will lead to a incoming wave which is basically the reflected wave which is been created after the incident wave has fallen on to this obstacle, but if there is no obstacle then there is no chance of that incoming wave to come.

So, similarly I wish to make this remark that the incoming wave solution is triggered only due to reflections from the boundary that is the crucial aspect and once the reflections from the boundaries are set in, then you are going to an incoming wave A, strange, it may appear strange that once you have this incoming wave as the wave will propagate, it will have the radial distance will go smaller and smaller, the r variable associated will go smaller for the outgoing wave, the r variable is going larger. As a result, it is going to attenuate whereas, for the incoming wave, as the wave is travelling, the r variable is going to get smaller and actually as the wave is travelling, it will increase its amplitude in the direction of its propagation because remember the direction of propagation of the wave, the incoming wave is inwards.

So, along the direction of propagation, the wave will actually increase its amplitude. So, that is as a little counter intuitive, but that is the way it is because you will understand that as the wave is increasing, it is getting focused into a more narrow region and therefore, the amplitude had to increase. So, this type of solution is very important in acoustics and this solution will lead us to what is called as the monopole and dipole which is what we will do in the next part of the course. With that we end today’s class.

Thank you.