Acoustics & Noise Control

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Module – 26
Lecture – 31
Source Impedance

In the class we introduced to you the notion of radiation impedance, in particularly it was emphasized that if actually we set the condition that acoustic pressure equals to 0 at the outlet, then that would mean perfect (Refer Time: 00:32) condition, and that would not lead to any transmission it would lead to a complete reflection.

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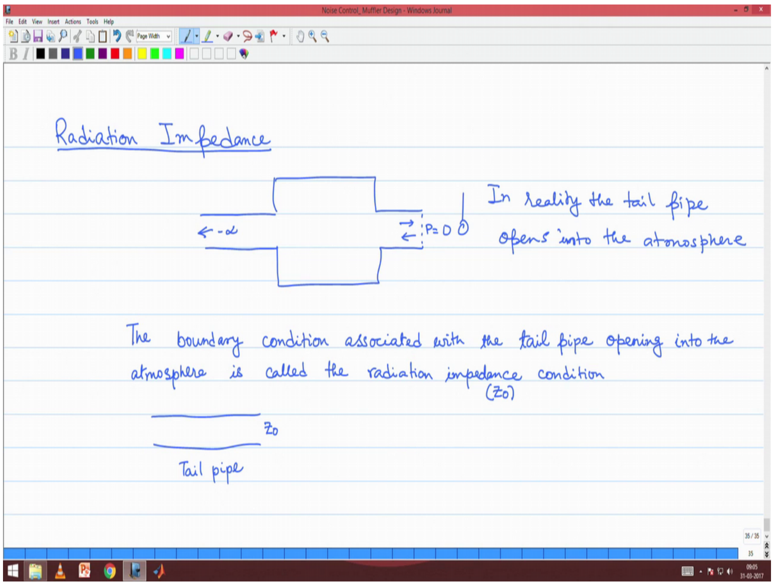

Radiation Impedance

In reality the tail pipe opens into the atmosphere

The boundary condition associated with the tail pipe opening into the atmosphere is called the radiation impedance condition (Z_0)

So, that is something that we need to make immense for. So, the boundary condition in particular, the boundary condition associated with the tail pipe opening into the atmosphere is called the radiation impedance condition.

So, the tail pipe when it opens in to the atmosphere, the more exact boundary condition as opposed to simply saying that pressure equals to 0, would be that there is a radiation impedance at the outlet and we will denote this by Z 0. So, Z 0 is the radiation impedance which is at the exit of the tail pipe, as it opens into the atmosphere. So, there are 2 forms of this radiation impedance, one which may seem little academic at this stage.

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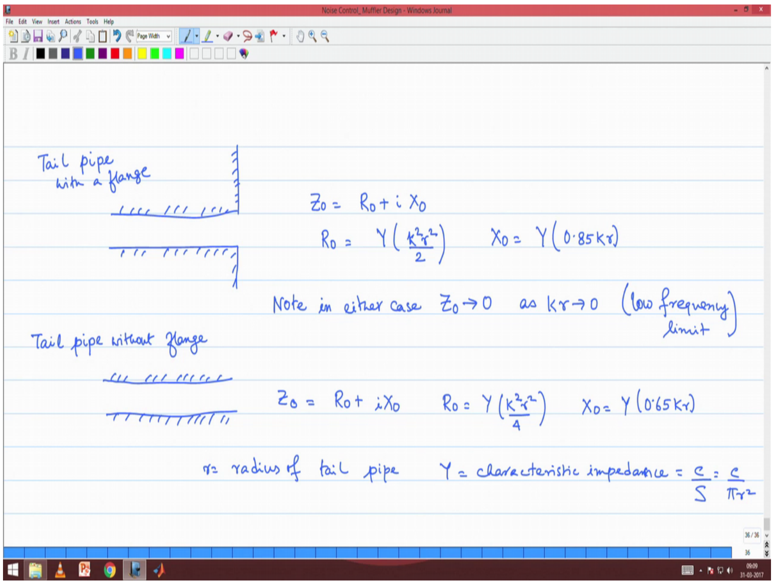

Tail pipe with a flange

$$Z_0 = R_0 + i X_0$$

$$R_0 = Y\left(\frac{k^2 r^2}{2}\right) \qquad X_0 = Y(0.85 kr)$$

Note in either case $Z_0 \to 0$ as $kr \to 0$ (low frequency limit)

Tail pipe without flange

$$Z_0 = R_0 + iX_0 \qquad R_0 = Y\left(\frac{k^2 r^2}{4}\right) \qquad X_0 = Y(0.65 kr)$$

r = radius of tail pipe $\quad$ Y = characteristic impedance $= \frac{c}{S} = \frac{c}{\pi r^2}$

But none the same it is important in some of the later derivations which we might do. So, if the tail pipe is having a flanged outlet, tail pipe with a flange. Usually you do not see such a tail pipe with a flange, but for a different reason we will see that this expression will also be useful.

So, when the tail pipe comes with a flanged outlet, then the radiation impedance Z 0 is having 2 parts R 0 plus i X 0, R 0 being the resistance and R 0 is given by Y into K square r square by 2 and x 0 the reactants is given by Y 0.85 K r. This is just a result I am quoting the derivation of this result is beyond the scope of this course, it is a result which is available in the literature and we will simply use this result ok.

So, this is the radiation impedance formula for the tail pipe which is opening into the atmosphere, but the tail pipe has a flange, and the more usual situation when the tail pipe just directly opens in to the atmosphere, and does not have any such flange artifact in this case. So, this is tail pipe with any such flange, in that case we will have Z 0 again to be given by a resistance and a reactance. The resistance this time will be given as Y, K square r square by 4 and the reactants will be given as Y times 0.65 K times r. K is the wave number r is the radius of the tail pipe, and Y is the characteristic impedance in terms of the mass velocity. So, it is actually given as C by S where S is the cross sectional area of the tail pipe which is pi r square right.

So, this with this formulas we now have a better description of the boundary condition associated with the opening of the tail pipe in to the atmosphere, and that is what is called radiation impedance, but what we did earlier till this point we were assuming that when the pipe is opening into the atmosphere we were assuming a 0 pressure condition. So, 0 pressure in turn would mean pressure by velocity is 0 that is the impedance was taken as 0; zero pressure means 0 impedance also. Looking at this formula you can realize that that is the situation which will prevail when K r is extremely small k r stands for wave number times the radius right. So, when wave number times radius is extremely small or in other words when the frequency is very small, then you will have this situation that this these impedances will go towards 0.

So, note that in either case that is whether you can flange or you do not have flange Z 0 tends to 0 as K r tends to 0 which is the low frequency limit. So, all that we have done till know is not simply trashed away we can say that that is applicable the boundary condition pressure equals to 0 is applicable, only in the extreme low frequency limit, but if you have if you want to do a more accurate analysis then you have to take in this factor K r. So, K r governs the radiation impedance ok.

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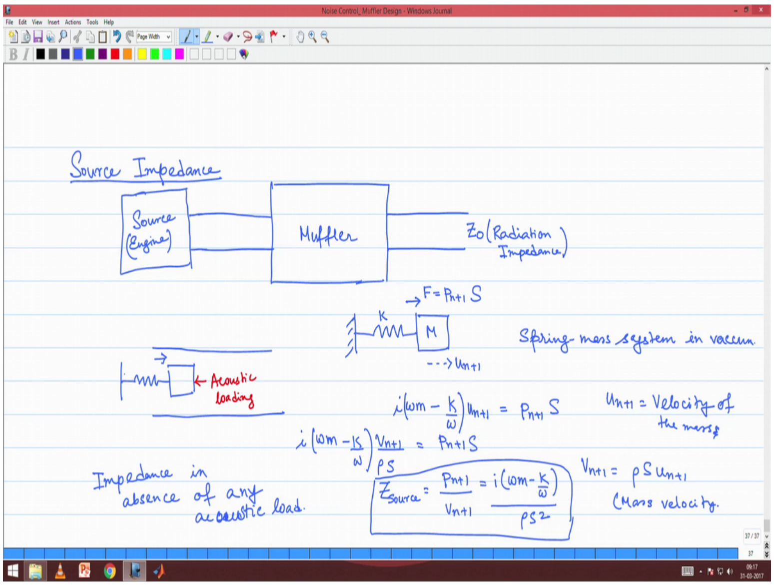

Source Impedance

$\rightarrow F = P_{n+1} S$

$\dashrightarrow u_{n+1}$

Spring- mass system in vaccum.

$$i\left(\omega m - \frac{K}{\omega}\right) u_{n+1} = P_{n+1} S$$

u_{n+1} = Velocity of the mass

$$i\left(\omega m - \frac{K}{\omega}\right) \frac{V_{n+1}}{\rho S} = P_{n+1} S$$

$V_{n+1} = \rho S u_{n+1}$

(Mass velocity.

Impedance in absence of any acoustic load.

$$\boxed{Z_{source} = \frac{P_{n+1}}{V_{n+1}} = \frac{i\left(\omega m - \frac{K}{\omega}\right)}{\rho S^2}}$$

Next we will move to one more aspect which is called the source impedance. So, as I said we were talking about a muffler within the ducting system and then the tail pipe, and now we understand the boundary condition of the exit at the tail pipe is going to be the radiation impedance Z 0, but we also need to ascertain what happens in the extreme upstream station. This is the extreme upstream station this is where the source is located it could be the engine. So, this schematically this could be the engine in the automotive example or sometimes in laboratory we can replace this engine with some sort of a speaker ok.

So, none the same there will be a certain source. Just like at the outlet we have replaced the boundary condition with a more accurate representation of radiation impedance, we need to replace the inlet condition also with an appropriate boundary condition, let us see how to find that. So, towards that end we first consider a duct where in the source of which is just a spring mass system which is driven by a certain force right. So, this spring mass system when it is driven by a certain force; obviously, the mass will vibrate. The mass is like that rigid piston once the mass vibrates it will set off the acoustic pressures within the duct right. But let us look at the converse picture once this mass vibrates and then there is acoustic pressure which is the which is produced in this acoustic duct, there is an additional force which is acting on this mass.

On top of this structural force indicated in blue there will be an acoustic loading also right because now the mass is subjected to 2 kinds of forces one is the structural and the other force is that due to the acoustic pressures, the acoustic pressure multiplied by the area of cross section will give the entire acoustic load. So, as result we need to understand this coupling between the vibration and the acoustic correctly, as to be able to model this situation.

So, let us do that, if you have a spring mass system alone in vacuum. So, this is a spring mass system in vacuum. There is no additional fluid around it right and then I am driving it with a force and I wish to call this force as a pressure times and area no issues. So, in that case I can always write j omega m minus K by omega which is the impedance of this spring mass system, multiplied by the velocity of this mass what is the velocity of this mass we will call it? V n plus 1, the nomenclature of m plus 1 will become apparent as we go along.

So, impedance into velocity should be equal to the force right. So, the force is Pn plus 1 i should call it as S because we have been using S for the cross sectional area Pn plus 1 into S right I should call this as U n plus 1 because this is the particle velocity right. So, this is what we will get by simply appealing to the theory of vibration type of an equation if we appeal to Newtonian mechanics, you will get a mx double dot plus K x is equals to force, force is pressure times area and then instead of working in terms of displacement we choose to work in terms of velocities and when you get that U n plus 1 is the velocity of the mass. But then if we want to convert to mass velocity we will need to account for rho times S times U n plus 1. So, this is the mass velocity

So, in other words I could write this situation as g sorry i am using j i should use i omega m minus K by omega V n plus 1 divided by rho S Pn plus 1. So, then you have the situation that Pn plus 1 divided by V n plus 1 is equals to i omega m minus K by omega divided by rho S square right there is an S which is multiplying that Pn plus 1. So, this is what we will call as the source impedance because this is the impedance of just the source, we are just treating that this spring mass system is what drives the acoustic waves with in the duct right. But as yet we have not accounted for the acoustic loading that is acting on to this mass right. So, this is the impedance in absence of any acoustic loading on the source.

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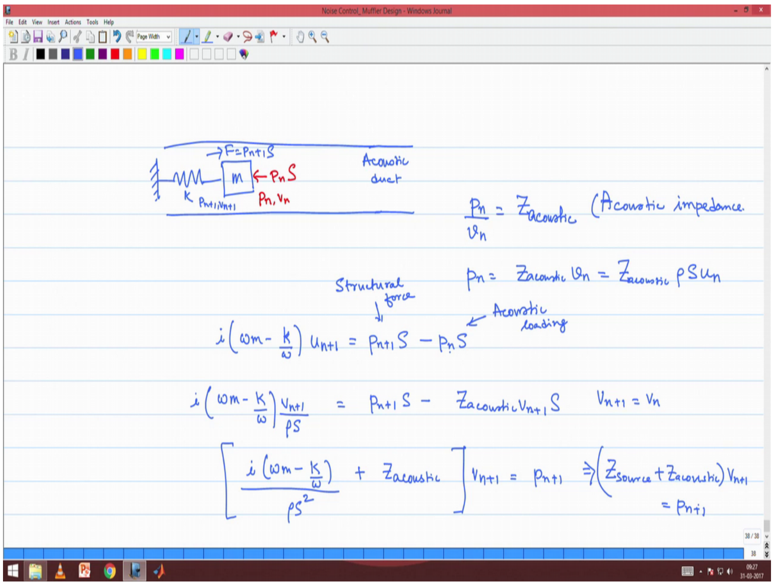

$$\frac{p_n}{v_n} = Z_{acoustic} \quad \text{(Acoustic impedance}$$

$$p_n = Z_{acoustic} v_n = Z_{acoustic} \rho S u_n$$

$$i\left(\omega m - \frac{k}{\omega}\right) u_{n+1} = p_{n+1} S - p_n S$$

$$i\left(\omega m - \frac{k}{\omega}\right) \frac{v_{n+1}}{\rho S} = p_{n+1} S - Z_{acoustic} v_{n+1} S \qquad v_{n+1} = v_n$$

$$\left[\frac{i\left(\omega m - \frac{k}{\omega}\right)}{\rho S^2} + Z_{acoustic}\right] v_{n+1} = p_{n+1} \Rightarrow \left(Z_{source} + Z_{acoustic}\right) v_{n+1} = p_{n+1}$$

So, this is the impedance in absence of any acoustic load. Now let us consider the situation where we have an acoustic duct and the inlet to the acoustic duct is this spring mass system, which is driven at f which is equals to Pn plus 1 times S, and the spring stiffness is K and the mass is m right. The moment you have an acoustic duct with any boundary condition any muffler or without muffler does not matter; that means, there is an acoustic wave which is set up to the right side of this mass that is pretty clear right to the right side of the mass there is an acoustic pressure field which exist. Now that acoustic pressure field multiplied by the area is going to be the force which is acting on the right phase of this mass.

So, if the acoustic pressure field exactly at this point is Pn right. The acoustic pressure fluid exactly to the at the right phase of this mass need not be the same as the pressure field to the left phase of this mass. Because the left phases of the mass is driven by a structure force right the structural force is scaled by the area is effectively a pressure. So, we are working in terms of pressure and mass velocities that is why we are flipping between force and velocities in the discussions, but the point is on the right side now once instead of a vacuum situation, you have a acoustic fluid contain in an acoustic duct then on the right side of the mass you are going to get an acoustic pressure. And that acoustic pressure in turn will hit the mass also right. The acoustic pressure together multiplied with the area is an additional load which is acting on the structure which now you have to consider.

So, carrying on with this derivation now the derivation in the previous case will just have to be repeated with by considering this additional acoustic pressure which is acting in the opposite direction right. So, all that we will know is that Pn by V n is going to be the Z acoustic that is the acoustic impedance. So, here just to the right of the right phase of the mass you have the situation Pn V n, and on the left phase of the mass you have the situation Pn plus 1 V n plus 1; obviously, because it is a mass V n plus 1 and V n is same, but Pn plus 1 and Pn is not the same there lies the trouble. So, Pn in other words could be written as Z acoustic into V n, and V n also could be as rho times S times U n right. Now if we look back at the equations of motion for the spring mass system i omega m minus K by omega into U n plus 1 must be equal to Pn plus 1 into S which is the structural force minus Pn into S which is the acoustic force. So, this is the structural force and this is the acoustic loading that is acting over and above the structural load right.

Now, let us do some simplifications i omega m minus K by omega, U n plus 1 is equals to Pn plus 1 S Pn can be now written as Z acoustic into rho into S into U n right or maybe I will put this as V n and I will change the U n plus 1 to V n plus 1 divided by rho S right. Also we have that V n plus 1 is equals to V n because the left side of the mass and the rights side of the mass should have the same velocities any way this is not a spring element this is the mass element which is driving these structures.

So, we are I mean driving the acoustic fields. So, V n plus 1 must be equal to V n as per the drawing the left phase, of the mass and the right phase of the mass has to have the same velocity. So, we will make this V n as V plus 1. So, in the next step we will bring i omega m minus K by omega divided by rho S square plus I missed out an S here. Pn times S Pn is Z acoustic into V n plus 1 and then there is a multiplicative factor of S right Pn into S Pn was Z acoustic into V n plus 1 and the S was missing. So, I will make amends for that. So, plus Z acoustic into S and all this should get multiplied with V n plus 1, and that should be equal to Pn plus 1 into S right. If I divide throughout by S then this S and this S will cancel and I will get a rho S square here right, but what is this term? This term is going to be the Z source right.

So, this term is Z source plus Z acoustic into V n plus 1 equals to Pn plus 1. So, in other words what we have here is that Pn plus 1 divided by V n plus 1 is going to be Z source plus Z acoustic ok.

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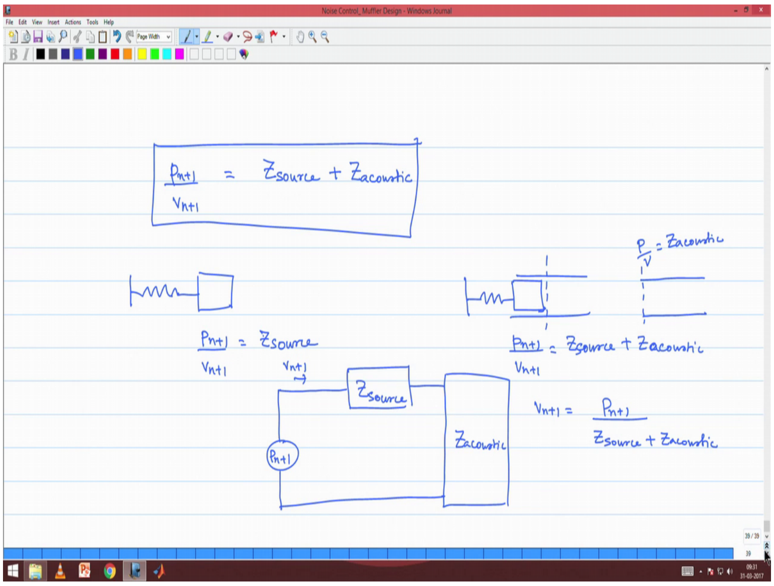

$$\boxed{\frac{P_{n+1}}{V_{n+1}} = Z_{source} + Z_{acoustic}}$$

$$\frac{P}{V} = Z_{acoustic}$$

$$\frac{P_{n+1}}{V_{n+1}} = Z_{source}$$

$$\frac{P_{n+1}}{V_{n+1}} = Z_{source} + Z_{acoustic}$$

$$V_{n+1} = \frac{P_{n+1}}{Z_{source} + Z_{acoustic}}$$

So, in presence of an acoustic medium which offers an impedance, which is Z acoustic the pressure by velocity ratio of my source is going to change Z. Now it is going to read as Z source plus Z acoustic let me illustrate to you in a different way what I wish to say.

So, let us say that in absence of any acoustic field, here Pn plus 1 divided by V n plus 1 is going to be equal to Z source, and when this spring mass system is put in an acoustic field with a dark with an expansion chamber with whatever you want if you put it, all that you have to remember is that the ratio here is now getting changed to Z source plus Z acoustic and what is Z acoustic? That means, if I just cut off at this dotted section and calculate the impedance of my acoustic line at this section, this ratio P by V is called the Z acoustic or the acoustic impedance right.

So, with this let us now come back to the case of our electrical analogy because that will things even simpler for us. So, the question is how we model this situation. So, to towards that extent we must understand this the following the acoustic field is created because of a certain excitation which is acting at the source, just imagine that the speaker is just like a membrane which could be simplified to a spring mass system which is driven by a certain force. And the force in turn is related to the pressure Pn plus 1 right. So, the question is the pressure or if we keep this force which drives the spring mass system if we hold this constant right then what happens to the acoustic field.

So, the driving factor for the acoustic field is the vibration of this mass, the mass in turn vibrates because of the force. So, the fundamental excitation source for both the vibration as well as the subsequent acoustic is that force which in turn is related to the pressure Pn plus 1 right. So, the fundamental source in our case is going to be Pn plus 1 right, and then if you have to incorporate this idea that there is a certain source impedance Z source, right and then there is a certain acoustic impedance in the line, we do not know whether it is in the shunt position or in the in line position, but we can always put it in this form this is perfectly general.

Now, if you have an electric circuit of this form then it is elementary to see that the total impedance will be Z source plus Z acoustic they are in series and therefore, V n plus 1 which is the current flowing in this circuit is exactly going to be related in that fashion. So, V n plus 1 is going to be Pn plus 1 divided by Z source plus Z acoustic. So, at this stage we have therefore; define the impedance at both the starting and the terminating conditions of our duct. So, once more finally, we will look at the schematic of our exhaust line.

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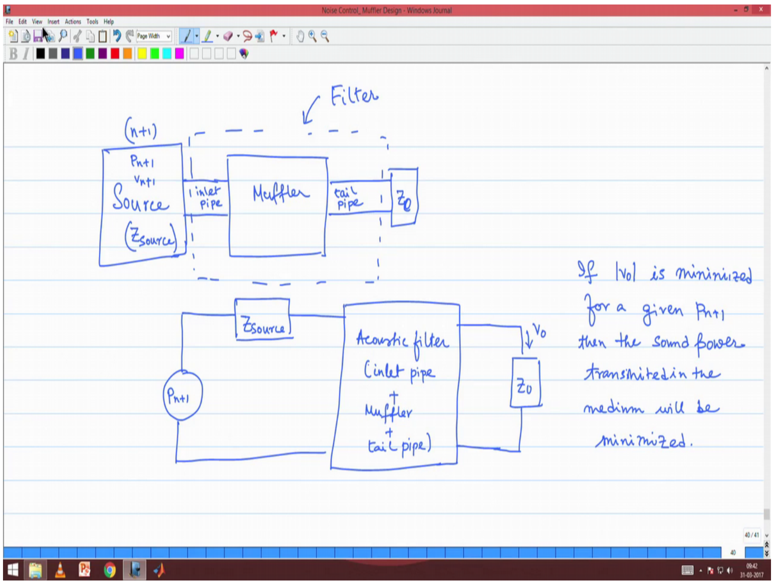

If $|v_0|$ is minimized for a given P_{n+1} then the sound power transmitted in the medium will be minimized.

So, this is a source which is the station n plus 1, and from here on words you have a muffler and this is the tail pipe. So, the muffler together with the inlet pipe and the tail pipe will constitute what is known as an acoustic filter. So, this is the muffler this is the tail pipe and this is the inlet pipe. All this together will constitute a filter right and finally, the impedance here is the radiation impedance Z 0 and the impedance here is Z source the source impedance, each station will be characterized by a certain pressure and velocity.

So, therefore, the circuit associated with this model is going to be the following Pn plus 1 Z source, then there is a muffler or acoustic filter which comprises of the inlet pipe plus muffler plus tail pipe sorry and then finally, you will have a radiation impedance. This is the configuration that we are interested in right this is the electrical analog of our acoustic transmission line you will have similar electrical analogs even for hydraulic transmission, but let us leave that aside. Once you create this electrical analog the picture here will enable you to draw lot of insights as to how to make the muffler design work let me give you some of those ideas.

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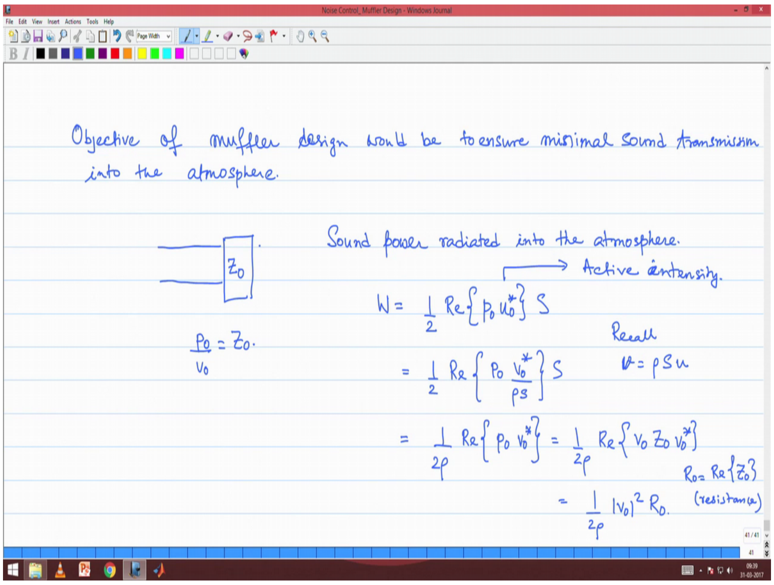

Objective of muffler design would be to ensure minimal sound transmission into the atmosphere.

Z_0

$\frac{p_0}{V_0} = Z_0.$

Sound power radiated into the atmosphere.

$\longrightarrow$ Active intensity.

$$W = \frac{1}{2} \text{Re}\{p_0 u_0^*\} S$$

$$= \frac{1}{2} \text{Re}\left\{p_0 \frac{V_0^*}{\rho S}\right\} S$$

$$= \frac{1}{2\rho} \text{Re}\{p_0 V_0^*\} = \frac{1}{2\rho} \text{Re}\{V_0 Z_0 V_0^*\}$$

$$= \frac{1}{2\rho} |V_0|^2 R_0.$$

Recall

$V = \rho S u$

$R_0 = \text{Re}\{Z_0\}$ (resistance)

So, the objective of muffler design would be to ensure minimal sound transmission into the atmosphere right the audible sound which escapes from the tail pipe should be minimal that is the objective we had in a in the earlier classes we looked at transmission loss, but here we will see that the transmission loss possibly does not capture this idea correctly.

So, let us look at it in a little more details. So, the question is how do we captured this idea that what is the sound transmission that is going into the atmosphere. So, remember finally, what happens is that there is a tail pipe which is opening at the atmosphere, which is taken as the radiation impedance Z 0 rights. So, the question is what is the sound power that is radiated into the atmosphere? The sound power that is radiated into the atmosphere w should be calculated based on the intensity calculations, what is the intensity expression telling us? The intensity tells us that it is the active intensity is going to be the real part of pressure time’s velocity particle velocity, and this you need to multiply with the area in order to get the total power right.

So, this is the active intensity. So, the active intensity multiplied by this cross sectional area will give the power which is escaping in to the atmosphere right. Let us redo this in terms of the mass velocities. So, that is no big deal, I simply need to relate the mass velocities with the particle velocities and the relation is nice and simple in of this kind right. So, remember recall V which is the mass velocity is rho times S times u right. So, therefore, this is the situation rho and S are; obviously, real quantities. So, I could as well pull them out, but S gets cancelled between the numerator and the denominator. So, we are left with this quantity, but then we know because we have a radiation impedance condition we know P 0 by V 0 is exactly Z 0 right. At the end of the tail pipe where the tail pipe is opening into the atmosphere we have argued that the boundary condition appropriate to the exit of the tail pipe is that of a radiation impedance denoted by Z 0.

So, therefore, I could replace P 0 as V 0 times Z 0 into V 0 star that would make it 1 by 2 rho mod V 0 square real part of Z 0 is what matters which is the resistance. So, R 0 is the real part of Z 0 which is the resistance. So, what we need to the objective in our muffler design would be to reduce this quantity which is V 0 square R 0. Now R 0 is something that you cannot play with it is sort of god given and you have the formula for R 0 which is what I wrote just a few slides back. So, these are the formulas for R 0 there is scarcely anything that you can change other than changing the dimension of the tail pipe, but let us say the dimension of the tail pipe is also frozen. So, R 0 cannot be changed, the only thing that you can change in order to minimize the sound power is the V 0, which means that in this part of the circuit what is V 0? V 0 is that which goes in the final part of the circuit, the extreme downstream part through impedance Z 0. So, if V 0 is minimized only then you will achieve minimal sound power transmission in the outlet of the tail pipe right.

So, this is a better way of bench marking a muffler. So, the conclusion would be that if mod V 0 is minimized for a given Pn plus 1 that what is Pn plus 1? Pn plus 1 is the structural force which basically excites the vibration of the source, and that vibration of the source in turn excites the acoustics in the medium right. So, the fundamental excitation source in this problem is that structural excitation which is denoted as Pn plus 1 into S, S is just a constant. So, we are not bothering about S, we are looking at it in terms of Pn plus 1. So, if V 0 is minimized for a given Pn plus 1, then the sound power transmitted in the medium will be minimized, and this should be the objective of a muffler designer right and how do you think you can achieve this objective of minimizing V 0? This time the circuit will help you rather than the acoustic or the mechanical analogy, looking at the circuit what should you do to minimize V0.

Student: Observe (Refer Time: 35:46).

Put something on shunt put a 0 impedance line in parallel make the larger part of the current go in the parallel path in the shunt path and minimal current should go on the.

Student: Z.

Load Z 0 that will save the power this is exactly what is done in electrical circuit theory also, but this approach of designing mufflers where mufflers where just called acoustic filters, we are just replacing the idea of an LCR circuit with the help of these mufflers, and we are drawing a complete analogy in a step by step by step fashion and we are lead to believe that in order to minimize our sound transmission into the atmosphere, we should be able to have a certain design objective and that design objective comes out neatly to be having some shunt path. Transmission loss does not give you this idea because transmission loss does not take into account either Z 0 or Z source.

So, transmission loss is like little fussy you should take it with not a pinch of salt, but a handful of salt right. So, I am telling it all the authority in my command that transmission loss is not the right benchmark for evaluating a muffler. It is the insertion loss which is more important. So, this idea will eventually lead to a better qualitative quantitative measure of the muffler performance which is loss insertion. Insertion loss will measure the sound power which is transmitted with and without muffler.

So, this is the situation what happens with a particular muffler, to reference this situation you will also do another calculation where what is the sound power that is transmitted without the acoustic filter. I should not say muffler I should say the acoustic filter because the acoustic filter includes not only the proper muffler, but also the inlet and the exhaust. Remember the inlet pipe and the tail pipe will have a certain transfer matrix and hence they will have a certain circuit representation right.

So, therefore, the V 0 that is getting computed, V 0 and hence the sound transmitted whatever computation you do we should non dimesnionalize that with respect to the sound power which is getting compute in absence of any filter right. And that ratio will lead us to insertion loss and that insertion loss is the measure for actually evaluating the muffler in a given situation with the given source and obviously, radiation impedance does not change. So, we will close the topic by having a brief look at the insertion losses in the next class fine.

Thank you.