Acoustics & Noise Control
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Module – 23
Lecture – 28
Electro Mechanical Analogies Simple Exam

In the last class we talked about electro mechanical analogies. In particular for the 3 fundamental mechanical elements namely mass spring.

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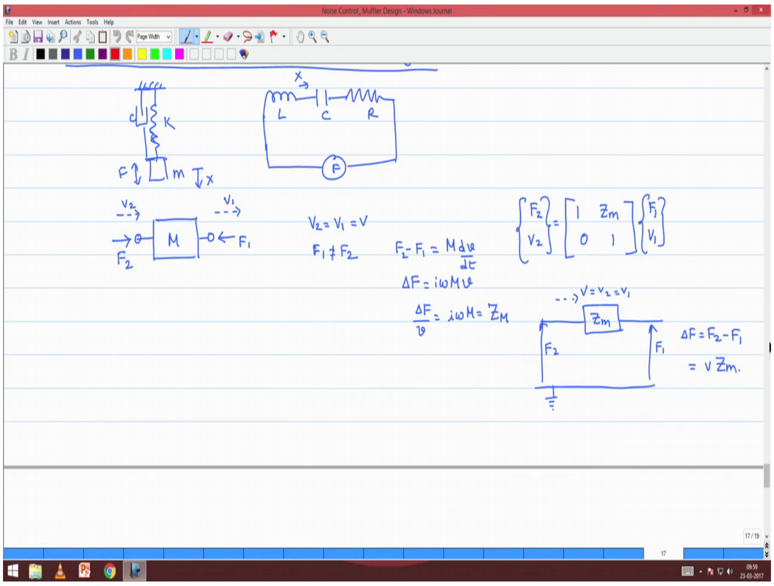

And the damper we were able to formulate both the transfer matrix as well as the equivalent electrical circuit. For the mass element we understood that the forces of the 2 sides need not be the same whereas, the velocities on the input and the output side has got to be the same accordingly, we obtained the impedance for the mass to be i omega m for the spring element.

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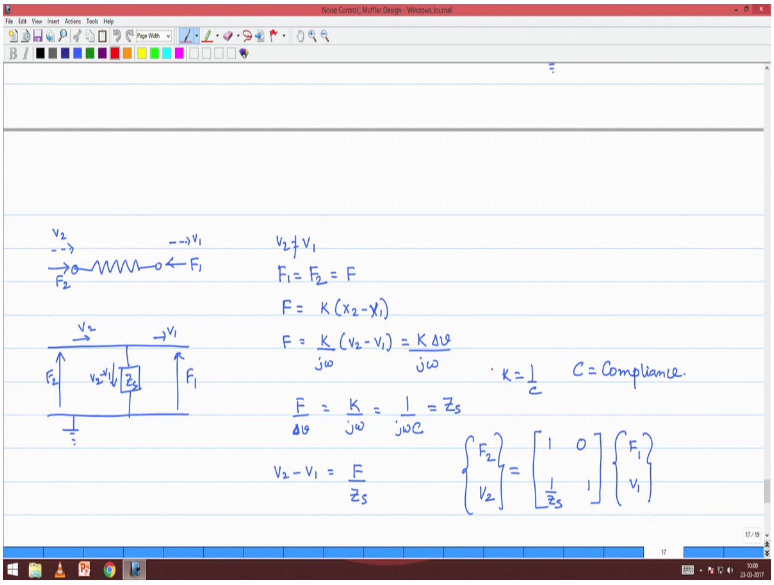

We obtained the equivalent electrical circuit to be an element of impedance Z s, which is placed in parallel in this fashion, and the transfer matrix was obtained in this way. Please note we have defined capital C goes to compliance, which is just the reciprocal of the stiffness k. So, capital C is the compliance.

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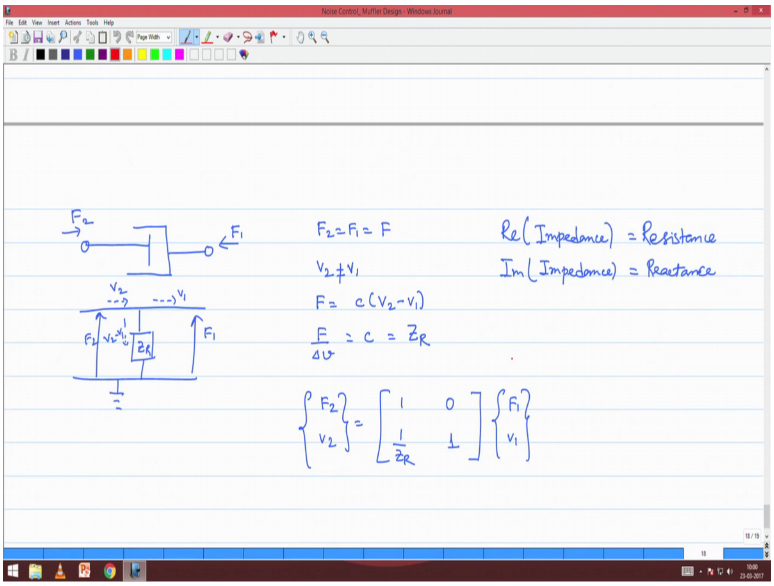

And lastly for the damping element also in similar fashion we understood that the force of the input and the output port in a damping element has to be the same. The velocities could differ so, accordingly the velocities here are V 2 and V 1 and this could be possibly be different. C in small alphabets denotes the damping coefficient capital C denotes the compliance, but c in the small alphabet denotes the damping coefficient.

And we were able to derive that the analogous electric circuit is also a parallel element of impedance Z R which is related to the damping coefficient and the transfer matrix was obtained in this fashion. Today we will look at how to assemble these fundamental elements in that were derived, and using these assembly of springs masses and dampers we should be able to construct any mechanical system. And accordingly we will define what is the electrical analogous circuit for this. And as also we will find out the transfer matrix appropriate to assemblage of such spring mass and damper elements ok.

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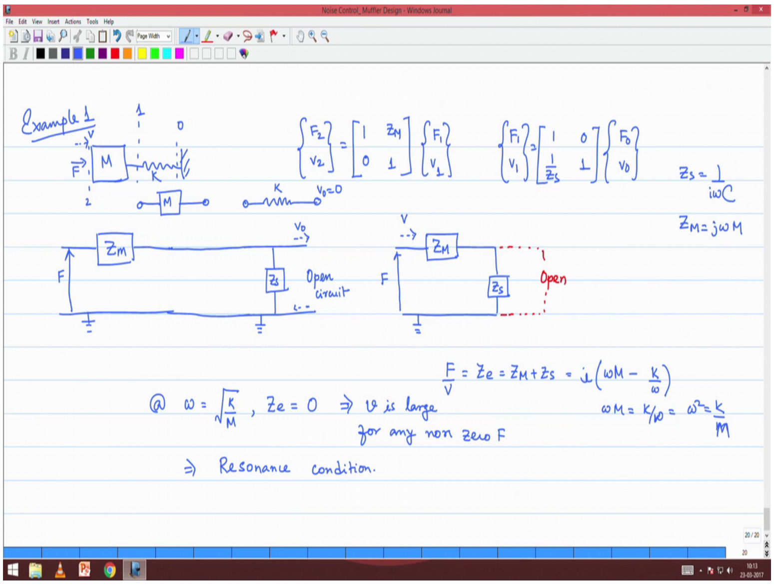

So, what we will do. Firstly, is to start with we will take our very fundamental example of just spring and a mass. So, this is example 1. So, as we know that for the mass element we will have an impedance of Z m. So, there is with reference to the ground line there is a force on the mass and that force is equivalent to a voltage, in the electrical circuit terminology. So, this is what it represents for the mass element alone. Similarly if we detach the spring element. So, what we do is that we detach the 2 elements the mass and the spring. So, the mass element corresponds to this electrical circuit, the spring element would correspond to this mechanical circuit sorry, the electrical circuit corresponding to the spring mechanical element will be this as has been derived in the previous class.

Now, what we observe in this case is this output of the spring is basically at a fixed position it is not allowed to move. So, this I will call this V 0 this point is not allow to move. So, that intern implies that there is no current which can go, in this part of the electric circuit. The output should not have any current. Now the output cannot have current only if we will have a open circuit configuration. So, this is basically an open circuit in case we considered this part of the circuit to be open which means we do not we just leave the wires hanging lose, we do not connect it right. That would be an open circuit. And that open circuit would essentially min, that the impedance is infinite for in circuit terminology, but in a more intuitive view we can understand that in this portion there is no current that will flow because the circuit is left open right. And a current being analogous to velocity that also means that V 0 is 0 which is exactly what we wanted.

Now, next what we consider is that between the output of the mass, and the input port of the spring. The same forces has to get transmitted which means the voltage between these 2 ends also has to be the same. So, this is the voltage which any way has to remain same, and the other line here was the ground line. So, this could also be connected. As a result what we have for this circuit is the following that there is and impedance associated with the mass, the value of which is i omega m. And there is an impedance associated with the spring which is basically in a shunt position, but then the other part which I am drawing it in red is basically needless to draw because this part of the circuit is open, nothing happens here. So, you might as well not draw it, but then the voltage between the ground line and the one which is hitting the mass, if this is the ground line this voltage is going to be effectively the force ok.

So, this is this completes the electrical circuit for very simple fundamental mechanical system that offers spring mass right. Now how does this boil down to in terms of the notations of the circuit? So, if you are equate to find the velocity, which basically means the velocity at the input end of this mass then, as per the circuit theory we understand F by V will be equal to the equivalent impedance of the circuit and since there are just 2 impedance Z m and Z s and they are in series here. So, that will be Z m plus Zs. So, F by V that is the ration of the force and velocity of the mass at it is input end is going to be Z m plus Z s. Which by definition is again i times omega m minus k by omega right.

So therefore, at omega equals to square root k by m, what will happen at omega equals to square root k by m? We will see that Z equivalent is going to, how much? 0 right. So, the impedance of the circuit is going to 0 at this condition, which basically has been obtained by setting omega m is equal to k by omega. And that intern would mean omega square is k by m right.

So, at this frequency omega equals to square root k by m, we see that the impedance of the equivalent electrical circuit is going to 0. So that means, it is virtually short circuited right. What happens when you have a short circuit in electrical terminology with a with a certain potential difference, if you short circuit large current will flow through, and large current essentially means large velocity of the mass point. So, that again boils down to our resonance idea, but here I am bringing the resonance idea from an electrical analogy rather than using the mechanical prospective. So, here what we see is that at Z e will become 0 and this would imply V is large for any non 0 F. This is the resonance condition. Where in you can have large response of this structure even though the forces are very small right.

So, or in other words the magnification from force to velocity is by a large amount. So, from here what we will realize is that the resonance is obtained. When you have the imaginary part of impedance to be going to 0 right. So, I mean here the impedance is completely imaginary anyway, but later we will see that there is a possibility of damping also that will bring in the real part, but maybe I will hold that for a while. So, now, for this very same system that of a spring mass system we will analyze the transfer matrix method. So, accordingly we have 3 station which are defined as 0 1 and 2 in this diagram. So, what we need to do is relate the forces and velocities between each of this successive station. Between 0 and one we understand it is going to be a spring element. So, F 0 V 0 is going to be related through F 1 V 1 by a spring element and the transfer matrix for that is going to be 1 0 1 by i omega 1 by Z s basically, this is going to be 1 by Z s and 1 where Z s is 1 by i omega s, i C capital C.

This is the transfer matrix relating, the variables between station 0 and station 1. Now in the next stage we need to relate the state variables from station 1 to station 2. So, we need to find out F 2 V 2 in terms of F 1 V 1. And that also is simple because that is just the transfer matrix associated with the mass element. That also was defined and that transfer matrix is given by this form, where Z m is j omega m right.

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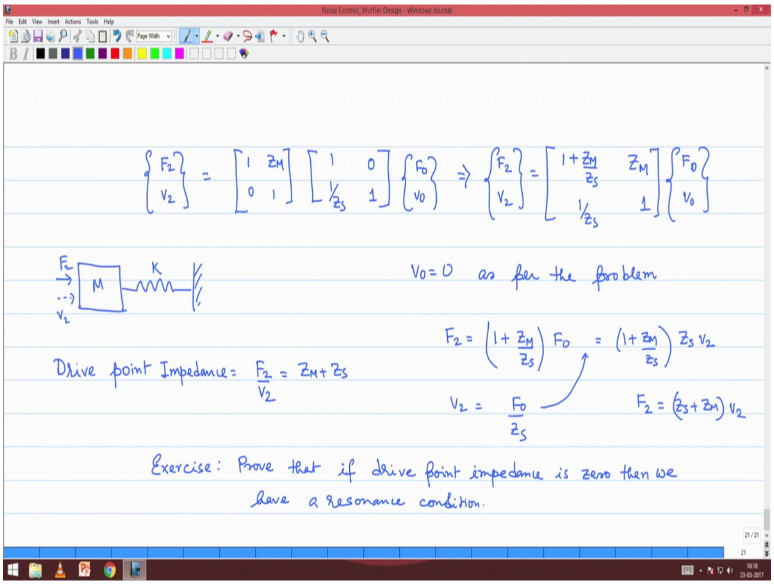

So therefore, it is trivial to find the transfer matrix relating station 2 to station 0 right. That will be the product of the 2 transfer matrices, which are given as 1 Z m 0 1 1 1 by Z s 0 1. And if you multiply them what we get is the following F 2 V 2 is given by 1 plus Z m by Z s. Then we will have Z m, and we will have 1 by Z s and finally, we will have 1, F 1 V 1 sorry, F 0 V 0.

So, this is the transfer matrix which relates the state variables between the upstream point and the downstream point. The downstream point is station 0 and the upstream point is station 1. Now from the transfer matrix itself again we can calculate what is known as the drive point impedance. By drive point impedance what we mean is the following, we will have the spring and mass. This is mass and this is stiffness it is being driven by a certain force, it has a certain velocity and as per our notations, this velocity is denoted by V 2 and the force is denoted by F 2.

So, the drive point impedance, will be given by F 2 by V 2. And again when this drive point impedance has shows a characteristics that it s nearly 0, then it will be associated with a resonance condition right. Because then even for a small force you will have large velocity of this mass right. So, in this case; obviously, the impedance will be completely imaginary as you can work it out, but the point is this how do we calculate drive point impedance in this case? That is very simple we have to realize that V 0 has got to be 0 as per the problem because the spring is fixed at one end.

So, V 0 has got to be 0 which means, from the transfer matrix as we can see F 2 is 1 plus Z m by Z s into F 0 right. And V 2 will have to be F 0 by Z s, there is no need to consider the effect of V 0 because V 0 is 0. So, from here if we make the substitution here we will get 1 plus Z m by Z s, instead of F 0 we might as well write Z s into V 2 right. So, that implies F 2 is equals to Z s plus Z m into V 2, or drive point impedance F 2 by V 2 is equals to Z m plus Z s which is exactly equal to the equivalent impedance of the electrical circuit. So, in this electrical circuit we have the equivalent impedance as Z m and Z s. And this F and this V is basically the input force and velocity that is being offer to the mass. So therefore, the drive point impedance is the equivalents impedance in the electric circuit, and that can also be calculated using the transfer matrix method as shown above. So, I will leave you as an exercise prove that if drive point impedance is 0, then we have a resonance condition.

Exactly what we did for this circuit analogy you just have to write the expressions for Z m, and Z s in it is entirety and multiply out those 2 terms and then again Z m and Z s, basically is what we have written. So, it is almost trivial exercise you could as well do it by yourself next.

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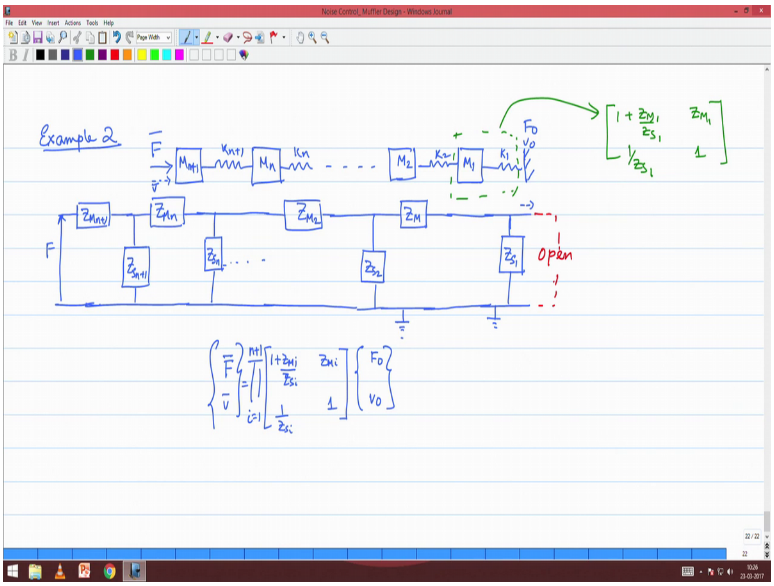

We will look at a second example actually for the first example. It is almost trivial because our if you start from the free body diagram and write the equations of this system, you could as well derive everything that you want from a completely mechanical prospective without going to this electrical analogy or transfer matrix or anything like that. But now the electrical analogy and the transfer matrix will become really handy, when you think about a large group of inter connected springs and masses. So, the idea here will be very simple as opposed to our free body diagram approach.

So, let us say you have as many as you can think springs and masses. The idea is just the same, start from the one which is closest to the fixed point there has to be some fixed point, otherwise is not a vibrating system otherwise it is a dynamical system. So, I will call this K 1 M 1, I will call this K 2 M 2, I should have putted, putted dotted line later. This is K 2 and this is M 2 and then things can continue for a while, and this is K n M n and then this is K n plus 1 M n plus 1 and so on. Finally, let us say there is a force hitting this structure ok.

So, then as we will now see the analysis here is really simplified. If you go by the electrical analogy or by the transfer matrix method. Remember for this system when n is large it is actually pretty cumbersome to do the free body diagram and write equations of equilibrium for each of the system or if you want to derive the lagrangean form that also could be a little more cumbersome, but in electrical analogy this will turn out to be really simple. So, let us start from the upstream point the upstream point is a spring with stiffness K 1. So, as we know the impedance associated with the spring is in the shunt position

So, this is the impedance associated with the spring right. Next is that there is a mass. So, mass will offer an impedance in electrical terminology which is in the inline position and this line is always the ground line right. Now again the same arguments will go through that is, the output end of the spring is not suppose to have any velocity which means, that this in the analogous electrical circuit they are cannot be any current in this branch, which can only happen if this circuit is left open. So, we are not going to close this circuit, it is going to left open which means as well we might not consider this part of the circuit. Next the force between the mass and the spring the same force has to get transmitted. Which means the voltage between these 2 ends must also be the same which means we can as well connect it right. And anyway these 2 are ground lines. So, there is nothing which prevents us to connect it right.

So, this idea will keep growing. So, we will have the next for the next spring mass system also we will have another Z s 2 and another Z M 2 and so on. This will keep happening till the nth level where in we will have impedance in the shunt position with the value of Z s n and the inline impedance would be Z M n then followed by that the upstream point will have an impedance associated with the spring n plus 1. So, Z S n plus 1 and then there is an inline impedance Z M n plus 1. And then finally, this mass is seeing a force which is capital F force is analogous to velocity in the electric circuit theory. So, that sorry force is analogous to voltage in the electric circuit theory, which means this point the input point of the mass will seen now a voltage which is F right. So, the circuit is fairly easy to draw. No matter how many springs and masses you give me all that I need to do is just keep arranging the springs in the inline in the shunt position and the masses in the inline position and complete this circuit.

Once the circuit is made I can take it to software such as simulink and simulate the system without worrying about how the equations have been formulated etcetera. So, any harmonic excitation problem can be done in this fashion transfer matrix will be even simpler. So, associated with each of this blocks we spring masses we have formulated the transfer matrix right. So, finally, if we need to understand what happens I will call this let us say F bar and I am interested in what happens to this V bar because I have lost a count of the number of stations. So, we will call this F bar and V bar this F bar and V bar has got to be related to F 0 and V 0, which is happening at the input end. Anyway V 0 is zero, but num the same we will chose to write the transfer matrix in a general form including the effect of V 0 right.

And then each of the transfer matrices starting from the upstream end need to be written 1 after the other and the product has to be taking. We know that associated with each transfer matrix associated with each of this spring mass system the transfer matrix I have already derived as 1 plus Z m by Z s, 1 by Z s, and Z m and 1 right. So, associated with the index we get an index one sitting over here right. So, each spring mass block is going to give you a transfer matrix with that specific structure. All that you have to do is to keep multiplying these transfer matrix found downstream end to the upstream end right. So, the generic form of the transfer matrix is this 1 plus Z M 1 Z s 1 sorry, I will call this i, Z m i on e by Z s i 1, and you will need to make a product of all this transfer matrices, i equals to 1 to n plus 1, in this case there are n plus 1 such transfer matrices. You will have to make this product. So, the transfer matrix can be readily obtained as a matrix multiplication. That is actually no big deal these are just 2 by 2 matrices no matter how many matrices are there you could almost do it in a pen and paper you could write a simple code also to do this matrix calculations multiplications in no big time right.

So, the point is when you have such assembly of springs and masses it is very easy to implement, the transfer matrix method or the electrical circuit, any one of them is going to be just as fine. The transfer matrix method though is what I will recommend in this course. So, next in the next example will put a damper.

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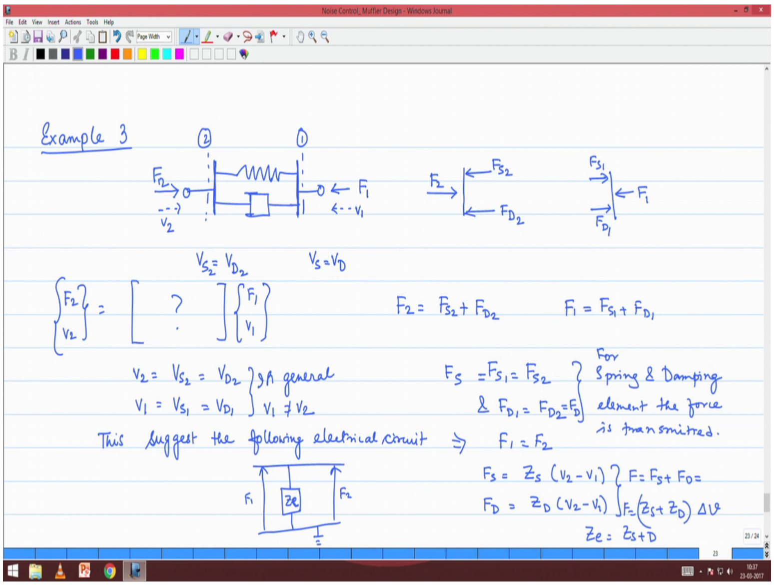

So, let us consider and assembly till now we will looking at assemblies which were in series. Now we will look at assemblies which are in parallel we will consider a spring and a mass sorry, spring and damper which are connected in parallel right. Please note carefully by this arrangement what is suggested is that the left hand of the spring and the damper are having identical position, the right hand of the spring and the damper is having identical position. So, they are locked because this plate or whatever you call it is a rigid plate right.

So, what I will do is that, I will call as usual this as station 2 and this as station 1. So, v so at this station 2 the V of the spring is equals to the V of this damper. Similarly at this station 2 the V of the spring is equals to the V of the damper right. So, at both this stations the spring element and the damping element have got the same motion, which means that kinematically they are equivalent right, but will the forces on the spring will the spring force and the damping force be same not necessarily right. So, the total force that comes here let us call this as F, F 2 I will call this. And I will call this as F 1. So, we need to calculate F 2 and F 1. The point is this the total force that you are putting in at the input of this assembly is getting divided into 2 parts, one in the spring the other in the damper right.

So, as a result if you take the free body diagram at station 2, what will you see? You will see that there is a force F 2 and there is a spring force resisting it there is also a damping force, which is resisting at station 1, what will you see with a free body diagram? You will see that there is a spring force which is actually the same as what we have written in the left hand. So, spring force and the damping force here will be resisting the force F 1 right. But we need to relate F 2 V 2, where V 2 is the velocity of this input port and V 1 is the velocity of this output port. So, what we need is a transfer matrix which relates F 2 V 2 to F 1 V 1 right. So, this transfer matrix needs to be found right. The what we have realize that thus far is V 2 is equals to V, V s 2 is equals to V D 2, which is the velocity of the spring at station 2 and velocity of the damper at station 2. The 2 points have identical motion. Similarly V 1 is V s 1 is equals to V D 1 right. This much is known that there is no ambiguity in the velocities, between the spring elements and the damping elements. But as for as the forces are concerned there can be a little issue.

So, F 2 is equals to F s 2 plus F D 2 right. And also F 1 is equals to F s 1 plus F D 1 right. Now please note that F s and I mean, F s 2 and F s 1 are same this is F s 2 F D 2 and this is F s 1 F D 1 right. So, we understand that for a spring element, the input and the output port of the spring element are at the same, force the forces cannot change. Between the input and the output of the spring element as well as for the damping element as what we did in the first lecture where we looked at individual elements. So, F s 1 must be equal to F s 2 and F D 1 must be equals to F D 2 right. So, this is because the spring for spring and damping element the force is transmitted. Just it would be completely transmitted. So, this in turn means this implies that F s 1 plus F D 1 must be equals to F s 2 plus F D 2 which means F 1 must be equals to F 2 right.

So, F 1 must be equal to F 2 that suggest that this the electrical circuit associated with this element is possibly in the shunt position, it is not in the inline position, but to verify or to confirm that it is indeed in the shunt position you have to still verify that whether the velocities at the 2 stations in that is the inlet and the outlet whether they are same or different. There is no restriction that V 1 has got to be equals to V 2 right. So, in general V 1 is not equals to V 2 right. So therefore, V 1 not being equal to V 2, but F 1 being equal to F 2 that suggest that whatever is the electrical circuit associated with this element must go in the shunt position right. So, this suggest the following electrical circuit. So, there has to be some equivalent impedance sitting here, there is F 1 on one side there is F 2 on the other side, this is the ground line. So, F 1 will be equal to F 2, but V 1 will be different from V 2 because there is an impedance which is in the parallel or in the shunt position. So, some part of the current will go in the parallel path leading to a difference in the current in the input side and the output side right.

So, the question is what will be this Z e. So, towards that end we realize that F s 2 or simply F s. Because F s 1 is equals to F s 2 we could as well say F s 1 is equals to F s 2 is equals to F s and F D 1 is equals to F D 2 is equals to F D. We already know that F s is equals to Z s times V 2 minus V 1 right. Delta V, we also know that F D is Z D times V 2 minus V 1. Remember V 2 and V 1 are same for both the spring element as well as damping element. So, no need of putting separate subscripts. We already have this relation that the force that is getting transmitted by the spring is related to the impedance of the spring multiplied by the relative velocity between the input and the output right. Same with the damper. So, these 2 relations now suggest that F which is equals to F s plus F D must be equal to Z s plus Z D into delta V right.

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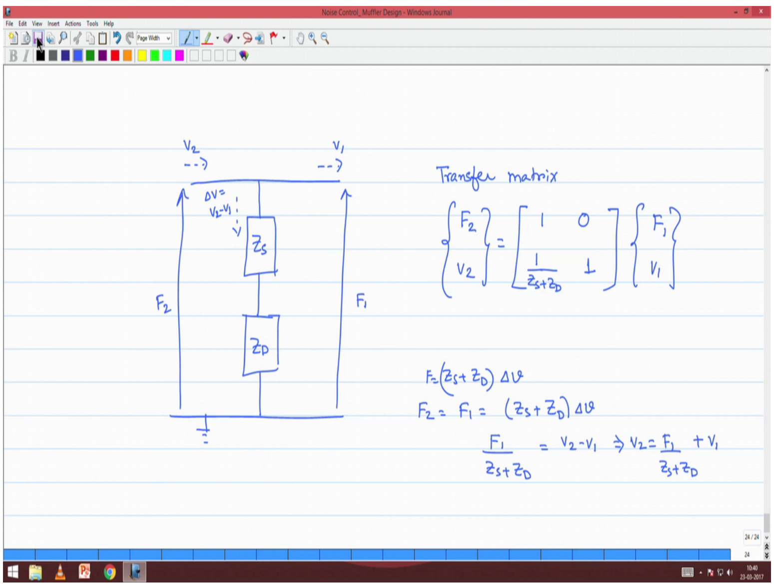

So, this means that the Z e basically is Z s plus Z D. To draw the picture you needs final form what we should have is this, the Z equivalent comprises of one Z s and one Z D. This is the ground line, this is the voltage F 2, this is the voltage F 1. So, what happens here is that V 2 current is going in, but only V 1 current is coming out the rest of it which is delta V. So, this delta V is going to be V 2 minus V 1. And this delta V multiplied by Z s is going to create the first potential delta V into Z s plus delta V into Z D is the total potential difference that will happen in this line which must be same as F 2 and F 1. This is exactly what you have, this is exactly what is what we have in this relation where we set F is equals to Z s plus Z D into delta F, into delta V. So, this F is basically the same as F 2 and F 1 because anyway in this situation we are saying that the input and the output force is just the same right. And what will be the transfer matrix associated with this system.

Transfer matrix will be F 2 V 2 has got to be related to F 1 V 1 right. F 2 has to be equal to F 1. So, the top row of this transfer matrix is trivial 1 0. What about the bottom row the bottom row? Need to have a capture the relation that V 2 minus V 1 has got to be Z s plus V 2 minus V F which is if I if I just copy and paste this relation. This is the relation that we had. So therefore, we also know that F is the same as F 2 or F 1. So, F 2 equals to F 1 is equals to Z s plus Z D times delta V right. So, F 1 by Z s plus Z D must be equals to delta V which is V 2 minus V 1 right. So therefore, V 2 minus V 1 has got to be or F 1. So, V 2 has got to be F 1 divided by Z s plus Z D plus V 1. We could write this up in this matrix form by noting that this element will be 1, and this element will be 1 by Z s plus Z D right.

So, spring damper combination element is almost the same as a spring element except for the fact that the spring element will have an impedance Z s the spring damper combination will have impedance of Z s plus Z D, both in the shunt position right. And now what you will see is that the impedance of now if you put this together with a mass.

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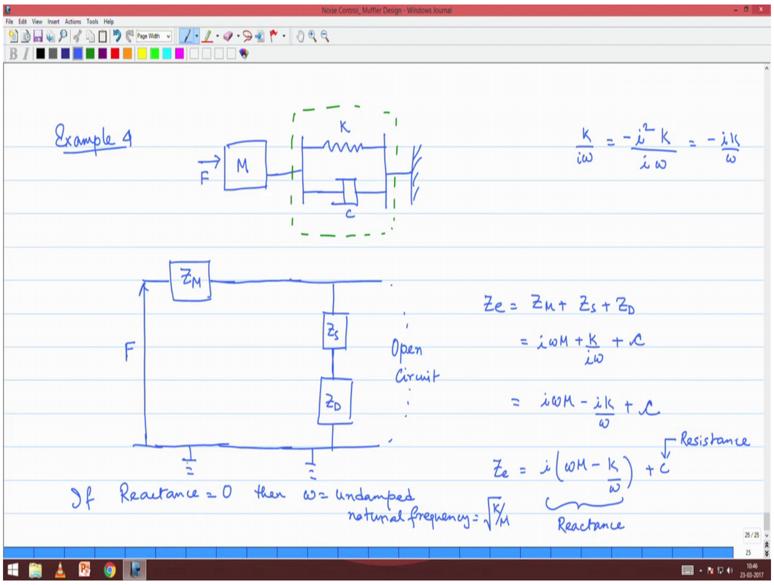

So, now, let us consider this as example 4. So, we will have a mass and then we will have that combination element which is a spring and a damper or maybe I will draw this in a little better way. So, this is the spring damper combination element that we have derived in the last page. And this is now connected with a mass, this is k and this is C. This is our good hold vibrating system which is the first starting point of theory of vibration ok.

So, now what we have understood is that for this part this spring damper system. We have been able to find the analogous electrical circuit which suggest that there has to be spring element and the damping element in the shunt position. And the output of this combination element must not carry any current right. So therefore, this has to be left open. So, this part is an open circuit we are not going to close this is not going to get closed right. And what is the upstream element? The upstream element is that of a mass which is placed inline right. And this is the ground and this is the ground.

So, as well we might join these 2, and since the entire force has got to be transmitted between the mass and the combination element, mark my usage of the word I am not saying the entire force from the mass is getting transmitted to the spring element or to the damping element. It is getting transmitted to the combination element which combines both spring and damper as given by that green block right. So, if the force has got to be transmitted this line can as well be joined. So therefore, there is no potential difference basically, between the output of the mass port and the input of the combination port right.

And finally, if this masses getting to see a force F that can be accounted for by saying that there is a force there is a voltage with respect to the ground. This is the electrical circuit associated with the spring mass damper element. But please note this spring and the damper are in parallel in the mechanical assemblage it is standing out that spring mass damper everything is in series in the electrical circuit. Do not get confused about this series parallel business look at it from first principles. And you will be able to argue please understand that fundamentally the spring and the damping element should come in the shunt position.

The fallacy happens because this shunt position is left open on it is other end right. Because it is being left open in the other end you are able to interpret this circuit where the spring mass and damping element are actually in series right, but fundamentally it is that the spring and the damping element are in shunt with respect to the mass right. So, now, you what you see here is that the effective impedance of this circuit Z e is going to be Z m plus Z s plus Z D. What is Z m j omega m? What is Z s plus Z s? We have evaluated as where was that?

Student: (Refer Time: 42: 37).

K by j omega. So, this was k by j omega and Z D was r right. K by j omega could be written as minus j. So, I could write k by j omega as minus j square k or why am I using j s sorry, I should use I this is k by i omega. So, this k by i omega could be written as i square with a minus sign minus i square is just plus 1. So, no issues if I write a minus i square. So, minus i square k by i omega and this in turn is minus ik by omega right. So therefore, I would write this as i omega m minus I k by omega plus r and eventually.

Student: Sir, in case of r into c.

It is capital small c sorry Z e is small c right. So, then I could as well take this i of an omega m minus k by omega plus c. So, this is the equivalent impedance this is the imaginary part of the equivalent impedance which is called as reactance, and this is the real part of it which is called the resistance. And what you see is that the undamped natural frequency is obtained when the reactance is set to 0. So, if reactance equals to 0 then omega is equal to undamped natural frequency which is square root k by m. I think we leave it here. We will take up a few more complicated examples may be next class. After this we will use the same analogy to look at acoustic resonators, Helmholtz resonators which are very important these are lumped acoustic elements. So, using this analogy it will be very simple to introduce that idea.