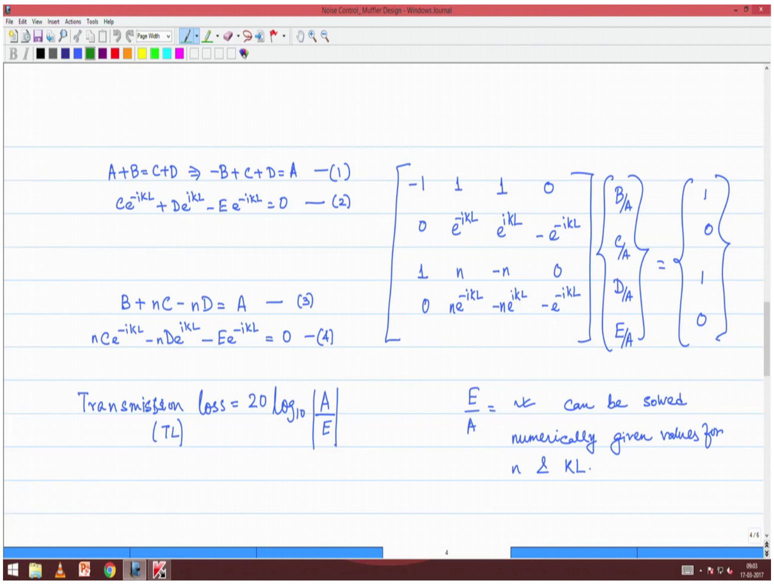
Acoustics & Noise Control
Dr. Abhijit Sarkar
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Module – 21
Lecture – 26
Transfer matrix method For expansion chamber muffler

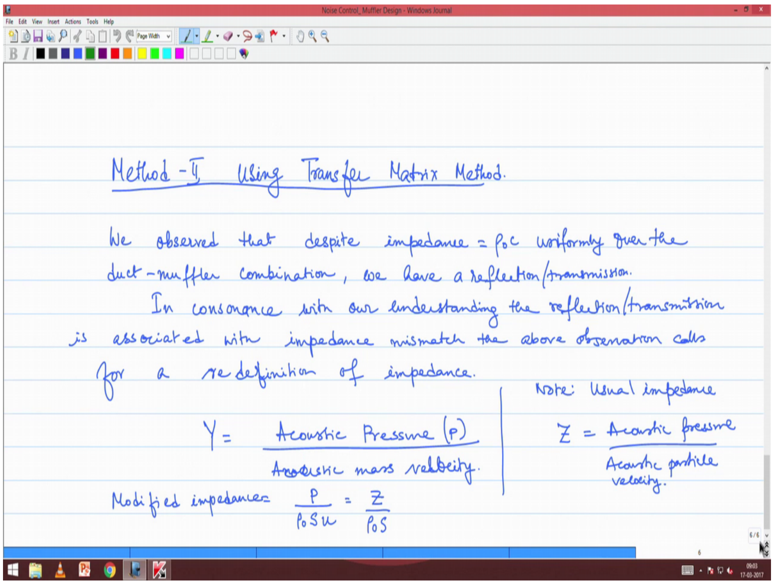
Today use the transfer matrix method to arrive at the transmission loss of the expansion chamber muffler, remember in the last class we talked about how to get it in the usual wave propagation scheme of analysis.

(Refer Slide Time: 00:27)



As remarked in the last class.

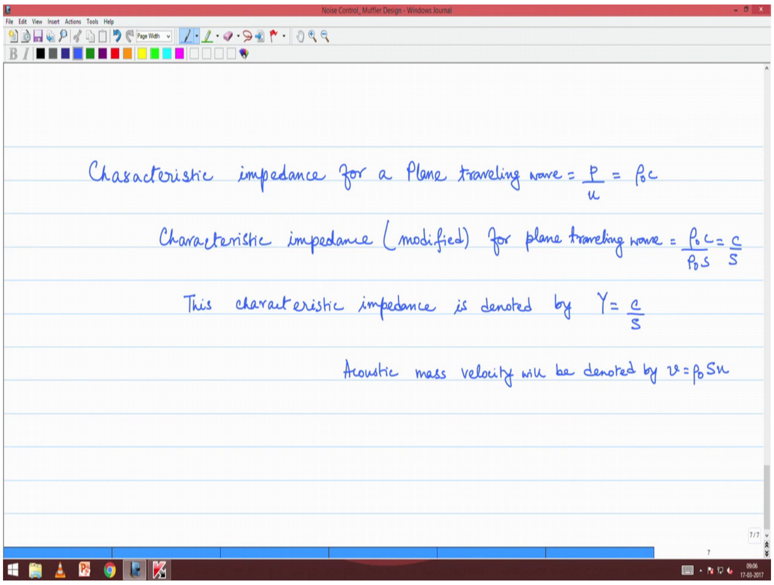
(Refer Slide Time: 00:31)



We are going to re do it in a little more elegant fashion using the transfer matrix method and when we left last time is that be defined a new kind of impedance I was just mode adopted to the scheme of duct acoustics, that with changing area of cross action and so on where in the impedance in terms of mass velocity. So, it is again the ratio of acoustic pressure by velocity, but the velocity has to be understood as a mass velocity rather than as the particle velocity ok.

So, towards this end you will recall that characteristic impedance for a plane travelling wave.

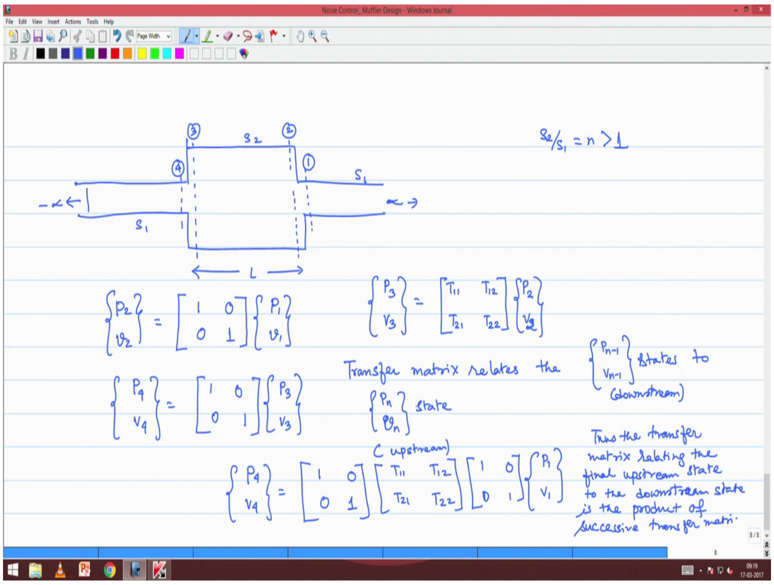
(Refer Slide Time: 01:10)



Characteristic impedance for a plane travelling wave that is the P by u was equals to rho naught c, u being the particle velocity right, but in terms of this modified impedance the characteristic impedance will read as modified for plane travelling waves, we know that the 2 impedance are related by this condition that we have to divide traditional impedance by rho naught s, s being the area of cross section right. So, if we do that we will get rho naught c divided by rho naught s. So, c by s is the characteristics impedance in this new definition for the plane travelling wave right. We will just abbreviate this by y. So, this characteristic impedance is denoted by y.

So, when we say capital y from here on it means c by s right. This is remember for the plane travelling wave characteristic impedance this is not for a standing wave situation and similarly we will define the acoustic mass velocity will denote by V . So, V stands for the mass velocity u stands for the particle velocity, and V is as we had said rho naught s u. So, that is how we are going to change a few notations from here on. Now let us look at the.

(Refer Slide Time: 03:43)



So, it is infinitely extended in both sides and this expansion chamber is of a length l and the area of cross sections are S 1 S 2 and S 1 again and we have taken S 2 by S 1 to be equals to n which is greater than one ok.

So, this is our geometry.

Now, we wish to relate the as I said the transmitted wave amplitude to the incident wave amplitude in this case. So, what happens in transfer matrix method is that we are going to look at individual pieces of this duct and we are going to construct a few hypothetical station. So, this is the downstream station, the station just upstream is this and then you have this station and then you have this station. So, we have this as our station 1, this is station 2, this is station 3 and this is station 4 and transfer matrix relates the state variables between each of these stations as simple as that. So, instead of trying to solve for in one short the entire 4 by 4 system or no matter how many variables you have, the approach is different here we are simply going to construct transfer matrices which relates the state variables between each of these stations right.

So, first we have going to relate how P 2 V 2 is related to P 1 V 1. So, at station 2 we where the elementary quantities of interest are the pressure and the mass velocities we are going to talk in terms of mass velocities. So, we are going to relate the upstream pressure to the downstream pressure and the upstream velocity, to the downstream velocity again I iterate when I say velocity from here all, it does not mean particle velocity it means mass velocity. So, we need to relate all the upstream and the downstream quantities with a certain matrix because up stream there are 2 states I mean every station there are 2 states pressure, and velocity we are we need a relation relating that 2 right.

So, the equation relating P 2 and P 1 V 1 is P 2 equals 2 P 1 that is it is 1 0. Similarly V 2 is equals to V 1 remember that is why we changed over from particle velocity to mass velocity V is the mass velocity, mass velocity on both sides is just the same. So, this is just an identity matrix no big deal right. Similarly it is easy to see that P 4 V 4 is same as P 3 V 3 just the same thing is happening right. P 3 V 3 is 1 0 0 1 and P 4 V 4. So, the pressure and velocity on either side of the expansion point and the contraction point is just the same velocity mean meaning mean velocity, all that remains is to relate between 2 and 3.

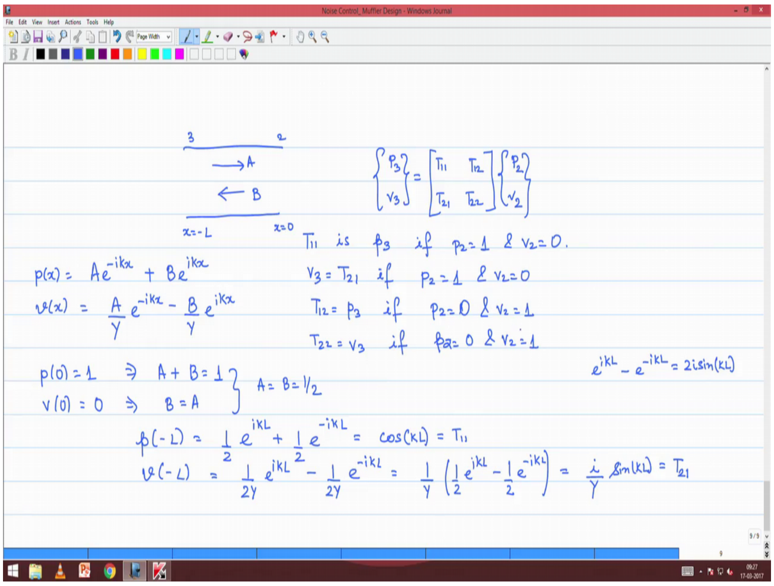
So, P 3 V 3 has to be related with P 2 V 2 also and we need to derived this matrix will call this for now T11, T12, T21, T 22. So, this is the generic form of a transfer matrix each of these matrices that we have derived or basically transfer matrices which relates the upstream and the downstream state variable. The state variables in this pardons is just the pressure and velocity, it is slightly different from state variable concept in dynamical system here remember we are talking with the harmonic assumption. So, when I say state variable it means pressure and acoustic mass velocity.

So, transfer matrix is relates the P n minus 1, V n minus 1 states to P n V n states P n minus 1 V n minus 1 is the downstream and P n V n is the upstream. Actually upstream downstream at present does not make much sense, but none the same we understand that the incident is from the left side and the transmission is from the right side right. So, we want we are going to number it such that at the end it is going to be 1. The point where the transmission will the transmission will actually come out is given the by convention a number one and each of its upstream successive states will be numbered accordingly in progressively higher numbers ok.

So, transfers matrix will be therefore, any matrix which will relate these upstream and downstream states. So, will anyway find out what are these forms of these T11 T12 T21 and TT2, but the advantage of this method if you realize is that if we simply multiply each of these transfer matrices we are able to relate P 4 V 4 with P 1 V 1 right. P 4 V 4 is this transfer matrix which basically is identity times P 3 V 3, but P 3 V 3 is this transfer matrix multiplied by P 2 V 2 and P 2 V 2 is this transfer matrix which again turns out to be identity times P 1 V 1 right. So, therefore, we can easily write P 4 V 4 as the multiplication of all these transfer matrices and here it turns out to be easy because the 2 transfer matrices. In fact, are identity, but in general it may not be identity, but none the same you could construct there transfer matrices in some way or the other.

But the crucial point is at the end of the procedure you could relate the upstream state, the final upstream state to the final downstream state simply as a product of transfer matrices and that is what makes things much simpler to handle. So, as I said in the last lecture that in transfer matrix analysis of these systems typically you need to have no more than 2 by 2 systems. At least for the simple ones for more complicated ones you can have more a greater size, but to understand this would be pretty simple and also to execute this would be pretty simple. So, thus the transfer matrix relating the ups final upstream state to the downstream state is the product of successive transfer matrices that is the key simplification that will happen ok.

(Refer Slide Time: 12:16)



With that let us now go to derive the transfer matrix for this section within the expansion chamber we named it as 2 and 3. So, this is station 2 this is station 3 and we understood that we could relate the states at station 3 to the states at station 2 using a transfer matrix of this kind what the job is to find out these values. The job is to find out these terms T11

T12 T21 and T22. So, towards this end we realize that T11 is equals to is P 3 if P 2 equals to 1 and V 2 equals to 0 right if you put P 2 equals to 1 and V 2 equals to 0 you will get P 3 is equals to you will get P 3 is equals to T11 right.

Similarly, V 3 will be equals to T21 if P 2 is equals to 1 and V 2 is equals to 0 right. So, in other words if we are able to solve in our usual plane wave analysis or using finite element method and what not that with this bound recondition the P 2 equals to 1 and V 2 equals 0 whatever pressure and velocity we get at station 3 is basically our quantities P 1 1, T11 and T21 right. So, that is precisely what will do, but before doing that let us also take the other picture that is T12 will be same as P 3 if P 2 is equals to 0 and V 2 is equals to 1 and finally, T22 will be equals to V 3 if P 2 is equals to 0 and V 2 is equals to 1 right.

So, under this special cases that is either P 2 is equals to 1 and V 2 equals to 0 or the other way round that is V 2 equals to 1 and P 2 is equals to 0 if we are able to ritrim what is the pressure and velocity of this upstream 0.3 then we should be able to find what is called the propagation matrix between the 0.2 and 3 right. So, that is how we will derive it and towards that end we will call this point as x equals to 0 and this point as x equals 2 minus l. The reason behind this is that we want simpler boundary the boundary conditions is given at the station 2 point. So, it is better to give the station 2 a coordinate of 0 because that will thing turn out much simpler and between these 2 points in the within the expansion chamber we realize there can only be an a wave and there can only be a b wave each of which are travelling in opposite directions right.

So, we understand that within this region P x will be given as A e to the power minus i k x plus Be to the power i k x and V x which is the mass velocity will be given as A by y e to the power minus i k x minus B by y e to the power i k x where y is the characteristic impedance in terms of the mass velocities not in terms of the usual velocities. So, basically c by s is which is basically constant within the expansion chamber remember we are now working out what is happening in the expansion chamber allowed we not worried about the other parts that is what is one very important simplification of the transfer matrix method it lets you work in individual segments rather than looking at the entire complicated picture in one shot.

The next step is pretty simple what we want to in the first case is that we want t at 0 to be. So, we want P at 0 to be 1 that would give us A plus B to be 1, we want V at 0 to be 0 that would give us A is equals to B right. So, collecting these 2 we can always say A equals to B equals to half is the condition that we have for the amplitudes of these 2 waves which are travelling within the expansion chamber. Now what remains to be found out is what is exactly the pressure at station 3, which is having a coordinate of x equals to minus l. So, P at minus L is a which is half into e to the power minus i k x is minus l. So, this will be e to the power i KL right plus b which is also half e to the power i k x x is minus l. So, that makes it e to the power minus i KL and as you very well know this quantity is given as cos of KL right. So, this basically is my T11 because as per the definition T11 is the value of pressured obtain at station 3 if somehow you are able to maintain the value of pressure at station 2 to be equal to unity, but the value of velocity at station 2 is maintain that 0.

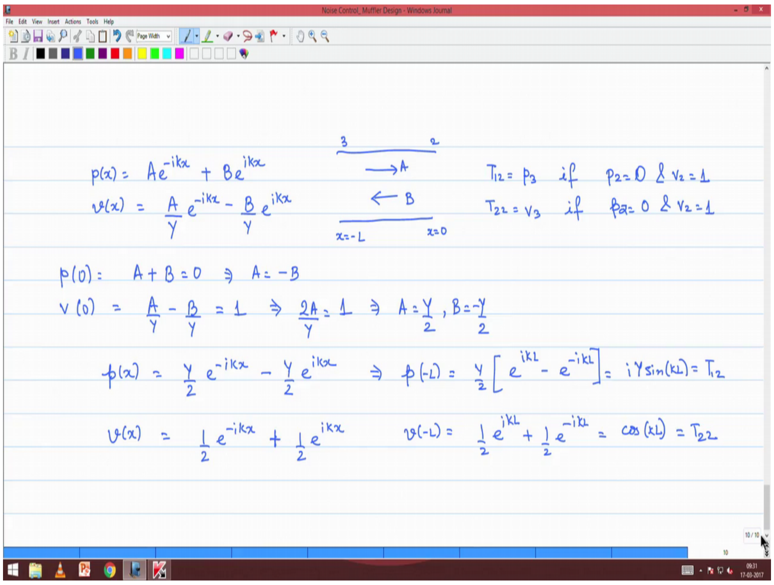
So, with these conditions we are actually able to solve for an obtain the value of pressure which is given by cos k l. So, they are for T11 is basically this value coming to next we need to find T22 one which is the value of velocity and at station three. So, velocity at station 3 is V at minus l which is given by one by 2 y e to the power i KL minus 1 by 2 y b e to the power minus I k l. So, that would read as that would read as one by y one by 2 e to the power i KL minus 1 by 2 e to the power minus i KL right you will know that 2 I sin KL is sorry e to the power i KL minus e to the power minus i KL would give us as 2 I sin KL right . So, therefore, we could make that substitution and we will have as following i by Y sin KL right and this would be equals to T12 fine.

Student: T 21.

T 2 sorry T21 T 2 1.

Next we need to derived what happens for the entries at the second column of our transfer matrix and for towards that in we need to solve for the condition that V 2 is equals to one, but P 2 equals to 0. So, which will do in maybe I will copy and paste this part so.

(Refer Slide Time: (20:49)

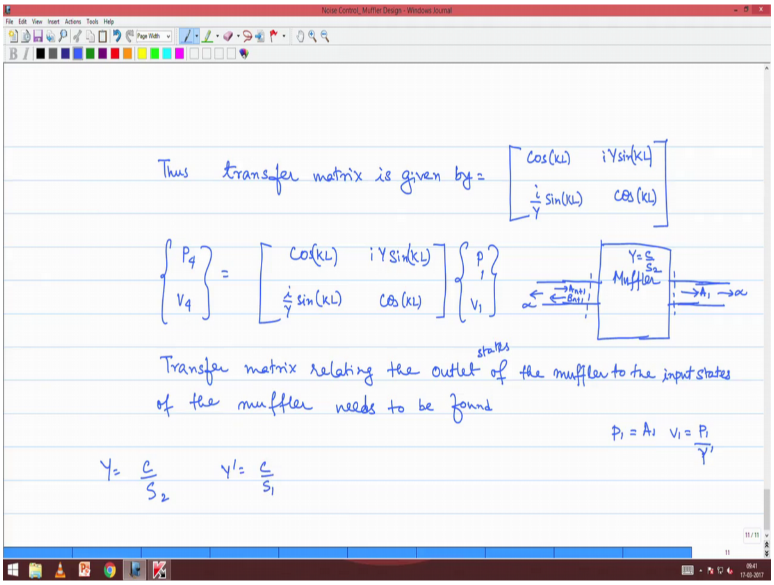


We need to find conditions where in I need to copy the picture also I guess yeah and finally, I would copy this 2 statements yeah. So, this is all that we need in the next part of our derivation. So, at x equals to 0 we want P 2 to be 0 this time. So, which means P 0 is given as a plus b and this has got to be 0 by our requirement which means that a has to be minus b right and V 0 is a by y minus b by y which has to be one.

But minus b is equals to a which means 2 a by y as better b equal to one which implies that a has to be y by 2 and that also implies that b has to be minus y by two. So, with this as a and b we would get P x as y by 2 e to the power minus i k x plus sorry minus y by 2 e to the power i k x and V of x is going to be half e to the power minus i k x plus half e to the power i k x right I am replacing a as y by 2 and b as minus y by 2 sins b is minus the sin of the expression is flipped in the second term so; that means, P at minus l will be y by 2 e to the power i KL minus e to the power minus i KL and this as be saw is I y sin KL this bracketed expression is 2 I sin k l. So, therefore, we get P at minus l as y I y sin KL right.

Finally V at minus l is going to be half e to the power i KL plus half e to the power minus i KL which is cos KL right. So, therefore, this is T21 or T12 sorry this is T12 and this is T 2 2. So, we have found all the 4 entries which are actually called forth pole parameters of our transfer we have got all the 4 pole parameters of our transfer matrix or let us rewrite transfer matrix in his grand entirety once again thus.

(Refer Slide Time: 24:29)



Cos KL i by Y sin KL i y sin KL and cos KL. This is the transfer matrix that we have got please note that it is not symmetric this is was it expected to be symmetry finite element matrices always turn out to be symmetry this is not finite element this is transfer matrix method here you are not relating P 2 P 3 with V 2 V 3 if you had done. So, you have got symmetric matrix, but here it is not about rerating the pressure states.

I mean the pressure variables at both the ends with the velocity variables and both the end it is about relating the pressure states and the velocity states at both the ends. And also please not that there is no discretization involved here this nothing about this is complete analytical expression, this is absolutely no errors that we have committed in arriving at this expression it is a complete analytical exact expression, there is no error at least in these stage. So, we will denote this by t. So, finally, what we had is that the upstream state at station 4 could be related to the downstream state at station one, using this very transfer matrix because the other transfer matrices turns out to be unity in this case right in a more complicated situation which hopefully we should be able to do we wills see that there are other forms of transfer matrices that are possible right.

So, now what will get is P 4 V 4 is therefore, given by a matrix multiplication of this transfer matrix i Y sin KL i by Y sin KL cos KL P 1 V 1. So, let us redraw the geometry once more. So, here is the input here we have a muffler and here we have the output do in the present example we had taken a simple expansion chamber muffler in general mufflers can be far more complicated than this simple expansion chamber muffler, but in whatever way we configuration of muffler you have, there are different ways to derive the transfer matrix method either experimentally or computationally or analytically, but the point is this you have to finally, arrive at this transfer matrix which relates the output of the muffler to the input of the muffler right.

So, finally, the transfer matrix relating the input of the muffler sorry relating the output of the muffler outlet of the muffler to the outlet states I should say of the muffler to the input states of the muffler needs to be found. In this case we have already done it, but no matter how general the muffler is once you get this transfer matrix which relates what happens to the outlet of the muffler to what happens at the inlet of the muffler, the subsequent innovations which we shall do now is again going to be remaining the same and this is going to give us what is known as the transmission loss. We have already seen transmission loss from the par dense of the regular way propagation idea here we are doing it in terms of transfer matrices and we will understand that this is much simpler and much more elegant to do. Remember the geometry that we are dealing with at present is that it is infinite both at the inlet as well as the outlet side at present right.

So, both where it is run into infinity, if it is running to infinity at the outlet side you do not expect any reflected wave there will be only a transmitted wave right. So, will call this wave as let say the A 1 wave right and will call this wave which is incident as a n wave which is the incident to the muffler right because there are let say n elements and accordingly infective there are n elements it will have to be a n plus 1 will call this n plus 1 there are a n elements in this which means they have basically n stations have to look at the station which is just ahead of this right, but at present it does not matter whether you call it a n or a n plus 1 does not matter really, but will there be a reflected wave here also it is going infinite in the inlet side also. But is there a reflected wave that is expected it should be right because if a n plus 1 is not equals to a n then where has the remaining gone the remaining has got to be reflected. So, the reflected wave here comes because of the reflection of the muffler not because of the infinite condition on the left end side.

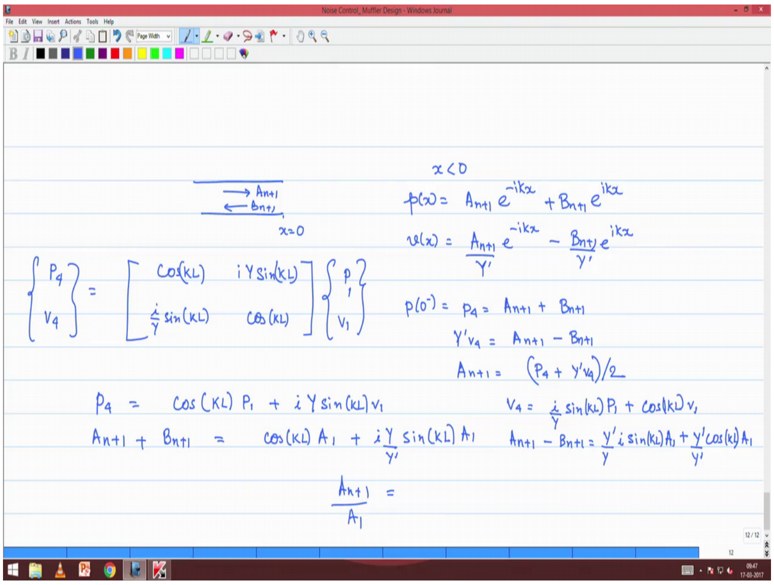
Because the muffler is creating a situation where the transmission wave is less than the incident wave, therefore the reflection wave the reflected wave has to get generated. And typically as we said what we are very interested to know is what is this ratio A n plus 1 to A 1 right the incident wave which is coming and the transmitted wave which is passing passed the muffler, and this ratio should be as large as possible and this is eventually in logarithmic scale is going to give you the transmission loss. So, let us see how from this transfer matrix we can nicely retrieve the this transmission loss characteristics. So, the remember the transmission the transfer matrix method works on the states it does not work on the waves it is working on the totals pressure P 1, which is created by possibly all the waves which are reciting in the exit part of the duct.

Similarly, P 4 in this case is quantifying the effect of the entire pressure due to both the incident as well as the reflected, but in some ways you need to sort of peel out the effect of the reflected wave and you should be able to be quantifying only the effect of the incident wave. P 4 is the pressure obtained because of both incident a as well as the reflected b wave right both of this ways will combine to give a total pressure P similarly a total velocity V, but transmission loss quantifies the incident on the inlet side to the transmitted on the outlet side right.

So, we need in some way to do this job for us. On the outlet site things are very easy that on the outlet side we know P 1 has got to be equals to A 1 right because on the outlet side there is only one wave right. So, on the outlet P 1 has got to be equals to A 1 there is no chance that there is any other way because anyway it is an unequate termination right and V 1 therefore, has got to be equals to P 1 by Y, but this Y is I will call this as y prime because I call Y corresponding to this expansion chamber we denoted it is by Y. Remember what is the expression for Y is given by C by S and S is the area. So, Y corresponding to this is C by S 2 and y prime will be C by S 1 has S 1 is not equals to S 2 y and y prime are not the same.

So, P 1 is equals to A 1 and V 1 is equals to P 1 by A prime. So, this is what happens at the inlet side, at the outlet side will have to do a better analysis.

(Refer Slide Time: 34:02)



So, at the outlet side we are having sorry at the inlet side we are having A n plus on and B n plus 1 as the condition. So, again let us take x equals to 0 here for this part of the analysis. So, therefore, P at x for x lesser than 0 we will have P at x to be given by A n plus 1 e to the power minus i k x plus B n plus 1 e to the power plus i k x, and the velocity will be given as A n plus 1 e to the power minus i k x by Y prime, minus B n plus 1 by Y prime e to the power i k x right that is the usual expressions that we have.

So, what happens at x equals to 0 minus at 0 minus we will have the quantity P 4 right this is what is hitting the inlet of the muffler. So, P 4 will be given as A n plus 1 plus B n plus 1 and similarly Y prime V 4 is going to be given as A n plus 1 minus B n plus 1. So, from these 2 equations it is very easy to recover what is A n plus 1 right. So, A n plus 1 is therefore, going to be P 4 plus Y prime V 4 right this is what we wanted we wanted to recover the incident wave which is incident on to the inlet side of the muffler in terms of the state variables and these state variables are already related to the output states and the output states are exactly given by one out going wave out going travelling wave. So, therefore, we should be able to do the ratio at this stage let us see how it comes through.

So, if you open up this expressions we are going to get P 4 is equals to cos KL P 1 plus I Y sin KL into V 1, but then P 4 is given as A n plus 1 plus B n plus 1 that should be equals to cos KL P 1 is A 1 i Y sin KL, V 1 is A 1 by Y prime right instead of V 1 I am writing A 1 by Y prime, and the next equation which reads V 4 is equals to i by Y sin KL

P 1 plus cos KL V 1 right. And V 4 by our calculation here is given by A n plus 1 minus B n plus 1 the Y prime that comes on the denominator I will shift it on the other side. So, this Y prime by Y the first term is Y prime by Y sin KL P 1 is A 1 right plus Y prime cos KL instead of V 1 I have to write A 1 by Y prime again right.

So, therefore, this y prime is going to, but you get the idea that at this stage at least with these 2 equation A n plus 1 divided by A 1 should be pretty easy to find right and that is exactly in logarithmic scale the transmission loss which we had derived. As a 4 by solution of a 4 by 4 system of equation, but as you see this can be almost done in pen and paper in reasonable amount of time, and we could also interpret the results nice interpretation will come out from this analytical formula which is why we will wait till the next class to give you the interpretation of this result of transmission loss do we have time I do not think we have. So, will keep it till here at this point will catch it from round about this part.