Acoustics & Noise Control
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Module - 13
Lecture - 18
Power Calculation

Good morning friends. So, today will look at acoustic intensity, the concept of acoustic intensity will be studied in today's lecture. Till now we had been looking at acoustic pressure and acoustic velocity as the 2 fundamental quantities of interest, but now we will look at the combination of them which will turn out to be acoustic intensity. But before getting in to this topic let me go back one step and do a similar analogy with I mean the formula that we will get in acoustic intensities, will try to just go back one step and re derive a similar formula which is the power calculations, for power calculations for a spring mass system.

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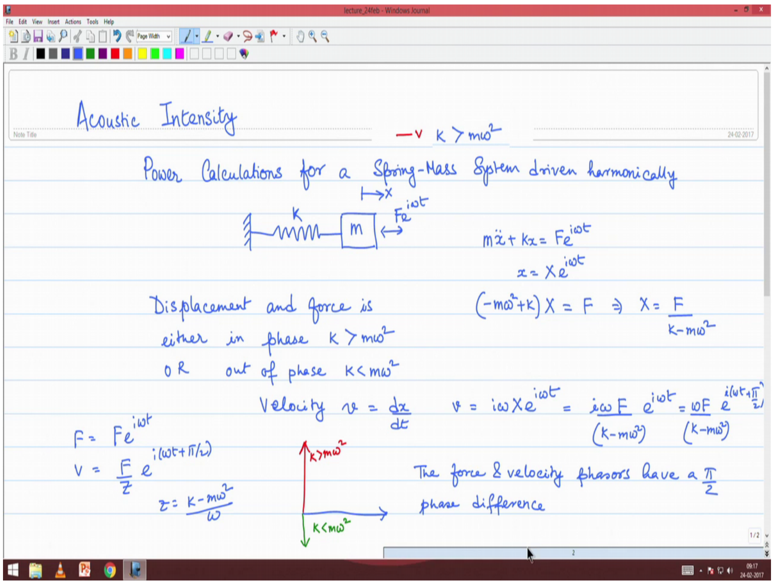

So, we will just this is technically not required for the acoustic calculation, but just to make a smooth entry in to this subject we will do revisit the power calculations for a spring mass system driven harmonically. So, the problem is the following you have a spring of stiffness K connected with a mass m and this is the displacement of, x is the displacement of the mass, it is driven by a harmonic force which as we know can be qualified as F e to the power i omega t.

So, the differential equation associated with this problem is well known m x double dot plus K x is equals to F, but now since F is harmonic we might as well call it F e to the power i omega t. So, we know for a linear system if it is undergoing an excitation which is harmonic the response might as well be harmonic. So, we choose small x equals to capital X e to the power i omega t and then if we make that substitution we will get minus m omega square plus K multiplying X is equals to F which means capital X is equals to capital F divided by K minus m omega square.

In other words the displacement amplitude and the force amplitude is related by a constant real factor right. So, the displacement and the force basically are in phase or out of phase depending up on whether this K is greater than m omega square or K is less than m omega square.

So, readily we see from this formula that displacement and force is either in phase which happens in the low frequency zone that is K greater than m omega square or out of phase which happens in the high frequency zone when K is less than m omega square. So, for a high frequencies the system actually behaves like a mass whereas, for the low frequencies it actually behaves like a spring because remember for a spring the displacement has to be in phase with the force. If you had only the spring K X equals to F which means the displacement and the force had to be in phase.

Now, what about the velocities? So, velocity v we know is dx dt right. So, therefore, velocity will be i omega times x in to e to the power i omega t right because small x has been found out as capital X times e to the power i omega t. So, therefore, capital X if we make this substitution we get i omega F divided by K minus m omega square e to the power i omega t and now I may group this i together as a phase factor that gives me omega F divided by K minus m omega square once again i omega t plus pi by 2 that is the factor within the exponential within the argument of the exponentials sign.

So, therefore, what we have this time is the velocity amplitude is omega F by K minus m omega square, but please note that now there is a phase difference between the force and the velocity and the phase difference happens to be pi by 2. In other words if the force vectors starts from the horizontal position the velocity vector will have a 90 degree lead if K is greater than m omega square and if K is less than m omega square then it will have a 90 degree lag. So, the velocity vector will lag the force vector if there is a, if the frequency is on the higher side higher than the natural frequency.

But the important point to note is that either ways there is a phase difference between velocity and force by an amount of 90 degree. So, let me write this point for you the force and velocity phasors have a pi by 2 phase difference which may be pi by 2 positive or pi by 2 negative depending up on whether the frequency is high or low.

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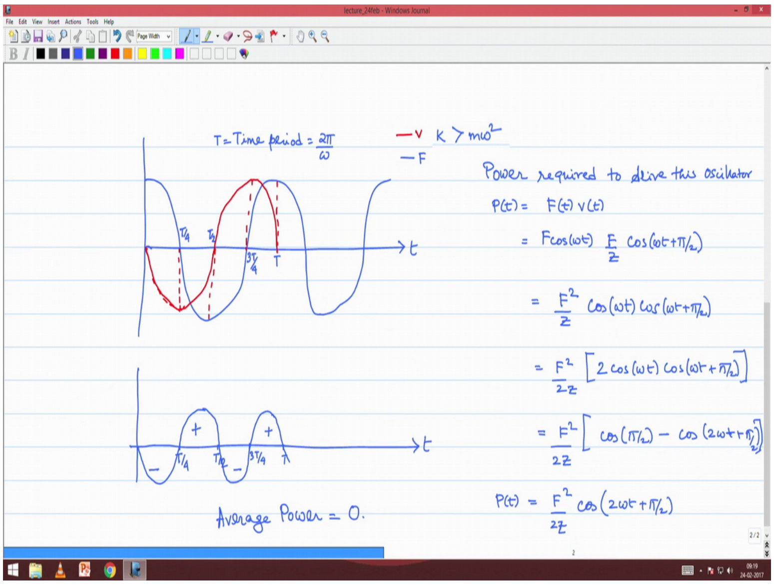

So, how does the graph therefore, look like? So, if we happen to plot the graph force against time let it be co sinusoid what would be the graph of the velocity, the velocity graph would be either a negative sign or a positive sign depending up on the phase. If K is greater than m omega square it has to be an negative sign right. So, that would be that graph would looks something like this.

Please excuse my drawing and then it will go to a peak value and then it will come down and this way, but the important thing to note in this overlaid graph is the following that wherever the force was going to 0 the velocity is going to its peak value. So, these peaks and the 0 values will be matching right. So, the red one is velocity for K greater than m omega square force that is all very fine right. So, now, if we wish to calculate the power what is the power that is required to drive this oscillator? If you think back the formula for power at any specific time is nice and simple F of t in to v of t that is it right. So, that is the instantaneous power. So, let us intuitively understand what is the power that is required to drive this oscillator from this plot, in this half so what I will do is here itself I will try to plot the power in this first half of the time period I will call this t by 4, this is t by 2 and this is again 3 T by 4 and this is T capital T is the time period of the oscillation which in turn is 2 pi by omega right.

Now, what happens from 0 to T by 4? Force and velocity I is having opposite sign. So, therefore, what is the power, the power is going to be negative in the first T by 4 quarter period of the time cycle. What happens in the next cycle, next quarter period for between T by 4 to T by 2 both force and velocity are negative. So, as a result the power that will come out is going to be positive.

What happens progressively in the next quarter cycle that is from T by 2 to 3 T by 4 force is negative, but velocity is positive as a result the power is negative and what happens finally, between 3 T by 4 and T, both are positive. So, finally, we get to see again a positive power. And this cycle will continue, between each consecutive quarter cycles you will now see that the instantaneous power is going to fluctuate inside the sometimes it is negative and sometimes it is positive.

What does the negative power and positive power imply? Negative power implies that power is actually not done by the external agent to drive this oscillator in fact, it is the oscillator which gives back the power to the external agent right that is the meaning of negative power because positive power means that is the power that is required by an external agent to drive the oscillator.

So, it is only between T by 4 to T by 2 and 3 T by 4 and T that you require an external agent to actually drive this system, but whatever the power is supplied by the external agent to the oscillator during these quarter cycle periods is actually returned back by the oscillator to the external agent between 0 to T by 4 and T by 2 to 3 T by 4 this is something similar happens even in an IC engine if you recall that during the power stroke only the IC engine supplies power right.

In the other strokes it is actually because of the inertia of the crankshaft it is able to execute the motion right if the power stroke alone supplies the power, but that happens to exceed the power required in the compression stroke that is why it is able to complete the cycle all by itself right.

So, powered that is required by the oscillator. Let us now return back to the oscillator case powered that is required by the oscillator is actually going to be 0 in an average sense because within certain duration it is actually the external agent which will supply the power. But in exactly the same duration of time in that is between 0 to T by 4 and 3 T by 2 to 3 T by 4 you are going to get the opposite effect where the oscillator is going to give back the power to the external agent and as a result it is this stage it should be obvious to you that the average power required will be 0 right whatever is going to the system is actually coming back right.

So, thus firstly, the instantaneous power will have a periodicity this time of T by 2 if you realize it is going negative it is going positive it is going negative it is going positive right. So, this time the power instantaneous power is not going to be of period omega, but rather it is going to be a rather of period to 2 pi by omega, but it is going to be a pi by omega. In other words the frequency has doubled right.

So, if you look at it in a more formal sense. So, we have the following expression capital F is equals to F e to the power i omega t and v is capital F by Z e to the power i omega t plus pi by 2 where Z is K minus m omega square divided by omega right. So, that being the case, but F of t what is, this is in the phasor notation what we have written for force and velocity. What will be the real part of this is of essentially what we are going to track. The real part of this force will be F cos omega t and the real part of the velocity will be F by Z cos omega t plus pi by 2 right and that gives us F square by Z cos omega t in to cos omega t plus pi by 2 right we can do a little more calculation by converting this as a sum of cos sin.

So, if we do that you will have F square by 2 Z cos of a minus b needs to be put here which is cos of pi by 2 minus cos of 2 omega t right, 2 omega t plus pi by 2 cos of 2 omega t plus pi by 2. So, what is cos pi by 2? Cos pi by 2 is 0. So, we will have F square by 2 Z in to cos of 2 omega t plus pi by 2. So, you see what I told you is in fact, correct that the instantaneous power has a frequency of double the frequency of the force and the velocity. This is the instantaneous power expression. So, the instantaneous power would go something like this, ok. But what is the average value of this power? The average value of this power is readily now seen to be 0 because it is going both in the positive direction as well as in the negative direction. So, the average power equals to 0 right because it is sinusoidally oscillating at a frequency of 2 omega between its positive extreme and negative extreme generating identical areas below the x axis; below the t axis and above the t axis.

So, therefore, they in an average sense the entire area is if calculated within a large enough time zone is going to be 0, right. So, therefore, we conclude that the average power required to drive the oscillator is 0 and this should be no surprise even from a physical principle because there are no energy dissipation mechanism that is present here, in this spring mass system there is absolutely no energy dissipation mechanism the energy dissipation mechanisms will be in the form of damper. Damper resembles are loss of energy.

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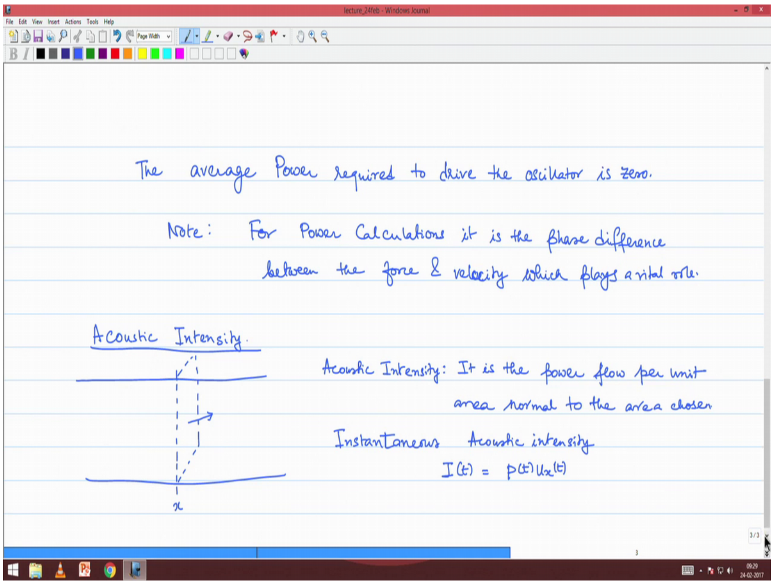

Accordingly if you addend a damper and redo this calculation you will be now able to show that the force and displacement firstly, will not always be in phase or out of phase there will always be an little bit of phase difference between the force and displacement which essentially means that the velocity and the force will never be allowed to have a pi by 2 phase difference. If the velocity and the force happens to have pi by 2 phase difference then you are going to see that the power required is going to be 0.

But once you put a damper you will get you will see that this condition will never be satisfied there is always going to be a phase difference which is different from plus or minus pi by 2 and that would essentially mean that on an average there is a certain power that will be require in order to drive the system and that make sense because the damper will sort of soak up the energy it is a dissipation mechanism. So, whatever energy the damper soaks up you need to supply that energy from the external source and thus make that oscillations happen.

So, the crucial point to realize is that the cos of this power being 0 is the fact that what is important for power calculations other than the magnitude of the force what plays a very crucial role is the phase difference between the 2 quantities which actually are inherent in the power expression. Power is a quadratic term it is a product of force and velocities. Not just the magnitude of force and not just the magnitude of velocity what is extremely important is the phase difference between the force and velocity which rules the power calculation. So, this is where I mean this will also happen in the acoustic case. So, that is why I took a few steps backward to illustrate this for you. So, for power calculations it is the phase difference between the force and velocity which plays a vital role.

So, now we will go to the acoustic case. So, till now we have been looking at acoustic pressure and acoustic velocity. So, let us again return back to our one dimensional case to start with. So, let us say this is a duct even if it is 3 dimensional space no big issues, but they start with a duct problem, so let us say at a particular section some x we know what is the pressure and at present we are dealing with plane waves. So, all across this dotted line the pressure is uniform right, that essentially means that the particles on the left of this plane are forcing the particles on the right of that plane with a value of force per unit area be the pressure. And what is the velocity at which this particles are crossing this dotted line the acoustic particle velocity right.

So, in other word similar to what is happening to a spring mass system I have just replace this mass with this dotted line right, what was happening in the oscillator situation is that I was driving the oscillator with the force the oscillator was moving with the certain velocity right. And what is happening here is that on this dotted line though it is not visible with our naked eye there are acoustic particles sitting, particles to the left immediate left of this dotted line are pushing this particles which is sitting on the dotted line with the force per unit area equals to pressure and these particles are now moving with a velocity v or u x if you may call it right. So, as a result the same situation is happening right. So, we could do a power flow calculation exactly in the same manner and that leads us to the concept of acoustic intensity.

So, acoustic intensity technically is defined in the following fashion. It is the power flow per unit area normal to the area chosen. So, here which is the area? The area is precisely the, this is the area the section right. So, what we are looking at is we are looking at the velocity of particles as they are crossing this area right and they are crossing this area because someone behind them is pressurizing them to cross this area is forcing them to cross this area. So, there are 2 situations that there are 2 quantities that are involved force and velocity just like we found in oscillator, force and velocity both are involved just that instead of force we will refer to as pressure because it is per unit area, force per unit area being pressure.

So, therefore, what is the intensity going to be the acoustic or firstly, just like we had instantaneous power expression. Firstly, we will define an instantaneous acoustic intensity which will be time dependent. So, the instantaneous acoustic intensity unlike the average value average value cannot be time dependent, the instantaneous value has to be time dependent it will be the power per unit area that is under investigation. For this present illustration I have chosen this unit area to be normal to the direction x right. So, therefore, the power is going to be force per unit area in to velocity in the x direction right. So, the pressure is given by p and the velocity in the x direction is u with the subscript x t right. So, this is the instantaneous acoustic intensity right.

Now, let us try to understand that I misspelled instantaneous, let us try to understand the average value of this intensity as it comes.

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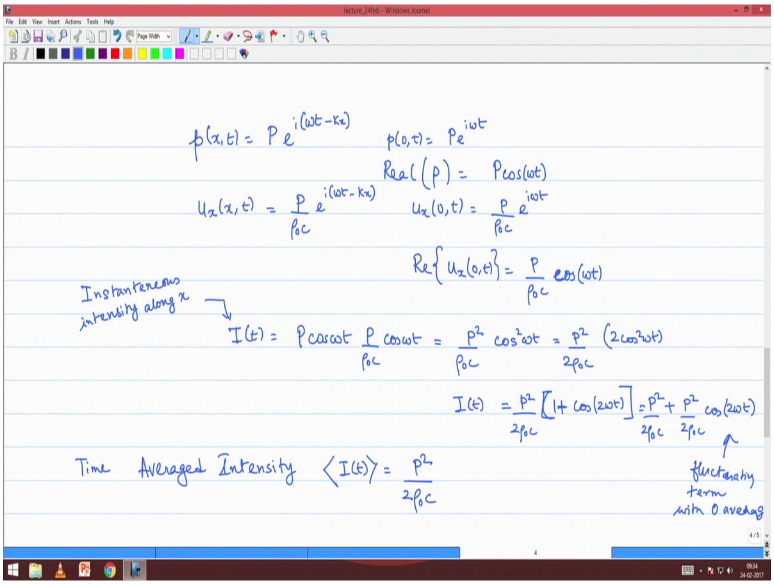

So firstly, let us take a travelling wave situation. So, for the travelling wave p t or p x comma t is going to be e to the power i omega t minus K x and then let us refer to capital P as its amplitude and we will call this section as x equals to 0. So, p t therefore, p at 0 comma t is going to be p e to the power i omega t. So, therefore, we are going to take the real part of it because that is what will be our quantity of interest right. So, the real part of acoustic pressure is going to be P cos omega t.

What about u x, x comma t? That is going to be p by rho naught c e to the power i omega t minus K x rho naught c being the characteristic impedance. So, u x at this same position is going to be p by rho naught c e to the power i omega t. So, the real part of this guy u x 0 comma t is going to be p by rho naught c cos omega t right. So, the instantaneous intensity therefore, is going to be p by p in to cos omega t; P in to cos omega t is the acoustic pressure that gets multiplied with p by rho naught c in to cos omega t which is the acoustic velocity along the x direction. So, this is the intensity, instantaneous intensity along x.

Please remember intensity is a vector quantity because intensity is pressure times velocity depending up on the direction of velocity the interpretation of intensity becomes vectorial pressure is scalar velocity is vector. So, depending up on the choice of your direction of velocity there is a direction attributed to the instantaneous intensity and also its time average counter point. But carrying on with this simplification what happens here is p square by rho naught c in to cos square omega t and that will lead to p square by 2 rho naught c in to 2 cos square omega t and that in turn is p square by 2 rho naught c 1 minus cos sin of 2 omega t or may be plus, 1 plus cos sin of 2 omega t. So, this is the instantaneous intensity, intensity which is varying with time as a function of time.

Here what we see is that the first term p square by 2 rho naught c. So, if I just open this out this will look like p square by 2 rho naught c plus p square by 2 rho naught c cos sin of 2 omega t. The first term is a constant value, but the second term will be fluctuating between plus and minus just like you had this fluctuation even for the oscillator right. So, this is going to be a fluctuating term with 0 average right.

So, therefore, if you take the average value, so the time averaged intensity which we will denote with a bracket of this kind that will be p square by 2 rho naught c this is a very important result. Now what we are saying that if there is a travelling wave and travelling plane wave to be more precise without choice of travel travelling plane wave we have actually seeing that there is a power flow happening at any arbitrary section and that is a time averaged power flow is there which is why we are getting this nonzero term which is p square by 2 rho naught c.

Let us look at another little more complicated example. So, what we have, what we had for the case of a plane travelling wave was this that the p by u ratio which was the impedance was perfectly real rho naught c right, but in general this is not true. You have seen examples where in like in the case of standing wave or even for the evanescent wave you will see that the p and u combination on the ratio of the p by u will not be real just like in the oscillated case we argued that if the ratio of p by v the force and displacement is real, you are going to get a 0 situation for the power calculations similar things will happen even for the case of acoustics. So, let me illustrate to you a more general situation, so easy in the previous problem.

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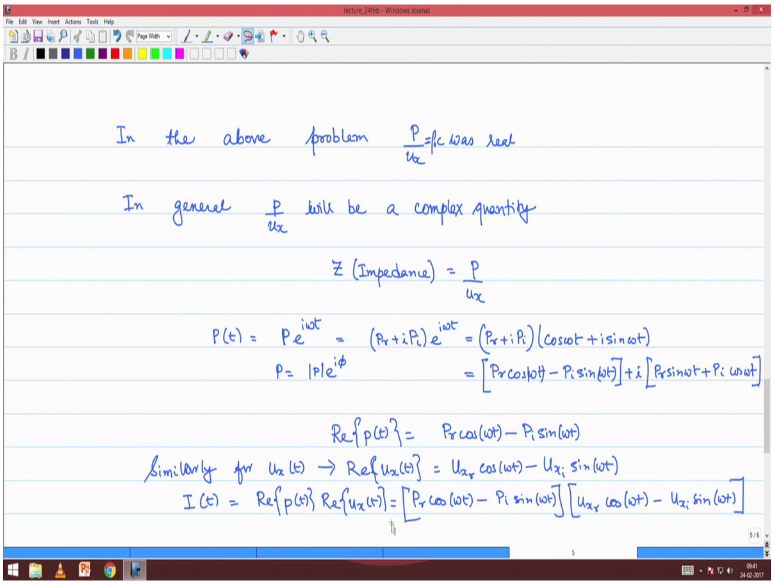

So, in the above problem p by u x was equals to rho naught c was real right because we were dealing with a plane travelling wave situation, but if there are multiple waves coming from different directions we know that you have a situation of standing wave and you cannot say that the p by u is going to remain real right it is in general a complex quantity.

So, in general p by u x will be a complex a quantity not necessarily real right. So, of this quantity will be called as impedance this is not the characteristic impedance though, characteristic impedance qualifies only for a single travelling wave in general if you wish to refer to this quantity which is the ratio of pressure by velocity you have to just call this as impedance. So, p by u x in general is impedance which in general again is a complex quantity. So, what happens to our intensity calculations let us look at it once more.

So, again we start with instantaneous intensity which is going to be a complex pressure amplitude multiplied with e to the power i omega t. Let us for once open out this complex quantity in to a real part and in to an imaginary part - e to the power i omega t right. So, I am just opening this p, I remember I told you that in general the amplitude associated with the phasor is complex just to keep track of your phasor. So, p here could is basically a certain magnitude into e to the power I phi where phi is the phase associated with the pressure right, but the intensity is not just p, but p multiplied with v. So, this formula as I have written it is just for p of t.

So, I will do a little more manipulation with this to get at this situation right and that in turn leads to Pr cos omega t minus Pi sin omega t that is the real part of it and the imaginary part of it is Pr sin omega t plus Pi cos omega t right. So, this is what we have in terms of phasor, but then we had taken the convention that the actual signal that will be obtained will be the real part of it is real positive x axis at time t equals to 0. So, the real part of this pressure is going to be Pr cos omega t minus Pi sin omega t right.

Similarly, you could do this calculation for u x also nothing changes basically. Similarly for u x of t you can do just the same calculations which p by u x. So, that will in turn lead to u x the real part of it in to cos omega t minus u in to sin omega t right and the instantaneous intensity will be the real pressure times the real velocity right. So, the instantaneous intensity will be real part of pressure times real part of velocity which in turn means Pr cos omega t minus Pi sin omega t in to u x r cos omega t minus u x i sin omega t right.

So, let us simplify this expression a little bit in the next page. So, instantaneous intensity is going to be this expression which I will copy and paste.

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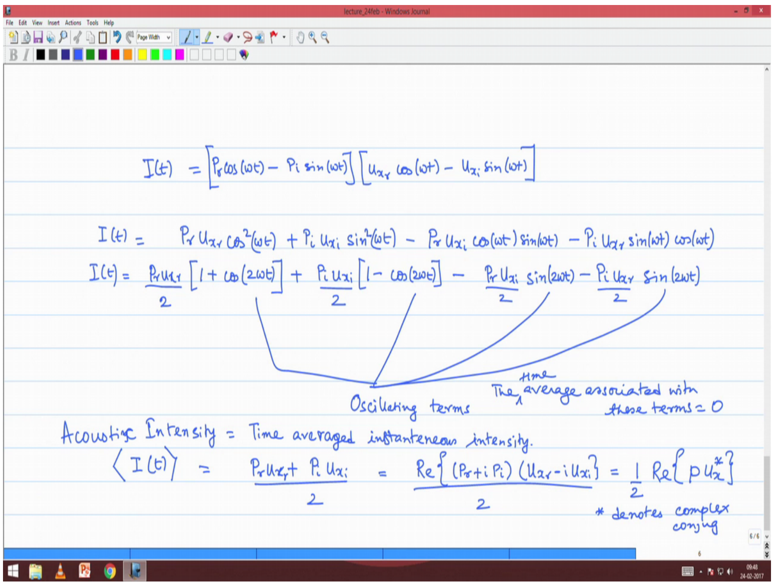

So, this is the expression for the instantaneous intensity, let us simplify this a bit further. So, what we have is Pr u x r cos square omega t plus Pi u x i sin square omega t minus Pr u x i cos omega t sin omega t minus Pi u x r sin omega t cos omega t right. So, in the next step will appeal to our good old trigonometry formulas and replace this as 1 plus cos of 2 omega t plus Pi u x i 1 minus cos sin of 2 omega t divided by 2 minus Pr u x I by 2 sin of 2 omega t minus Pi u x r by 2 sin of 2 omega t. This is the instantaneous intensity.

Again you get to see that the instantaneous intensity has got a frequency dependence which is 2 omega. Instead of omega dependence you are getting a 2 omega dependence this is actually obvious because intensity is a product of pressure and velocity pressure is both of them have got an omega dependence which is why the product has got a 2 omega dependence. But other than the fact there is a cos sin of 2 omega t and the sin of 2 omega t dependence there is also a constant term which you see here, these 2 terms are associated with this one value or constant whereas, this terms all this terms are fluctuating terms right all these terms are oscillating terms. The average associated with these will be 0 or the time average associated with these terms is plain and simple 0, right.

So, what is the average of these instantaneous intensity? It is going to be Pr u r plus Pr u x r i am sorry plus Pi u x i divided by 2 only the terms associated with these 2 ones will be responsible for the average quantity all other terms involving the trigonometric quantities will not lead to any average value right. So, this is what we will referred to acoustic intensity in general, when we say acoustic intensity we generally mean as time averaged as the time averaged instantaneous intensity. Because that is what is physically more meaningful because that gives us an idea that over so efficiently long duration of time what is the energy or what is the power that is getting transferred from the left side of that point under the consideration to the right side right.

So, whatever analysis we did for 1 d is actually true for even a higher dimensional space its actually true for even situations which do not have plane wave because even if you do not actually the plane wave assumption was only used in this formula where we said p by u x is equals to rho naught c. When we made it general, so in general as I said in general there will be a complex impedance. So, even if it is a spherical wave even if it is a cut on wave any kind of wave even if it is an evanescent wave any wave with any wave you can find p and u in a particular direction, you can find the ratio of these 2 p, these 2 quantities it may be real it may be imaginary, but definitely it has to be complex right.

So, we have taken a perfectly general approach here where we have taken that the ratio of these 2 quantities is complex the quantities themselves are complex has been taken though we could have taken one of them to be real without loss of generality. But just to be perfectly general here what we have said is the 2 quantities have been taken as arbitrary complex numbers and then we have been able to find out the time average in instantaneous intensity which is abbreviated as a single in the single term as acoustic intensity and that is given by the following number.

These could have been written much in a much simpler way if we said it is real part of Pr plus i Pi in to u x r plus i u x i by 2 right, if we take the real part of the complex pressure amplitude and the complex velocity amplitude we will end up with getting the same answer right sorry, with the this will come with a minus sign this will come with a minus sign. So, if you take the real part of the complex pressure amplitude and the complex conjugate of the velocity amplitude right then you will get exactly this same answer you can verify that Pr u x r is real Pi u x i it comes with a minus i square minus i square is plus 1. So, the real part of it is Pr u x r plus Pi u x i right.

So, this is in fact, leading to this answer. So, therefore, in short form what is often quoted in text book as the formula for intensity is real part of the complex pressure amplitude multiplied by the complex conjugate of the complex velocity amplitude. So, star denotes complex conjugate.

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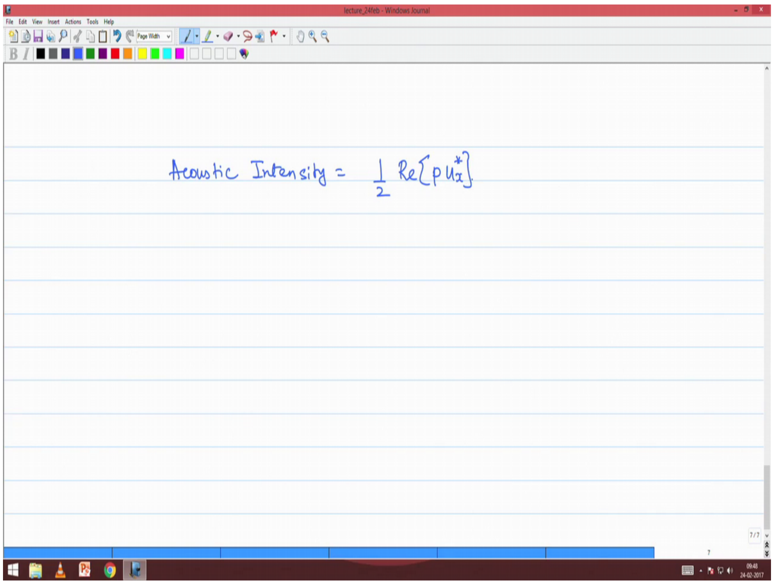

So, the working formula is actually pretty simple for intensities, so the acoustic intensity working formula is simply half of real part of pressure times u x star right, it is u x if you are interested to calculate the power flowing through an area which is normal to the x direction.

If you are interested to know the power flowing through an area which is normal to the y direction you have to replace u x with u y, if you are interested to know the power flowing through an arbitrary area which is arbitrally aligned then you have to find the normal to that area and find the velocity in the direction normal to that area right.

So, appropriately you need to replace them, but this is very important that we are getting half of real part of pressure times velocity conjugate as the formula for intensity which is what you will see is often quoted in different textbooks, but usually most of the textbooks that I have come across does not give this background for the derivation of this formula which is what we did here. So, we will take it up from here in the next class.

Thank you.