**Acoustics & Noise Control**
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**Module- 12**
**Lecture - 17**
**Cuton Waves in duct**

Till now we have been looking at plane waves in our discussion on Acoustic Wave Propagation. Today we will look at one variant of plane wave. It is technically called the cut on waves. These are not really plane waves, but then we will understand these waves as super position of plane waves. So, let us take the problem head on.

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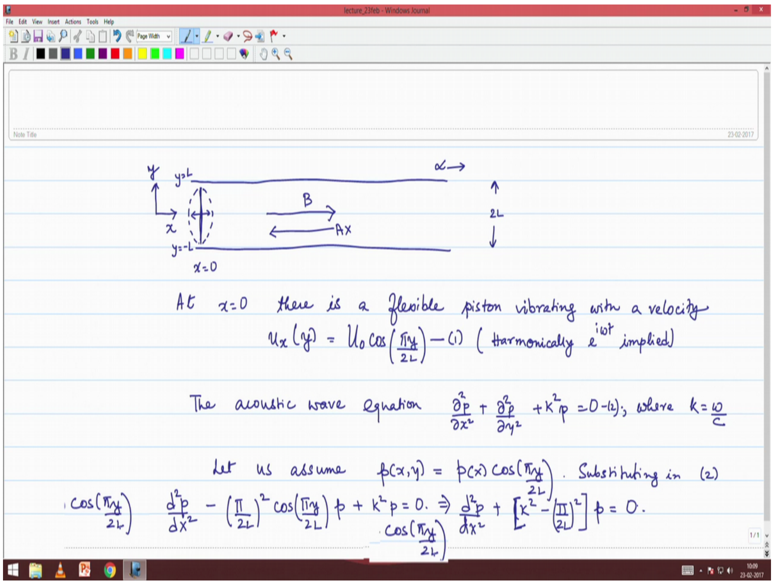

So, the problem that we will be interested looks pretty similar that we have been analyzing earlier also. So, again to start with, there is a long infinite duct and we have already seen that sort of a problem and the similarity also lies in the fact that it will be excited by a piston, but the piston till now we were taking as a rigid piston which would oscillate to and fro in the following fashion. So, these are the extreme configurations within which we were assuming that the piston would be oscillating. Now, we will take things in a different fashion. So, we will no longer say that the piston is completely rigid. We will rather think that the piston itself is a flexible structure and therefore, it vibrates not like a rigid to and fro motion, but it will vibrate like a sinusoidal fashion. So, this is how it vibrate. So, each and every point of this piston is going to move up and down between these two dotted lines as is indicated. So, the piston will be assume to lie at x equals to 0. So, this is our coordinate axis. So, the piston is at x equals to 0 and as usual this is an infinite wave infinite duct to start with. So, here we will refer to this problem which is called the cut on problem in acoustic ducts. Acoustic cut on modes is what specifically we wish to study.

So, from here on we will need to appreciate the fact that in contrast or previous derivation, we were saying that nothing changes in the y direction. Here we are expecting a particular change because right at the location x equals to 0, we do not, we are not expecting that all particles to be moving at the same velocity because the velocity of the piston after all is in a sinusoidal fashion or may be the displacement. You will refer to is as shown in this figure is definitely in a sinusoidal form. So, therefore, you do not expect all the particles which are sort of kissing the surface of this piston to have identical motion.

So, therefore, we will not expect a one-dimensional wave, definitely not a one-dimensional plane wave to be induced in this situation. So, to start with, we will have that at x equals to 0, there is a flexible piston vibrating with a velocity u x. It is in the direction of x as a function of y because this piston does not have uniform velocity across all its points. So, it is a function of y and we will take this point to be y equals to plus l and the bottom is to be y equals to minus l which means that the width of this duct is 2L. So, the from of this function is such that it has a maxima at x equals to 0, y equals to 0 and it is falling to a zero level at the edges.

So, we can assume this form to be cos of pi y y 2L, right and there is certain amplitudes. So, we will call that may be capital U, capital U0. So, the piston is vibrating in this form and obviously, it is vibrating harmonically. So, that goes without saying e to the power i omega t is implied. So, the question is what are the acoustic response at various points that as set up due to this condition, right. So, towards that end we firstly realize that the velocity of the piston which sort of excise the acoustic response is having a specific form along the y direction which is given in this equation.

So, to recall the acoustic wave equation that we need to solve is given by del 2 p del x 2 plus del 2 p del y 2 plus k square p equals to 0, where k square or where k is omega by c.

Omega is the angular frequency of the oscillations and c is the sound speed, right. So, here we have to solve this equation with the understanding that the velocity at x equals to 0 should exactly confirm to the velocity of the structure, the equation for which is given in equation 1, right. So, let us assume a specific form for this p.

This time we are already having a lead to the form of the y dependence in this acoustic pressure function p. The form of y dependence should possibly be at this stage. I cannot say should be, but I will say qualify that as should possibly be of the same form as the velocity because y dependence of the velocity is showing a form which is like cos sin pi y by 2L. So, let us bet on this function p of x, y is some function on x times cos of pi y by 2L. At this stage, it is just an assumption, but we will prove that this assumption is in fact holding good, right.

So, we will see that in as we go along, so if we make this substitution in to the governing equation, substituting this in equation 2 where this is my equation 2, what is it we will get? We will get the following cos of pi y by 2L del 2 p del x 2 can now be written as d 2 p d x 2 because that is the only variable of x. Being the only variable that is left independent, the y dependence has been assumed to be cosine.

So, if you now take second derivative of this function with respect to y, you are going to get a minus pi y by 2L whole square cos pi y by 2L in to p plus k square p cos of pi y by 2L, right pi by 2L. There is no y here this pi by 2L, right. So, now this in other words can be written as d 2 p d x 2 plus k square minus pi by 2L whole square p equals to 0, right.

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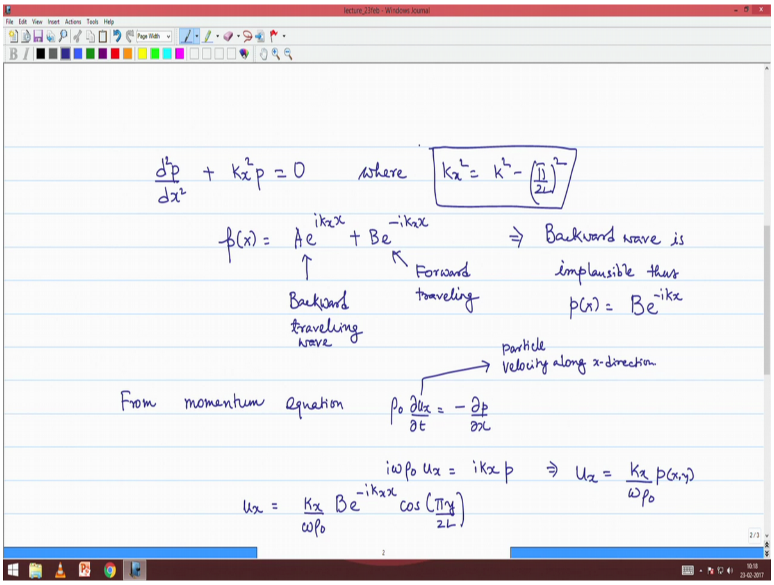


So, in the next step what I will do is, I will just rewrite this in the following form d 2 p d x 2 plus k x square p is equals to 0, where k x square is k square minus pi by 2L whole square. That is exactly what we had right, but the solution for this is already known. The solution for this is p as a function of x will be a e to the power plus i k x in to x plus b e to the power minus i k x into x, right.

This is the same old situation as the one d acoustic plane wave equation. Just that we have k x appearing instead of k and this being A1, this equation being identical to the one acoustic plane wave equation except for the fact that k x has taken the place of k. This solution can simply be coated as the sum of two complex exponentials. Each of them has a physical characteristic associated with it. This represents an inward or a backward travelling wave and these represents a forward travelling wave because the time dependence is e to the power i omega t.

So, we have thus argued that a wave is an incoming wave which is coming into the point x equals to 0, but we realize that in this situation, this a wave is not plausible because it is an incoming wave. There is no reflection that can come in because we are at present dealing with the situation that the duct is infinite.

So, the only wave that is expected because of an excitation at x equals to 0 is a forward way within the duct. So, accordingly we will choose only the forward wave to be the solution. So, backward wave is implausible, thus p x will be chosen as b e power minus i k x in to x, right and now for the value of b, for that we will refer to the moment, the Euler equation or the momentum equation, so that you will recall is given by the following rho naught del u x del t is equals to minus del p del x right, where d u x is the. So, this u x refers to the velocity, particle velocity along x direction, right.

So, i omega rho 0 u x has got to be with the minus sign. This will become plus i k x in to p which in again gives us to the equation that we had already obtained earlier that u x or the particle velocity would be k x divided by omega rho 0 into p, right. So, therefore, the particle velocity is k x by omega rho 0 into b e to the power i minus k x x, but then there p is a function of both x, y right and x, y is p x into cos of pi y by 2L. So, we can bring that in.

So, that is cos of pi y by 2L which will multiply this factor, but then we want by kinematic continuity condition, we want that u x and x equals to 0 should exactly match the velocity conditions of the piston which is now assumed to be flexible.

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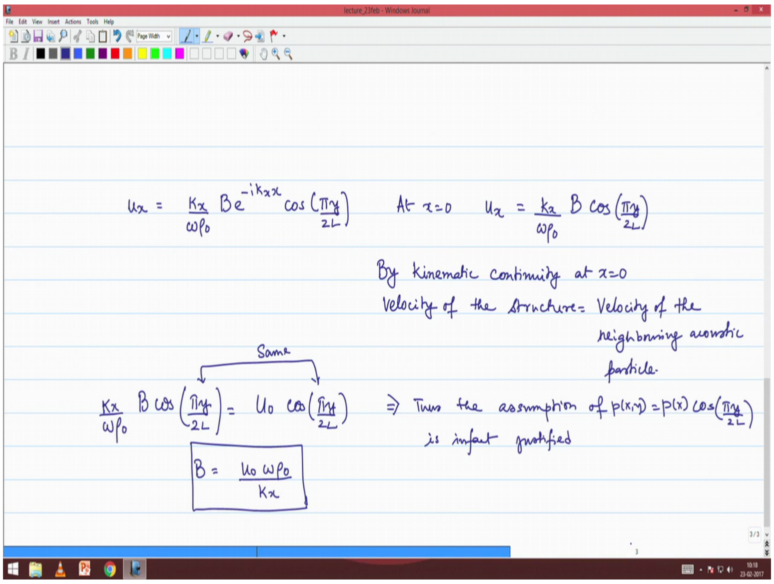


So, we had this equation. So, at x equals to 0, we will have Ux to be given by k x by omega rho 0 b cos of pi y by 2L, but by kinematic continuity at x equals to 0 velocity of the structure must be equals to velocity of the neighboring acoustic fluid particles. So, at x equals to 0. What we must have is k x by omega rho 0 b into cosine of pi y by 2L must be equal to u 0 cosine of pi y by 2L which was exactly the form of the piston velocity that we have assumed we had given a certain piston which is moving in a particular way cosine pi y by 2L and the amplitude being U0.

So, now we are substituting that and with that we should, we make two conclusions. One is that our choice of at the remember at the opening we said that let us assume that y dependence associated with the acoustic pressure function bared the same cos sinusoidal dependence and that assumption is now proved to be correct because with that assumption, we are led to a velocity form which exactly matches to the velocity form of the driving structure, right.

Therefore, the fact that these two velocity forms are same: thus, the assumption of p x y being a product of A p function of an s function, sorry together with the same y function as was embedded in the velocity of the piston, we are now seeing that assumption is in fact justified because it is only with that assumption we are getting 2; the final conclusion that the acoustic fluid particle is having the same velocity as the neighboring structural particles.

So, had you taken any other functional dependence in y, you would not have got to this conclusion, right. So, therefore, this is correct and this also means that B is U 0 omega rho 0 divided by k x. So, we have been able to find the amplitude. Also the amplitude of the wave is given by this formula, where k x is given by this formula, right. Now, other than facts that the mathematical derivation is over, let us try to once more understand the physical implications of this solution.

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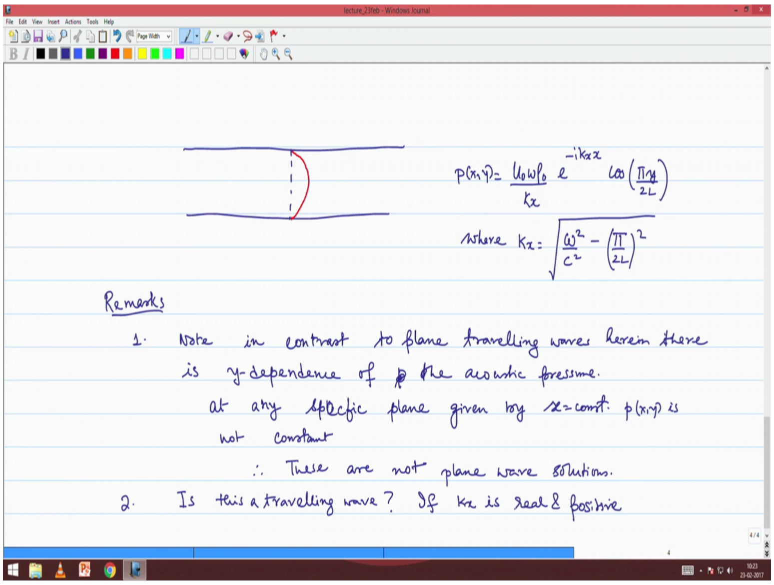


So, we are having a duct wherein we are getting the p x, y to be some amplitude which is U0 omega rho 0 by k x in to e to the power minus i k x x into cosine of pi y by 2L, where k x is actually omega square by c square minus pi by 2L whole square square root of it, right. So, the first question is, do we always get a travelling wave solution? What do we mean by wave? This time I must, may clarify that once again by the very form of it. So, I will put a few remarks.

The first remark is that note in contrast to plane waves, plane travelling waves herein there is y dependence of acoustic pressure. By definition a plane wave is a wave wherein a perpendicular to the direction of travel we expected all the particles to have identical acoustic pressure. Identical acoustic velocities, everything had got to remain constant over planes which are perpendicular to the direction of travel, but now noting from the formula that we have derived mathematically we are saying this is not true. We cannot identify the plane perpendicular to the x direction to be the plane wherein nothing changes no things are actually changing at any y.

So, at any specific plane if you see the pressures are not constant, so in other words at any specific plane given by x equals to constant, any constant you choose at any specific plane pressure p x y is not constant. So, therefore, these are not plane wave solutions. So, this is our first encounter with waves which do not have a plane wave type of characteristics, but the next question is, are they travelling wave. So, here specifically the form that happens on along any x equals to constant line, the acoustic pressure will have the exactly same dependence as the velocity profile. So, if you take along this section, if you happen to compute what is the acoustic pressure, it will have the same sort of dependence as you had for the velocity of the piston, right.

So, it no longer is a plane wave, but none the same it has a particular profile, right the profile of which is as indicated in this red line. So, the next question which we turn is that is this a travelling wave? May not be a plane travelling wave, but is this a travelling wave? You still have an e to the power minus i k x sort of dependence. So, e to the power i omega t into the power i minus k x x is still there.

So, the point is if k x is real and positive, so then what we are getting is p x, y is U0, U0 omega rho 0 by k 0 k x into cosine of pi y by 2L into e to the power minus i k x x, right.

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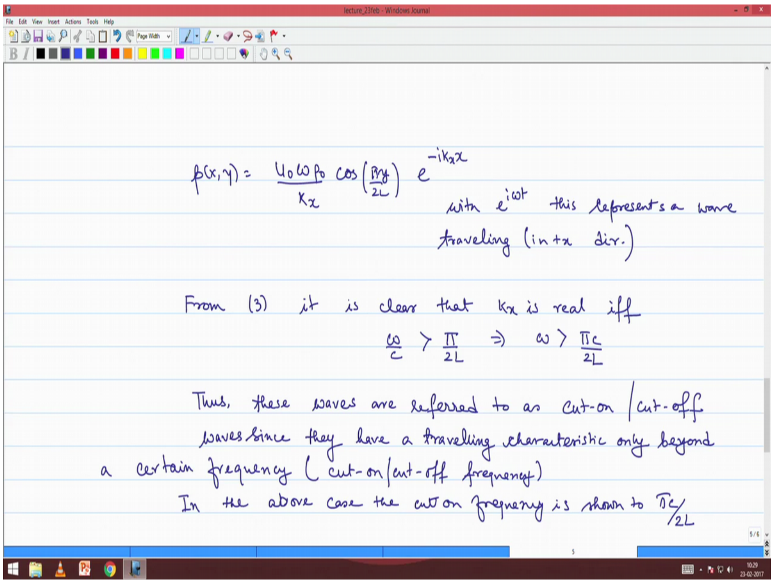


So, if k x is real and positive, then obviously this is a travelling wave because with e to the power i omega t, this represents a travelling in plus x direction represents a wave travelling in plus x direction, right. So, that part is easy to conclude, but then there is a big if is k x real and positive look at the expression for k x. K x will be real only if omega by c is greater than pi by 2L. So, this wave will be a travelling wave provided omega by c is greater than pi by 2L. So, from this equation which I guess I should call it 3. So, from 3, it is clear that k x is real if and only if omega by c is greater than pi by 2L or omega is greater than pi by c pi c by 2L, right.

So, in other words, yes this is a travelling wave, but only for frequencies higher than a particular value, right. So, this is why it is called a cut on wave because beyond the frequency it is a travelling wave. So, it is like a switch which is switching on at a frequency beyond this value pi c by 2L and only beyond this value, the solution is that of a travelling wave, right. Thus, these waves are referred to as cut on. Sometimes it is also interpreted the other way round cut off waves. So, what is the interpretation of cut off? So, definitely they exist at high frequencies, but when you are lowering your frequencies below a critical limit, the waves will no longer have a travelling nature.

So, therefore, in that sense it is a cut off wave also. So, cut on and cut off is interchangeably used if you take this idea that you are sweeping the frequency range, then the point is this sort of a wave will exist only when the frequency is high enough the critical frequency being given by pi c by 2L, right. So, that is one aspect of it. So, this is y and it is called cut on and cut off wave.

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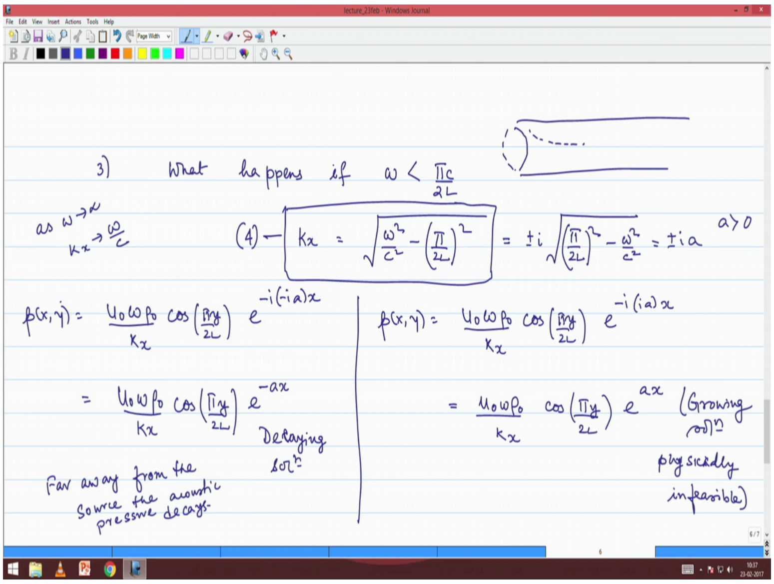


So, the question is what happens? The third remark that I would like to place is what happens if omega is less than the cut on frequency by the way pi c by 2L is called the cut on frequency, I should make that comment also. So, the frequency I will just elaborate this point a little more. These waves are referred to as cut on waves since they have travelling characteristics only beyond a certain frequency. This frequency is referred to as the cut on or the cut off frequency.

So, in the above case, the cut off frequency is shown to be pi c by 2L, right. So, this wave mode will be a travelling wave mode only beyond that particular frequency. So, the next question is what happens if the frequency is lesser than that? So, if the frequency is lesser than that, k x a square root omega square by c square minus pi by 2L whole square and in this case, you could write this as plus or minus i square root pi by 2L whole square minus omega square c square because pi by 2L is more than omega by c. We have just shuffled this around and what is the form of p x y that we are looking at. We have already derived p x y to be of this form. So, let me cut and paste this formula.

So, this is the formula that we have already derived. So, let us take this as plus or minus i into a. So, basically a is this under root quantity and it is positive because it is just what you get by using your calculator. So, now this k x here, I would like to substitute it with let us say if I substitute this with i times a, what happens is I get a minus i square and minus i square is plus 1, right. So, therefore I will get a growing solution. In this case, e to the power minus i square which is plus 1 a x and by our assumption that a has been chosen to be greater than 0. This implies a growing solution which again is physically infeasible.

What about the other choice of sign? The other choice of sign if we now choose k x to be minus i times a into x, I have just replaced k x as plus i times say in on the right hand side and I am now replacing k x as minus i times a in this part of the derivation. So, that will possibly through up more interesting solution. This part remains the same. So, this time you have e to the power minus a x as the solution. This is decaying solution. So, it decays at as it travel. So, here is your duct. So, some excitation is given at x equals to 0. Here what you see is that the response along x is decaying.

So, after a certain distance away you are not going to get much of the acoustic pressure. So, whatever is happening is happening in the region near to the source. So, again the situation of a near field or evanescent wave comes in wherein you have the acoustic pressures to be active only near to the source something which we had seen even in the previous problem, where we took a bending wave to be travelling and we saw that below the coincidence frequency that there is only a near field region, where the acoustic particles are active and you are getting an acoustic pressure, but the acoustic pressure rapidly decays in the direction transverse to the bending source.

Here the interpretation of the source is that of the vibrating piston which is vibrating in a sinusoidal fashion and here you have seen the same physical implications coming out that it is only below a particular frequency which is technically turned as a cut on frequency that you are getting an evanescent wave, wherein the acoustic pressure waves are rapidly decaying within a region very close to the source, but far away from the source there is no response.

So, here what we see is that far away from the source and by far away I mean I choose x, such that a times x is large, right. So, if I choose again a times x is like 10, then I would get a decay which is e to the power minus 10 which is pretty large number, right. So, that may be a pretty large value, large decay compared to what is happening at the source.

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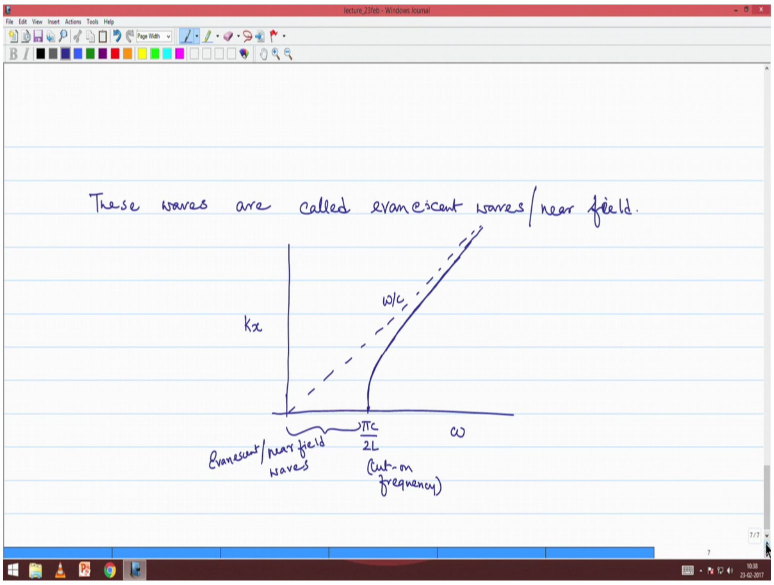


So, far away from the source, the acoustic pressure decays, right. So, these waves are called evanescent waves or near field waves, right. So, we could possibly draw a similar dispersion curve also, but this time it is with respect to k x and omega. So, k x and omega plot would emerge. So, the critical value is this value which is pi c by 2L. The cut on frequency is as we have derived is pi c by 2L for frequencies which are more than pi c by 2L. We expected travelling wave characteristics and accordingly k x is real positive and that is why it leads to a travelling along plus x direction type of a solution. So, if this equation is now plotted, I call this equation as the dispersion equation. So, these equation which is let us say equation 4 as of today. So, if this equation 4 is plotted, it would look something like this.

Firstly, the plot is only valid for values of omega which is greater than pi c by 2L. It is not valid for the other part of it because then it will be imaginary. We are not drawing the imaginary k x numbers. So, it will look something like this. Remember if omega is very large, then this quantity or pi by 2L will have minimal effect. So, it will tend to, as omega tends to infinity, k x will again tend to omega by c because pi by 2L that time will look like a very small number in comparison to omega by c and we will not be able to nullify much of the effect. The negative nullification, the partial nullification I should say is minimal.

So, therefore, this plot would finally tend to approach the omega by c curve at large values of omega. So, this is like a curve which starts out at pi c by 2L, but tries to latch on, but will never be actually hitting that value, but it will sort of asymptotically touch the curve omega by c. This is the dispersion curve corresponding to the cut on mode and this frequency is called the cut on frequency and this region, this frequency region you will get near field waves or evanescent waves, right. Let us have another interpretation now associated with this solution. So, this will be my remark 4. So, again I will copy and paste may be this equation. Firstly, I need to select, ok.

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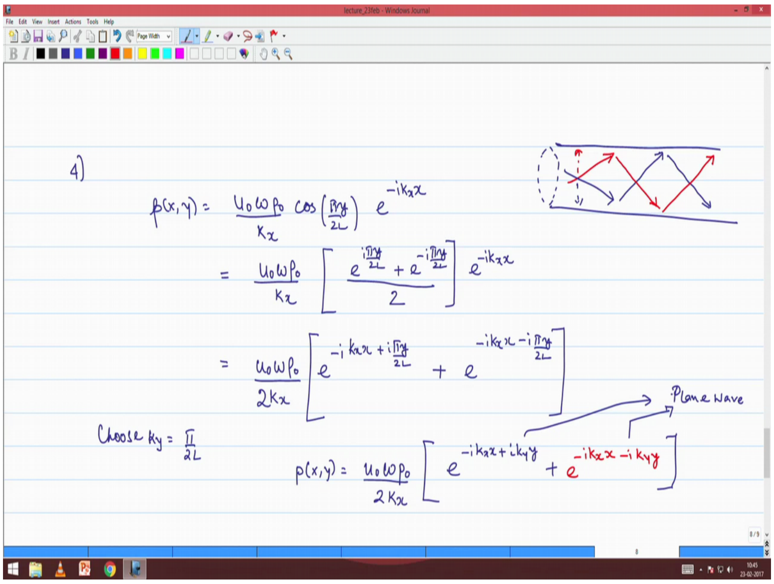

So, we will have another interpretation of this solution which is possibly, which should possibly tie things up with respect to our plane wave analysis. So, we did make a remark that this sort of a solution is not a plane wave because it does not satisfy the conditions of, it does not satisfy the conditions that you know sections which are normal to the direction of travel should remain at the same invariant conditions. All particles should be having identical response. It does not satisfy that. So, this does not qualify to be called as a plane wave, but it is possible to decompose this solution as a super position of two plane waves. Let me show you how.

So, I could always decompose this cos quantity as the sum of two exponential quantities, right. I could write this as e to the power i pi y by 2L plus e to the power minus i pi y by 2L divided by 2. That is exactly cause of pi y by 2L and e to the power minus i k x x is as usual. So, this in the next step, I will just rewrite this solution in the following form. So, we will have e to the power minus i k x x plus i pi y by l is 2 here which I should write plus e to the power minus i k x x minus i pi y by 2L, right. So, if I choose let us say k y equals to pi y, sorry pi by 2L, so if I choose k y to be pi by 2L, then this solution is written as p x y equals to u 0 omega by rho 0 divided by 2 k x e to the power minus i k x x plus i k y y plus e to the power minus i k x x minus i k y y.

In this form, we realize that though in its entire, in its totality p x y is not a plane wave, but what we have essentially done now is that we have decomposed the solution, the response in terms of two plane waves. We already know that both of these are plane wave solutions, but they have an important feature. One, both of them are travelling in the positive x direction, but one of them is travelling in the positive y direction. The other is travelling in the negative y direction, right. So, accordingly if we have to draw a diagram, then this wave I may change the color may of this. So, I will do this in red, correct: E to the power minus i k x x min minus i k y y given in red.

So, what is the wave? What is the direction of travel? For this wave with k x and k y both positive, it is going to be in the first quadrant. It will travel in the first quadrant. What about the other wave here? This wave will travel in the fourth quadrant, right. So, what you have here is that the total acoustic pressure is now seen to be the super position of these two waves: one traveling in the first quadrant; the other traveling in the fourth quadrant together.

As you can understand the component of these two waves are traveling in the opposite directions with regard to the y direction, right so in the y direction you have a component of the red wave to be going in the upward direction whereas, the component of the blue wave is travelling in the negative direction. So, in the y direction you have two waves which are travelling in the opposite direction and thus, they create a standing wave.

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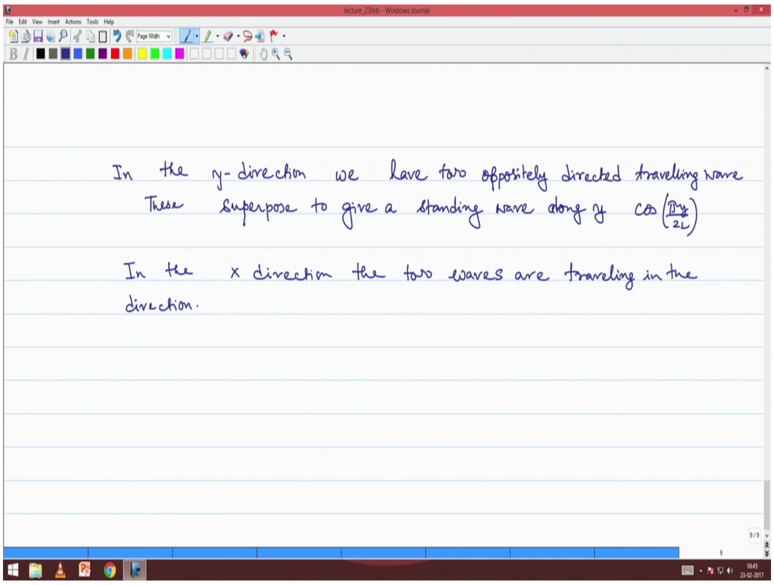


In the y direction, we have already seen standing wave in the context of acoustic ducts. So, here again what we see is that in the y direction, we have two oppositely directed travelling waves. These super pose to give a standing y wave along y which is precisely this form cos pi y by 2L. So, cos pi y by 2L is like a mode shape, but that is only along y. This observation does not hold for the x direction. In the x direction, the two waves are travelling in the identical direction. So, that is why it is travelling in x, but it is standing in y.

So, there is a standing wave profile which is generated in the y direction which travels along x. That is the interpretation of the cut on waves that I would like to bring up out. So, basically what happens is that once you have an excitation which is anything, but like of a rigid piston, not only this cosine you can try it with anything else also. What will happen is that plane waves actually will emerge, but they will keep getting bouncing on and off. So, what will happen is that this will get reflected.

Now, you already know that the angle of incidence, angle of reflection formula if you apply, what happens is that this will have a multiple reflection, there are plane waves, but these multiple reflections will keep happening and these multiple reflections note from the schematic figure itself is clear that these multiple reflections are not going to have any travelling characteristics in the y direction, but in totality the wave or the acoustic pressure profile will have a travelling wave characteristics in the y direction, in the x direction, but in the y direction, it will remain stagnant or it will remain standing wave because it cannot escape in the y direction. It is always going to get reflected back within the domain and as a result in the standing wave, there is a standing wave which is generated in the y direction, but in the x direction, it is going to be a travelling wave.

So, these are very important class of waves which we have now encountered. These are called cut on waves. So, we will meet again in the next lecture and then, we will take it up from here.

Thank you.