Acoustics & Noise Control
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Module – 09
Lecture – 14
Plane Waves: Reflection and Transmission

Good morning friends. In the last class we have been saying the different studies related to one dimensional plane wave, so, recall.

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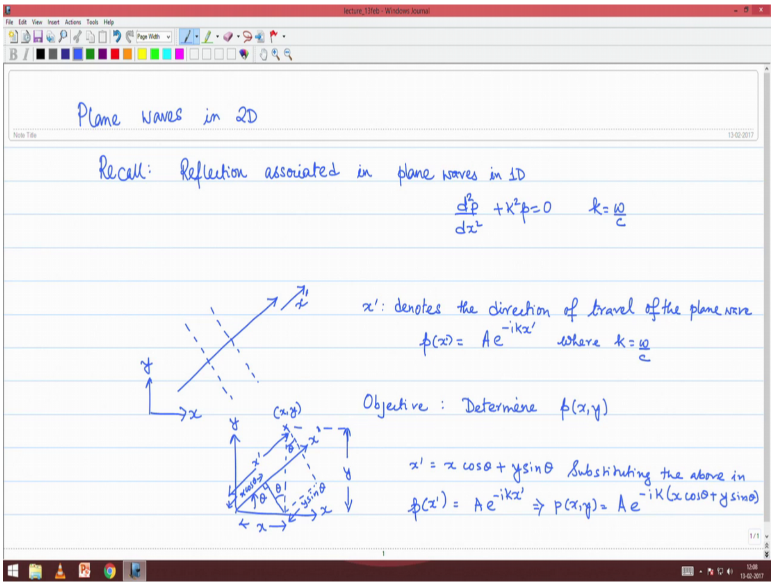

So, we had looked at the reflection phenomena associated with plane waves in 1 D. So, when I mean 1 D the governing equation for this plane wave was d 2 p, d x 2 plus k square p equals to 0 and k was equals to omega by c. So, this is the equation that we solved because we had essentially one independent special variable which was x the time variable again I iterate that by taking the harmonic assumption the time variable is k.

Today what we will look at it is we will look at plane waves in 2 D. So, it is not true that plane waves occur only in a one dimensional setting, you could have plane waves in high dimensional sitting also. So, that is what we will look at it. So, towards that and let us understand what we are trying to mean. So, plane waves as we understand is by definition those sorts of waves where in the direction of travel or planes perpendicular to the direction of travel will have identical pressure and velocity conditions right. So, if we take the direction of travel in this fashion then all the planes which are perpendicular to this direction of travel will have identical condition in terms of pressure and velocity.

So, in the previous approach we would have align our axis along the direction of travel of this wave right, but now we wish to adopt a global x y axis let us say and do an analysis where the direction of travel of the plane wave is not in alignment with our global coordinates system. So, this is the problem and that will lead us to the 2 dimensional plane waves and there on it is another simple generalization to even 3 dimensional plane waves. So, what we need to do is we will now transform the coordinates from the x prime coordinate system. So, x prime denotes the direction of travel of the plane wave and a plane wave itself will have an expression of the form A e to the power minus i k x prime right where k is omega by c omega is the angular frequency and c is the speed of sound.

We already know that in the x prime coordinate system it is nice and simple it just a one dimensional plane wave which is propagating the expression for which is well known which has already been derived e to the power minus i, k x prime within arbitrary constant which denotes the complex amplitude of this wave. So, what we wish to do is now to arrive at the expression for I should call this p in terms of x prime not in terms of x as here. So, we are having the following objective we wish to determine p x comma y, we are given p in terms of x prime we have to determine in terms of x and y which is the global coordinates system not in terms of x prime.

So, that should be simple enough if you recall that this is just a question of transformation of the coordinate axes. So, if we have an x prime axes which is aligned let us say at an angle theta with the x axis then if you look back at how we arrive at the transformation of any point. So, let us say this is a point of interest this point will have a x value which is marked off here and the y value would be this right. So, the coordinates of this point being x and y the x and y distances are as labeled in this diagram. So, if you drop at perpendicular in this fashion, the x prime distance is supposed to be this this is x prime distance.

So, x prime is x cos theta what is x cos theta this is x cos theta right and if this is theta this is 90 this is this also becomes theta which means this is theta. So, if this is y then this distance is y sin theta right. So, therefore, x prime is x cos theta plus y sin theta this is just a simple application of geometry and trigonometry the transformation of variables. So, we can simply substitute x prime is equals to x cos theta plus y sin theta in the expression that we have derived for p x prime.

So, we can now say that we will substitute the above relation in p x prime is equals to A e to the power minus i k x prime, and that would give us p in terms of x and y because we are changing over from x prime variables to x y variables using this relation that we have just now derived, and that leads us to A e to the power minus i instead of minus i times k instead of x prime we now wish to write x cos theta plus y sin theta.

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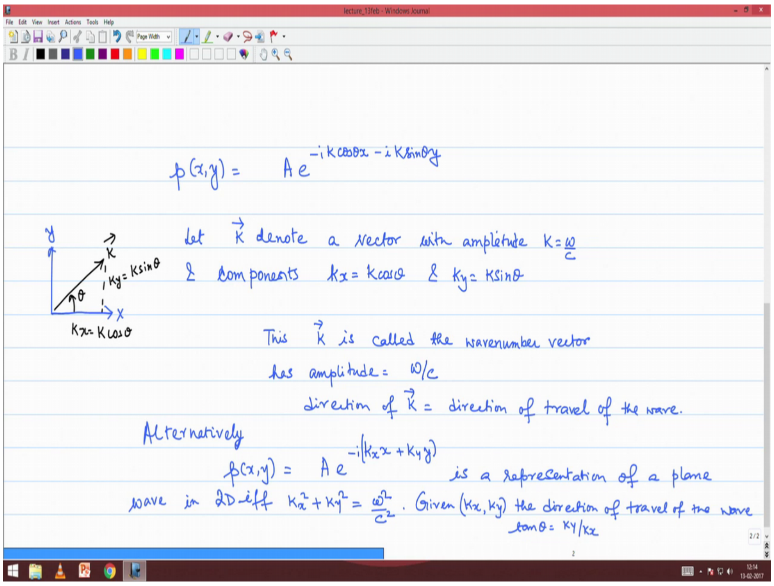

So, in other words p x comma y would be written as A e to the power minus i k cos theta in to x minus i k sin theta into y.

So, what we note is that let k vector this time denote a vector with amplitude k which is equals to omega by c and components k x which is equals to k cos theta, and k y which is equals to k sin theta. So, what we are asking is that this is our x and y coordinate system, we are defining a new vector call it the k vector which is having components k x equal to k cos theta and what is theta? Theta as us been marked of in the diagram is the angle between the x direction and the direction of travel of the wave itself. So, k x is k cos theta and k y is k sin theta.

So, in other words what we have done is that we have chosen a vector the amplitude of which is equal to the definition of wave number, but the direction of which is aligned to the direction of the travel of the wave right, the direction of the travel of the way was in the x prime direction which is at an orientation of theta counter clockwise with the x axis. So, we have defined a wave number vector.

So, this k vector this k vector is called the wave number vector generally when we when we say wave number we it is supposed to be a scalar which is having the value frequency angular frequency divided by sound speed, but if we explicitly say that this is a vector that means, it is magnitude is remaining the same that of the wave number, but the direction it is associated with it is same as the direction of the travel of the wave. So, wave number vector has amplitude equal to direction of travel for the wave and direction of this k vector is same as the direction of travel for the wave sorry the amplitude is omega by c the amplitude is omega by c, and the direction of k is same as the direction of travel of the wave.

So, now what is the benefit we get we say that p x comma y is A e to the power minus i k x x plus k y. So, now, we are getting to see 2 components of the wave number. So, any 2 k x and k y will be a 2 dimensional wave if they do satisfy the condition k x square plus k y square is equals to omega square by c square because this k x and k y are virtually the components of the wave number vector along the direction of x and y, but the amplitude of this wave number vector is supposedly constraint at the value of omega by c. So, this k x k y cannot be just arbitrary chosen, k x and k y has got to be such that k x square plus k y square is k square which is omega square by c square.

So, this is the alternative viewpoint. So, alternatively p equals to this is a representation of a plane wave in 2 D, if and only if k x square plus k y square is equals to omega square by c square. If you choose k x and k y to be such that it is the sum of the squares is becoming a constant which is omega square by c square then also such choices will lead to plane waves, but given k x and k y you could always determine the direction of travel of this wave by a simple trigonometry again. So, given k x comma k y the direction of travel of the wave is tan theta equals to k y by k x. Just from this diagram if you say that the direction of the wave number vector is equal to the direction of travel for the wave, and the direction of any vector can be determined using this sort of a formula tan theta is equals to y component by x component.

So, what we are doing is that we are escalating this wave number concept to sort of assimilate the idea of a wave number vector the wave number vector will denote not only the magnitude of the wave number which is omega by c and that is what we have been looking at wave number from this perspective, but the direction part also could be associated by aligning this wave number vector to the direction of travel of this plane wave.

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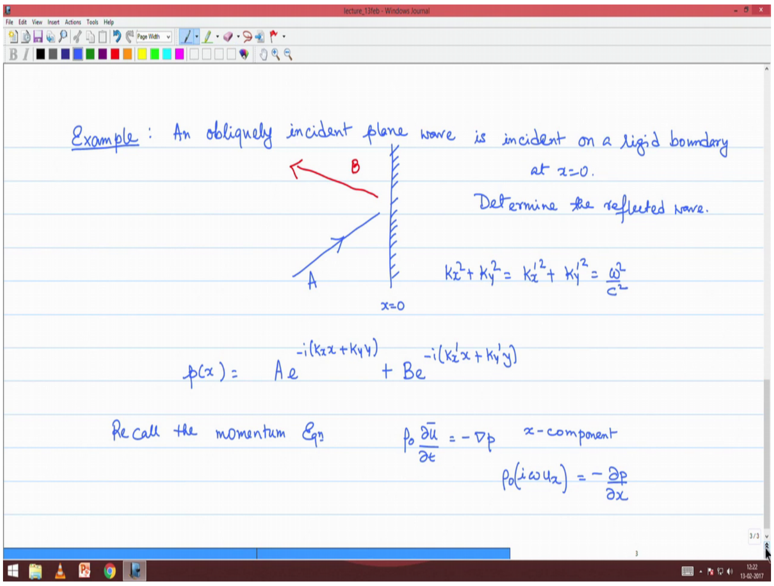

So, with this introduction let us now look at an example where we can illustrate the applicability of a higher dimensional null wave, high dimensional plane wave. So, in the last class we have seen that if a normally incident plane wave is incident on a surface where the velocity is maintained at 0 then it will lead to reflection. We will complicate that problem just a little bit by saying that this time it is an oblique incidence. So, there is an amplitude a wave of complex amplitude A which is obliquely incident on the same rigid surface. So, at x equals to 0 we have an wave an incoming wave which is incident on it the question is what happens after this wave is incident what is the reflected wave in specifically we wish to find out.

In this case there is no transmission because it is facing a hard boundary there is nothing which will go transmitted because of this hard boundary. So, the objective is an obliquely incident wave obliquely incident plane wave is incident on a rigid boundary at x equals to 0. So, determine the reflected wave our intuitive filling in geometric optics tells us that we could possibly determined this wave by simply using the relation such as angle of incidence equals to angle of reflection, but we will in fact derive this same result from our completely different perspective using the framework for of wave propagation as we have developed.

So, let us see how it goes. So, p x is an incoming wave which is A e to the power minus i k x into x plus k y into y that is the incident wave, but we are expecting that there will be a outgoing or a reflected wave also which we call it has b. So, there will be a b wave could possibly have a different wave number components, the wave number would possibly be different because obviously, the wave number components denote the direction of travel of the wave, now the direction of travel of the a wave and the b wave cannot be the same because one is incident as shown in the diagram and the other is reflected.

Therefore, the reflected wave and the incident wave has got to be different at least in direction. So, therefore, let us choose a component which is different right. So, the components of this reflected waves I will denote it by k x prime and k y prime, but then k x prime and k y prime cannot be arbitrary, k x prime and k y prime are components of the vector k whose magnitude is omega by c. Similarly k x and k y cannot be arbitrary they are also components of a wave number vector, the magnitude of which is constrained to be k x k square which is omega square by c square.

So, the relation that we will have to keep in mind is k x square plus k y square must be equals to k x prime square, plus k y prime square which is omega square by c square right. So, the magnitude of the 2 wave number components is such that these the square of it is the magnitude of the 2 wave number vectors has got to be the same. So, this is the relation that we have and the relation for pressure is given by this. Now what we need to do is we need to ensure that at x equals to 0 surfaces, the velocity perpendicular to the surface is going to be 0. We should not care much about the tangential velocity because we have already assume that the acoustic fluid is in viscid, for an in viscid fluid the tangential velocity will not match on the rigid surface you are allowing the slip to happen because you have taken the fluid to be in viscid.

So, therefore, there is no chance that u y will be 0 at x equals to 0 surface, but u x is supposed to be 0 at the rigid surface x equals to 0. So, again we now need to find the relation between the pressure and the velocity. Last time in dealing with plane waves that was simple it was just a division by rho c, but remember that was the formula that we derive for one dimensional plane wave we now need to rework that formula for 2 dimensional plane. So, let us say what that goes as. So, recall the momentum equation or the Euler equation is given by rho naught del u del t and u is a vector is equals to minus gradient of p.

So, the x component of this relation is going to read as i omega u x into rho, rho naught into i omega u x is equals to minus del p del x right. Just as I told you again I will repeat myself whenever you take time derivative because you have taken the harmonic resumption all time derivative can be replaced by a product with i omega, any quantity is having a harmonic time dependent. So, therefore, the time derivative of any quantity is i omega multiplied by that quantity itself.

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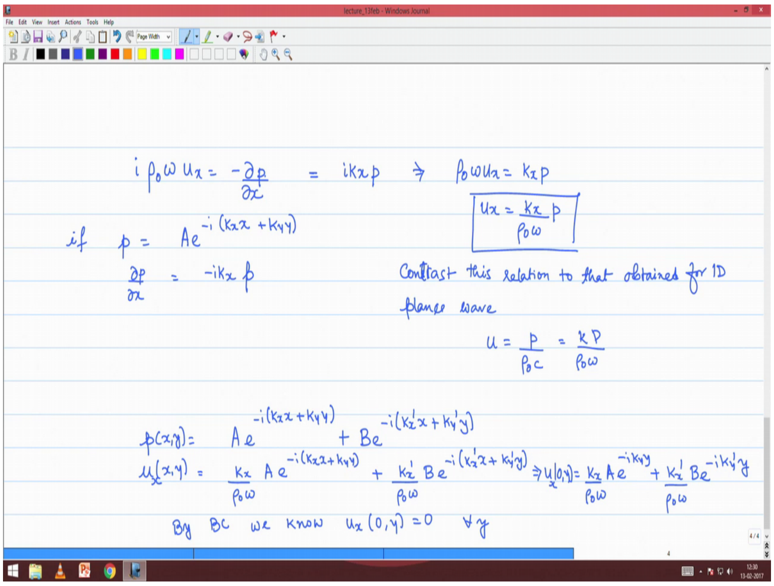

So, now coming to del p del x. So, I think I have to go to the next page. So, rho naught i omega u x is equals to minus del p del x. Now if p is equal to e to the power minus i k x x plus k y y right then what is del p del x? Del p del x will again bring the factor minus i k x along with it and then entire part will be repeated which means p itself right. So, this will lead to if we substitute this back it reads as i k x the minus will get cancelled of i k x times p right that if we reduce it a little bit we are going to get rho 0 omega u x is equals to k x times p, which means u x is k x divided by rho 0 omega into p.

So, this is the relation between the speed of the particle velocity along the x direction to the pressure right. If you contrast this relation with that for 1 D plane wave, for contrast this relation to that obtained for 1 D plane wave if you recall the relation was u equals to p by rho naught c and rho naught sorry c could be written as omega by k. So, the difference between these 2 relations the one that we obtained for the one dimensional case and the one that we are seeing now is in the fact that the wave number in one dimensional case is replaced by the component of the wave number along the direction that you are looking at. So, the subscript it appears both in u as well as in k in the higher dimensional analogue.

In the one dimensional case you did not write a subscript associated with u because you know it is one dimensional there is only one special variable of interest which we took it as x right. So, that is why u was written as it is without mentioning the subscript as u x, similarly k does not require you to give a component because you are dealing with one dimensional wave the only component is along the x component and there is nothing else, but now that we are dealing with a 2 dimensional plane wave there are 2 components of k, one coming in the direction the other coming in the y direction.

So, well what that we see now is this fact that the velocity to pressure relation remains almost the same except for the fact that you have to correctly account for the direction; because now you have 2 directions in your coordinates system x and y, u along the x direction will be given in terms of k along the x direction similarly u along the y direction will be given for with respect to k along the y direction right and the factor in the denominator rho 0 omega is just the same.

So, we have this relation which tells u x is equals to k x by rho naught omega. Remember this happens because we had taken a negative sign associated with the k x, x phasor. So, the sign of k i mean the complex exponential here is e to the power minus i k x it had been plus i the sign would have been reversed and that is again easy to understand from the perspective that which direction you are wave is going along that direction you have to align your velocities. So, now, we get back to the relation that we had developed for px. So, what I can do probably is cut and paste this relation. So, I will copy this and I will bring it back once again here. So, this is where what we had already derived.

So, now what we will do is, we will now derive u x associated with this pressure waveform right. So, the u x associated with the A wave will be k x divided by rho naught omega into p. So, the p is just the same A e to the power minus i k x into x plus k y into y right. And for the B wave just you have to replace k x and k y with k x prime and k y prime. So, that should be easy that gives us k x prime rho naught omega B e to the power minus i k x prime x plus k y prime y.

Now what we will need to enforce is that we need to enforce the condition at x equals to 0. So, at x equals to 0 we have the following k x by rho naught omega is A e to the power minus i k y y plus k x prime rho naught omega B e to the power minus i k y prime y and we demand that u 0 has got to be 0 for all the values of y. By boundary condition we know I should have put a subscript x here to denote that this is the velocity along the x direction.

So, if velocity along the x direction at any x and I think I have been sloppy with this notation. So, this is a function of both x and y. So, I will make that correction here since it is a function of both x and y and similarly this is a function of both of only y x has been kept at 0 and y could be anything. So, by boundary condition we know that this quantity u x 0 comma y should be 0 obviously, for all y right. It is it is important to understand this concept because here the surface itself extends to all y values this is our x direction and this is our y direction. So, the surface in itself extends to all y values.

Now, if you want this summation to go to 0 right the you see the 2 functions have got the form that in which we have written it has 2 different dependencies in y right, one in both are harmonics in space in in in the variable y, but one of those phasors is rotating at a speed k y and the other of those phasor is rotating at a k prime y right, but then if you have to really enforce the condition that u x at all ys for x equals to 0 has got to be 0, then there is no way in which these 2 functional dependencies can be different. If it were to be true that k y is not equals to k y prime then there is no way in which this condition can be achieved because you would be dealing with 2 phasors which are rotating at different speeds at some time they at some values of y they may cancel each other, but at other values of y they will not cancel each other.

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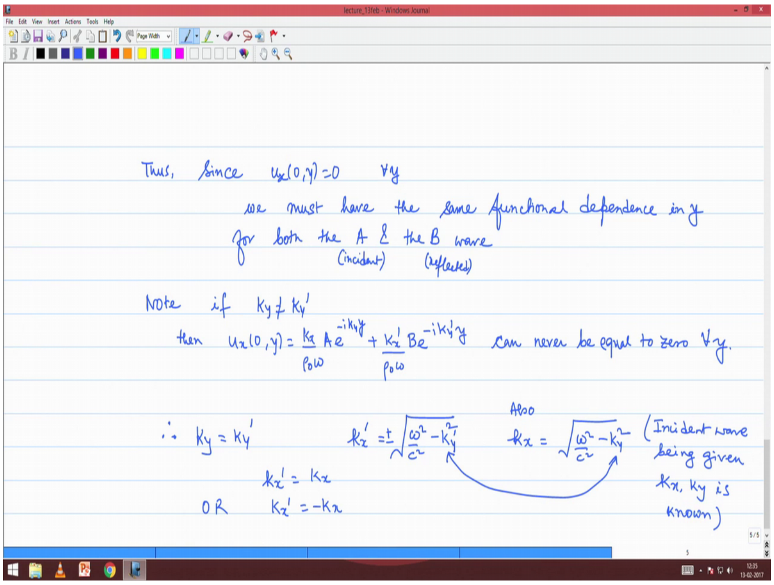

So therefore, by our demand that this boundary condition hold good for all the values of y we are led to believe the following; thus since u x 0 comma y must be 0 for all y we must have the same functional dependence in y for both the A and the B waves. There are 2 waves that we are dealing with the A wave or the incident wave and the B wave or the reflected wave right both of them must have the same functional dependence I will just created what I said by noting that note if k y happens to be not equal to k y prime then u x 0 comma y which is given by this I can again cut and paste that I guess, u x at 0 comma y which is given by this can never be equal to 0 for all y that is not possible, because you are having 2 phases in y which are rotating at different speeds, it is similar to saying that sin omega t plus sin 2 omega t, there is absolutely no way in which this summation will go to 0 for all times right at certain times it might, but for all times it will never go to 0.

So, the only possibility that is left with us is to choose k y is equals to k y prime. So, therefore, this is what k y prime has got to be. Now if k y prime is equals to k y we also know that k x prime has got to be square root omega square by c square minus k y prime square, but this with the square root there can be a positive or a negative sign associated now which is the sign that we should selected whether positive or whether negative. If we select the same sign of k x prime as we had for k x; what was k x by the way? K x also had to satisfy this sort of an equation right plus or minus omega square by c square minus k y square.

But we have said that the incident wave is given to us, given the incident wave we are trying to determine the reflected wave. So, we are assuming the sign of k x is given because the incident wave is given to us. So, the incident wave being given k x k y is known k y prime also has got no freedom to choose any other values k y prime also is constraint to have the same value as k y itself right. Now coming to k x prime if k x prime is chosen be of the same sign as k x, remember because k y prime and k y are actually equal right. So, there are 2 possibilities k x prime is actually equal to k x or k x prime is the negative of a k x, there are 2 possibilities there is nothing else.

But now let us analysis these 2 possibilities carefully, if we choose k x prime to be equal to k x we already know k y prime is equals to k y then we will conclude that the reflected wave travels in the same direction as the incident wave, there is no difference in direction between the reflected wave and the incident right. So, that seems quite a distance possibility I mean that cannot happen.

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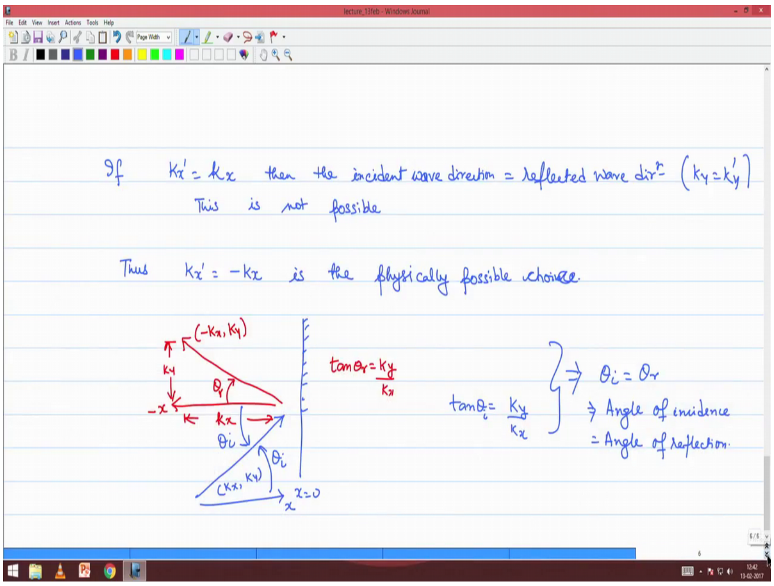

So therefore, the other possibility looks better that we choose the sign of the ref k x prime to be opposite to that of the sign of k x of the incident wave right. So, if k x prime equals to k x then the incident wave direction must will be same as the reflected wave direction, not that k y is already equals to k y prime. So, this is not possible you are expecting a change in direction between the incident and the reflected wave. So, thus k x prime equals to minus k x is the physically possible choice.

Now, let us once more come back to our geometry. So, this is x equals to 0 we had an incoming way or an incident wave with the wave number vector k x, k y which means this angle theta if you like this theta is given by tan theta is equals to k y by k x, and what we are now concluding is that we are going to get a reflected wave and that reflected wave will have the wave number components as minus k x and k y right. It makes sense if you can realize that the phase associated with the change in the positive y direction look at what happens in the positive y direction this wave the blue wave is having a certain phase change in the positive y direction. So, it is travelling in the component of the blue wave is actually traveling in the positive y direction, the component of the red wave is also travelling in the positive y direction.

Therefore, you see that the wave number components along the y direction are actually identical; where as the component of the incident wave along the x direction is in the positive x direction, where as for the reflected wave we are saying that the component of the reflected wave along the x direction is directed towards the negative x axis. So, it makes sense. So, the wave number vector will be here this flip in sin. So, the minus k x component associated with the reflected wave is actually denoting the fact that the wave as it travels it travels along the it has a component the wave number as a vector is aligned with the wave in reality, but this wave number vector the component of this wave number vector along the x axis is directed towards the negative x axis rather than towards the positive x axis, which is why it bears a negative sign.

So, therefore, the theta associated with this wave. So, this theta is taken with respect to the minus x axis where as the theta that we took here was with respect to the positive x axis. So, how will you calculate let us let me call this theta r so that we can distinguish. So, how will we calculate this theta r? So, this theta r would be given by exactly the same relation tan theta r is k y by k x; because the component of this wave number along the x direction is k x and the component along y is k y. So, tan theta r remains as k y by k x and tan theta I could be more specific by calling this theta i. So, tan theta i is k y by k x. So, this theta i by alternate angles will also become this theta i.

Therefore, what we say is that tan theta i and tan theta r both are having the same numerical value, which essentially means the value of angle of incidence is same as the angle of reflection. So, what we have to conclude from here is that theta i is equals to theta r which implies the angle of incidence equals to the angle of reflection. So, using a wave propagation approach we have now been able to show that for plane waves the angle of incidence has got to be the same of angle of reflection, this is not true for spherical waves this may not hold for other types of waves right this is only true for plane waves.

Now, in optics what does happen is that you are having a wave length which is pretty small, which means far away from the source as I keep on saying you will have a plane wave condition. So, in optics all that we see in at least day to day life is this fact that the light has gone has travelled quite a few wavelengths from the source which means plane wave conditions as selected in and therefore, we have this very nice theory in optics that angle of incident as got to be angle of reflection. And there are different ways otherwise also to prove this fact that comes from the minimization of time statement and you would have done problems in calculus associated with that, but please understand here we have just use the framework associated with wave propagation and simply by enforcement of boundary conditions, we are lead to the same conclusion that for plane waves angle of incidence will be equal to angle of reflection. We will carry on this idea a little more in the next problem.

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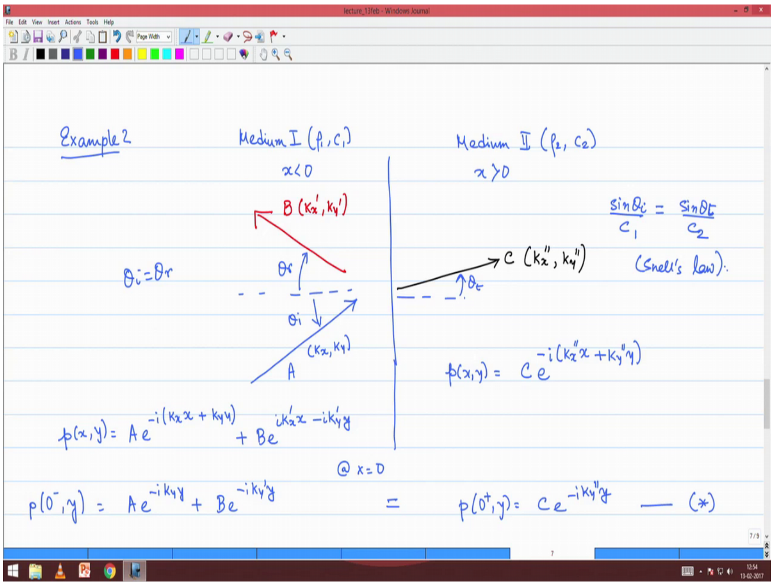

So, here we will now consider 2 medium. So, this is medium 1 and that is medium 2; and medium one we will take the properties to be rho one c 1 and medium 2 we will take the properties to be rho 2 c 2. As usual we know that there is an incident wave a which is incoming and that we will take it as having the wave numbers k x and k y. So, everything is known about this wave; we are interested to find that what is the form of the reflected wave and the associated wave numbers, and this time we are expecting a transmitted wave also because some part of this incident wave will get reflected, but some other part should get transmitted.

So, we are expecting that there will be a transmitted wave c, and this time with k x double prime and k y double prime. So, this is what you would have the analogy in optics again is similar problem in optics would lead us to the result of snails law right where which is what we will derive right away. So, this region we will refer to as x less than 0 and this region we will referred to as x greater than 0. So, the pressure in the x lesser than 0 region p x comma y, in the x lesser than 0 region would be given as a e to the power minus i k x x plus k y y plus b by now we know that k x prime would have the opposite sign as k x.

So, therefore, we can save our effort by saying that the complex exponential dependence would be flipped in sign as for as the x variable is concerned, but for the y variable it will preserve the same sign as the incoming wave. So, the incident wave is having minus i k y y dependence, the reflected wave will also preserve that same sign minus i k y prime y right. But in the x direction i have chosen to flip the sign because by now i am sure that the component of the reflected wave will definitely be align towards the negative x axis rather than the positive x axis; because the incident wave is align to the positive x axis. So, that reflected wave as got to be aligned with the negative x axis.

Therefore, I have chosen to flip this sign if you do not choice to flip the sign at this stage you will be working it and again you will have to argue it out like the wave we have argued in the previous example that k x prime must have the opposite sign of k x right that part should be and here what we have in the x greater than 0 region is a complex amplitude c, that should get multiplied with the complex exponential e to the power i k x double prime x with the negative sign plus k y double prime y both of them have got the negative sign.

Now, at x equals to 0 what happens let us see for the 2 pressures; at x equals to 0 if we have to take the 2 pressures p 0 comma y from the negative side p 0 minus comma y that has to be A e to the power minus i k y y plus B e to the power minus i k y prime y and from the positive side it has got to be C e to the power minus i k y double prime y right. Now again the same story repeats you must have the same pressure also on the 2 sides of the fluid otherwise you will have a loss of the continuity.

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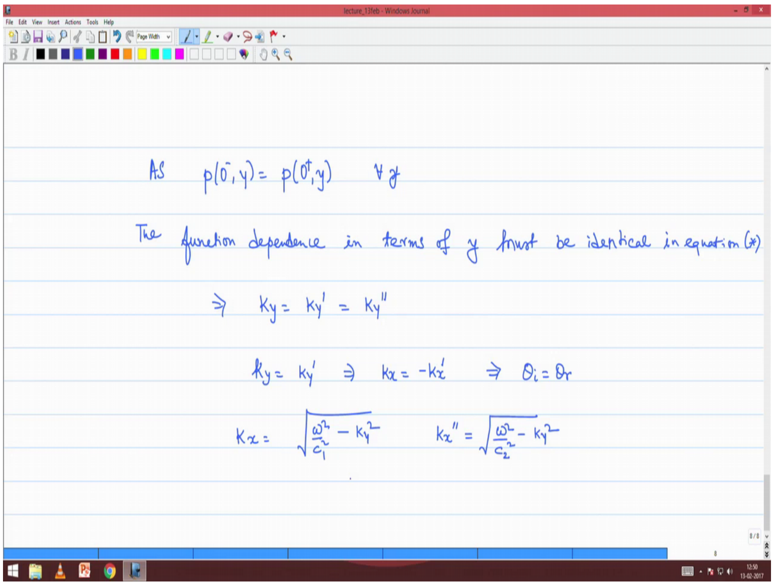

So, the pressure at the interface coming from both the 2 fluids should be just the same which means k y k y prime and k y double prime all have to be same.

So, thus as p 0 minus y is equals to p 0 plus y for all y the functional dependence in terms of y must be identical for all the waves I mean for this one. So, these 2 conditions must be equal which means the functional dependence of these 2 functions that I have written in terms of y they must be identical. So, that is what I meant by saying the functional dependency in terms of y must be identical in equation star and this is my equation star there cannot be any different functional dependence. So, this in turns implies k y must be equals to k y prime must be equal to k y double prime.

Therefore, k y equals to k y prime is exactly the same as we have got for reflection right. So, k y equals to k y prime implies k x equals to minus k x prime, and you will have the condition theta i is equals to theta r right as was shown in the previous derivation. So, this is theta i and this is theta r. So, you will have k x prime to be flipped in sign with respect to k x it will be at the opposite side, but then k x double prime will be slightly different. Please note that k x is going to be omega square by c square minus k y square and k x double prime will be this is for medium one and for medium 2 we should have omega square by c 2 square minus k y square.

So, the fact that the 2 mediums all different is now playing it is role here; because the relation between though the 2 k y s are same for the incident wave and the transmitted wave, but it happens to be that the wave number in the x direction will change. So, we could turn this around.

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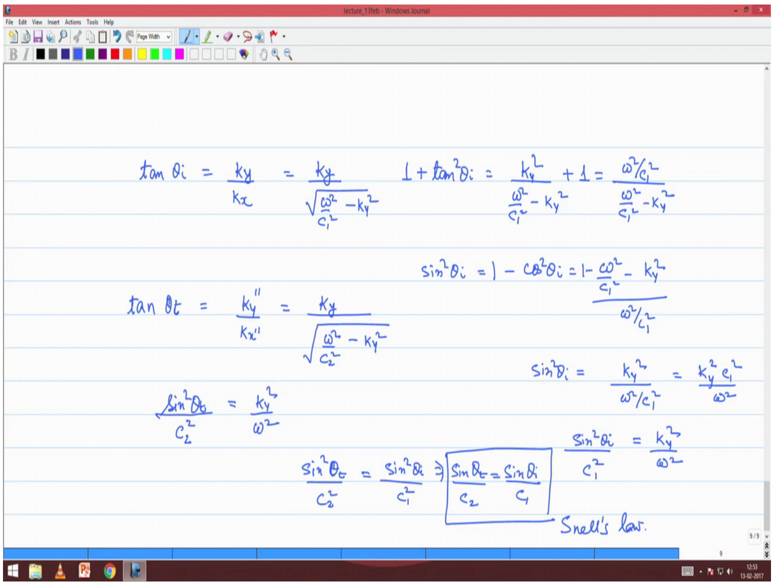

And if we know which to determine tan theta i that is given by k y by k x, and k y by k x could also be written as omega square by c 1 square minus k y square right and similarly tan theta t which is the angle for the transmitted wave. So, this is my theta t variable. So, tan theta t would be k y double prime divided by k x double prime and k y double prime is k y that has been proved k x double prime will be omega square by c 2 square minus k y square right.

So, with this as tan theta let us try to evaluate what is sin theta, but we will do that in steps. So, that tan square theta i is k y square divided by omega square by c 1 square minus k y square and 1 plus that is x square become. So, that will become omega square by c 1 square divided by omega square by c 1 square minus k y square. So, cos square theta would be omega square by c 1 square minus k y square divided by omega square by c 1 square. So, therefore, 1 minus cos square theta is sin square theta which will read has k y square divided by omega square by c 1 square.

So, sin square theta i is k y square c 1 square divided by omega square. In other words sin square theta i divided by c 1 square is equals to k y square by omega square. If we redo this same trigonometry here we are going to get sin square theta t just instead of the subscript i we will use the subscript t other than that everything is same and we have to carefully note that c comes with the subscript 2 here, where as c came with the subscript one in the expression that we have derived. So, sin square theta t divided by c 2 square is going to be the same ratio which is k y square by omega square.

So, therefore, we will have sin square theta t divided by c 2 square to be the same as sin square theta i divided by c 1 square, which would imply sin theta t divided by c 2 should be same as sin theta i divided by c 1, and this you will recall is nothing, but snails law. So, this relates the angle of incidence to the angle of refraction or the angle of transmission.

So, it turns out here theta i has to be equal to theta r angle of incidence equal to angle of reflection and it turns out that sin theta i by the speed in the incident region which is c 1 must be same as sin theta t divided by the speed of the waves in the transmitted region. So, this is what is usually referred to as Snell's law ok.

Thank you, we will pick it up from here then.