Fluid Dynamics And Turbo Machines.
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Part B.
Module-1.
Lecture-3.
Integral Analysis.

Good morning and welcome to the 2nd week and this is the 3rd lecture of the 2nd week, in the last 2 lectures we have introduced the integral analysis of fluid dynamics. In the first lecture we derived the Reynolds transport theorem and then in the 2nd lecture we applied the Reynolds transport theorem for some specific applications and also we found how does the Reynolds transport theorem looks like in case of mass and momentum conservation. Today we will see how the what form the Reynolds transport theorem takes for angular momentum conservation, so, because this has very much an application in the case of Turbo machines. So, the 3rd part of this week's lecture deals with the angular momentum conservation.

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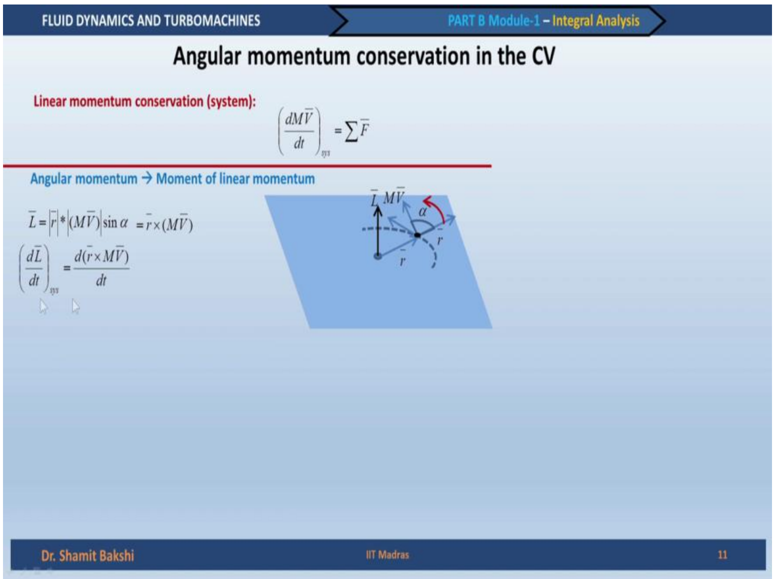

So let us go to the slide. In the case of angular momentum conservation, so we are applying these conservation equations to a control volume just to remind ourselves that we are dealing with control volumes as opposed to systems in the case of fluid. Now, let us relook at what form the linear momentum conservation for a system looks like, not for a control volume, for a system how it looks like. So, this is basically Newton's 2nd law of motion that rate of change of momentum is sum of all the forces. Again this is a vector equation which is indicated by the F bar and V bar here which are the vector quantity. Now, this linear momentum conservation we have derived for the control volume also. Let us see what is angular momentum before going into the angular momentum conservation equation.

So, angular momentum is defined as moment of linear momentum. Let us see what do we mean by moment of linear momentum. So, this is linear momentum conservation equation which is shown here and angular momentum by definition is moment of linear momentum and we will see why is that the angular momentum is also conserved like the linear momentum. So, let us write a mathematical expression for the angular momentum. To write that expression we take help of this figure, this figure actually shows a point moving along the trajectory and then we want to find out what is the moment of momentum of this particular point which represents a particle actually. So, let us see how the, how to find the angular momentum of this particle about this point. So, when we are talking about a moment, it has to be about a particular point.

So, let us see how the, what is the expression for angular momentum of this particular about this centre point. So, the particle is located at a distance R bar, the position vector of the particle is R bar, so that is given here and the moment, the linear momentum of the particle is given shown by this arrow M V bar. Now we have to find the moment of momentum, linear momentum, so we have to, we extend this position vector of the particle with respect to this particular centre point about which the moment has to be taken. So, this is basically the same position vector and the angle between the position vector and the momentum is Alpha. Let us take that angle is Alpha.

This MV bar also lies in this plane which is shown here. So, it has an angle with R bar which is given by Alpha. Now MV bar has 2 components, one component is tangent to the trajectory of the particle at this point, so one component is perpendicular to the position vector R bar, so that is this one, this is the component which is perpendicular. And there is other component which is along R bar. Now when we are finding a moment of the linear momentum, we have to understand that the moment only applies to the component which is perpendicular to this vector. So, if we want to write now the expression for angular momentum which is indicated as L bar in this expression, it is given as modulus of R bar.

So, we are talking about, this is exactly not a, I mean the expression is not exactly a vector quantity, so this should be modulus of L bar, that will be given as R bar multiplied by MV bar sin Alpha, so the component of MV bar along this direction. Now the cos Alpha component does not give any moment. So, this is the expression. Now we can write this as R bar cross MV bar. Okay, so basically this in vector notation again, L bar is a vector, so this is given as R bar cross MV bar, so basically it is a cross product of the position vector and the momentum, linear momentum. Why have we taken it as R bar cross MV bar instead of MV bar cross R bar, that is because of the Convention.

The Convention is according to the right-hand rule, if we write it like this, that means R bar cross MV bar, that means if we place our hand somewhere here and then we point the direction of rotation with 4 fingers other than the thumb, then the direction of the thumb will indicate the direction of L bar or the angular momentum. So, the direction of L bar is this. So, L bar has a direction of this and the magnitude of this now, we have to see whether this quantity is also conserved. So, this is basically the destination of angular momentum, it is moment of linear momentum. We can find out whether this quantity is conserved or we can get a similar equation like we have got here for angular momentum.

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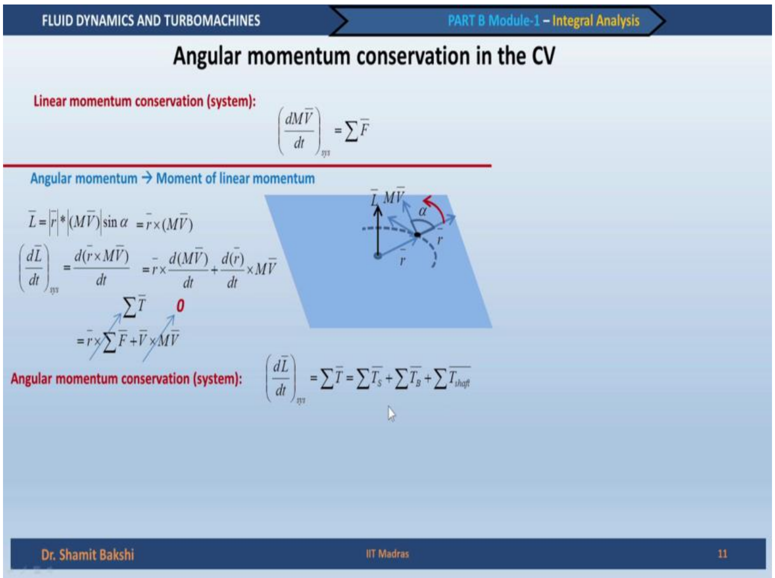

So we take derivative of this L bar with respect to T for a system. So, just to remind ourselves, we are in this, so far we are only talking about system and we have not gone into the control volume. So, for the system we can write this as d dt of R bar cross MV bar and we can apply the laws for a cross product we can directly apply the law of different very, finding the derivative of 2 products. That means this will be equal to R bar cross dM V bar by dt and dR bar by dt cross MV bar. So, this is allowed to actually break it into these 2 components. Now, if you write it like this now, what is the first component? The first component we can see dM V bar by dt appearing here which is there in the linear momentum conservation equation also. So this can be replaced with the force or the force vector.

So, we can write this as R bar cross the force vector, so we can write it as R bar cross the force vector and this as V bar cross MV bar. This part dR bar by dt is actually the velocity. So, this is V bar cross MV bar. Now, this part is equal to the first part is actually the torque. This is the moment of course, like the moment of momentum is angular momentum, the moment of force is torque and this can be replaced with torque. And V bar cross V bar is actually zero, so the cross product of a vector which itself is zero because the angle between them is zero and sin zero is zero. So, this is zero, so what we are left out with is the conservation equation for angular momentum for a system.

So, this is angular momentum conservation for a system is basically dL bar by dt of a system is sum of torque. Of course this is well known to us that like the rate of change of for causing a rate of change of momentum we need a linear momentum, we need a force and for producing rate of change of angular momentum we need a torque. Now this torque is also given as sum of all the torques. So, this is also a vector quantity, it can be, you can have torque about all the 3 axes. We have like we wrote in the case of force as the sum of forces as the sum of surface forces, sum of body force and the sum of surface and the body forces.

Here also we can write it as sum of the torque produced by the surface forces like the friction force, the pressure force, etc. and the torque produced by the body forces like the torque produced the gravity. Apart from these 2 part we also have the torque produced by the shaft. So, this is basically the new component which is important for an angle of momentum conservation of asystem. We will see more about this shaft torque when we write the angular momentum conservation for a control volume.

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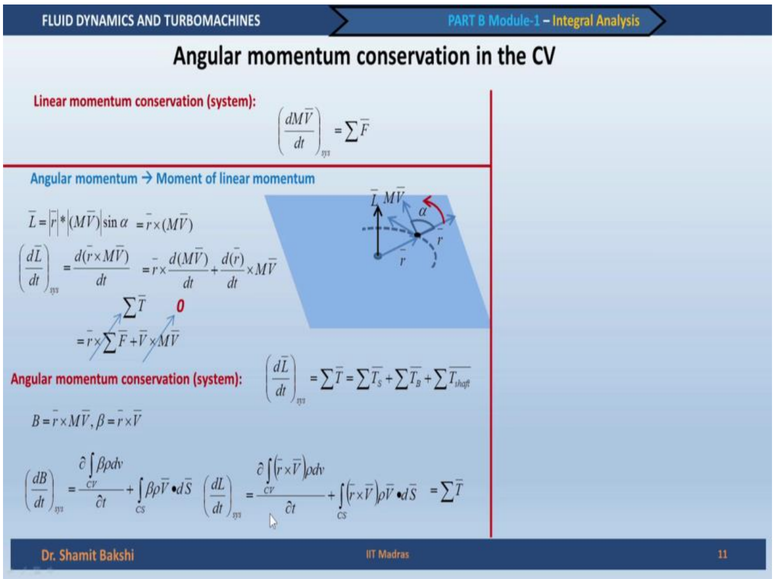

So, let us move onto, this is the angular momentum conservation for a system, let us move on to angular momentum conservation for a control volume now. So, if we remember the Reynolds transport theorem is given like this. So, rate of change of, so basically it relates the time derivative of any quantity for a system with the quantities for a control volume, this is the control surface. Now, what was beta defined as? Beta was basically the property B per unit mass, B is any quantity. So, we have seen that this as in the form of mass, we have seen it in the form of momentum, now we are seeing this quantity B in the form of angular momentum. So, B is R bar cross MV bar, so beta will be, if we divide this quantity with mass, beta is basically R bar cross V bar.

So, we can plug-in this R bar cross V bar into this equation, what we get is basically the angular momentum conservation for a control volume. So, dL by dt for a system is this derivative of this, beta is replaced with R bar cross V bar in both the places, so this is basically equal to sum of all the torque. This is the angular momentum conservation in the case of a control volume. So, if we just remember in the case of linear momentum conservation, we had beta as V bar, this has been replaced by R bar cross V bar in the case of angular momentum conservation, that is the only difference.

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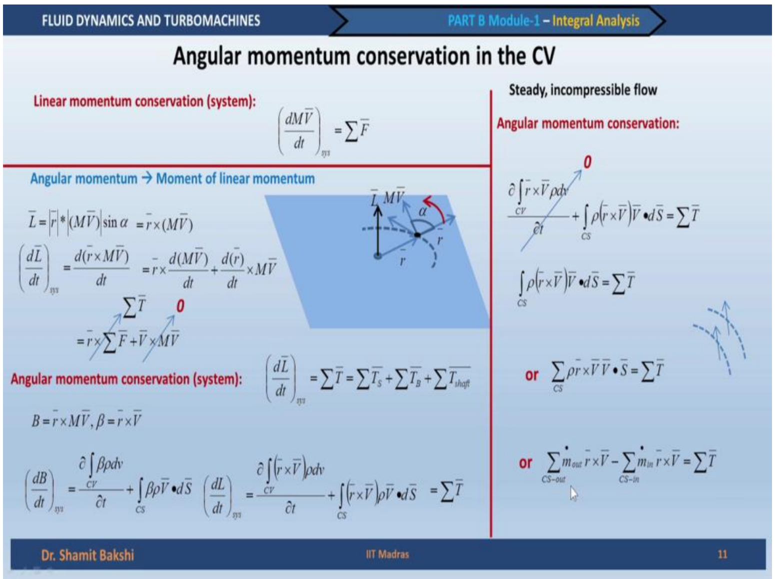

Now let us see, like we saw in the case of linear momentum conservation, we applied this equation, the situation as such, so this part of the equation actually is the angular momentum conservation for the control volume. This looks complicated because it has integral, it has differentials, derivative terms, let us write it for simple conditions like a steady and incompressible flow so that we can simplify, write this equation in a simplified form which is the case in most of the applications which we will be looking at. So, steady means it is time invariant and incompressible means there is no density change in the… We have already defined this. Now, for the angular momentum conservation for such a system can be written, this is the general angular momentum conservation, for a steady incompressible flow we can further say that this component is zero because this is time invariant.

So, any derivative of any quantity with respect to time is zero and we can write that 2^{nd} one, the 2^{nd} quantity that integration of the angular momentum over the entire control surface is equal to the sum of all the terms. So, this can be further written, suppose we have, when the quantity angular momentum is continuously varying cross through the surface. So, whatever is coming in through the inlet or exit, that also varies along the inlet or the exit. Now, that may not be the case for many situations, we can have a case like this where the angular momentum coming in through this control surface is constant and coming, going out angular momentum of the fluid going out through this exit control surface is also constant, it is not varying along the control surface.

If that be the case then we can write, simplify this equation further and write it as a summation of this quantity, that is the net exit of the angular momentum from the control volume is equal to sum of torque. Now you can have multiple inlet and exit. If you have one inlet and one exit, it will be the angular momentum of the fluid, the rate at which the angular momentum of the fluid exits the control volume subtracted from the rate at which angular momentum of the fluid enters the control volume, that will be the torque transmitted. Now, that is what is written here, so we have written this in the form of rho, in the form of M dot out and M dot in.

So, as we can see here rho V bar dot S, S is the surface area of the control surface and the dot product signifies that the velocity perpendicular to the surface is responsible for the mass flow going out into the surface. So, rho V bar dot S can be clubbed together and written as M dot and then this full equation can be written as M dot out multiplied by R bar cross V bar - M dot in from all the control surface from which fluid is entering M dot in multiplied by this angular momentum for unit mass, that should be equal to sum of torque. So, basically this is a more simplified equation than what we started for a case where it is steady, incompressible and we have surfaces, along the surfaces there is no variation of velocity or angular momentum.

So, under that condition we can write this equation in a more simple way like what is shown here. This is very useful for us because for most of the cases we will be using this for of angular momentum conservation equation. This simply says net rate of angular momentum exiting the control volume is equal to sum of all the torques. If the net rate of angular momentum exiting the control volume is negative, that means that the, there is a net inflow of angular momentum into the control volume. So, that means that torque is negative or reverse. So, you will see this further in the next slide.

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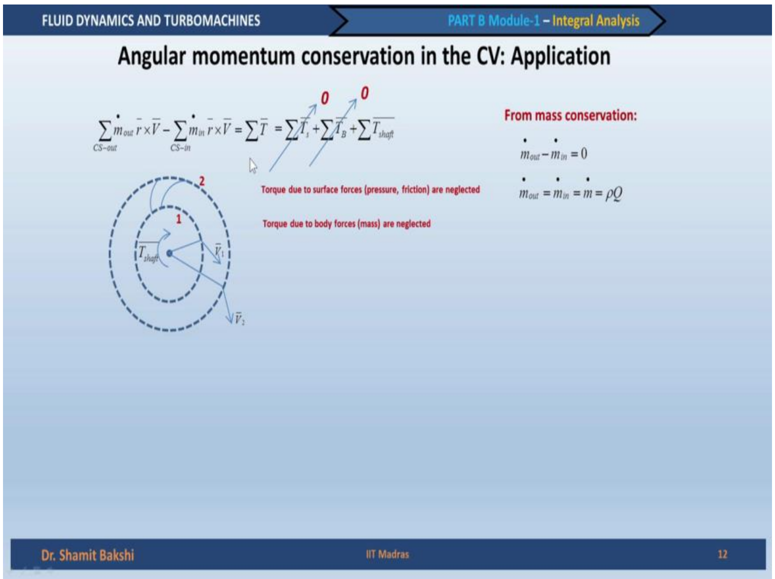

So, in the next slide we make, we demonstrate the application of the angular momentum conservation for a control volume. So, we have taken 2 control surfaces, this is a typical Turbo machine kind of application, this surface, control surface 1 is the inner surface and the outer surface is named as control surface 2. Now let us say at a particular point or through this surface, the fluid is coming into the control volume, control volume is the annular space between these 2 control surfaces, so it is coming in so this control surface 1 and going out to the control surface 2. So, it is coming in at the velocity and with certain angular momentum and it is going out at the velocity given the velocity vector is given as V 2 bar at a certain angular momentum.

So if we apply, for this situation if we apply the angular momentum conservation equation, let us see what we get. So, we start with the last equation which we got in the previous slide that the M dot out and the angular momentum per unit mass exiting the control volume and M dot in multiplied with angular momentum per unit mass coming into the control volume, for that is sum of torque. For this particular application again we can also use the, so sum of torque, then that torque is given as sum of the surface torque due to the surface forces, due to body forces and shaft. Now, this part, the first part, torque due to the surface forces, that is the forces like pressure force or friction force on the whole, let us neglect that part.

So we will make this part zero for this application. Torque due to body force, that means the weight of the fluid in the control volume, let us also make that part zero, let us ignore that part. We are left out with the shaft torque only. The shaft torque is the torque supplied to the fluid in the control volume in the case of a pump or the torque output, torque coming out from the control volume in the case of a turbine. So, that is basically the meaning of this shaft torque. If it is a pump, we can imagine before going into simplifying this equation further, we can easily understand that if this shaft torque is positive, that means this difference will also be positive, that means the angular momentum of the fluid which is coming out of the control volume is more than the angular momentum of the fluid which is coming into the control volume.

So, this is the situation in the case of pump. The pump actually introduces transmits angular momentum to the fluid in the control volume so that the angular momentum of the exiting fluid is more than that of the fluid coming into the control volume. It is reverse in the case of a turbine. So, what does the turbine do? So, in the case of a, let us say in this expression, if the, sum of the shaft torque is negative, so that means the angular momentum of the fluid going out of the control volume is less than the angular momentum of the fluid coming into the control volume. Or in other words, the fluid is losing angular momentum as it crosses the control volume. And this is the momentum which is transferred as torque to the turbine.

So, in the case of a turbine, the shaft torque is negative which in other words mean that the angular momentum of the fluid exiting the control volume is less than the angular momentum of the fluid coming into the control volume. Now, let us simplify this left-hand side of the equation little more. So we also know from mass conservation for this particular case that we can directly say that mass flow rate out of the control volume is same as this is, we can again get this from the, by applying the Reynolds transport theorem. So, that means the mass flow rate out is equal to the mass flow rate coming in which we can just write as M dot or density into the volume flow rate.

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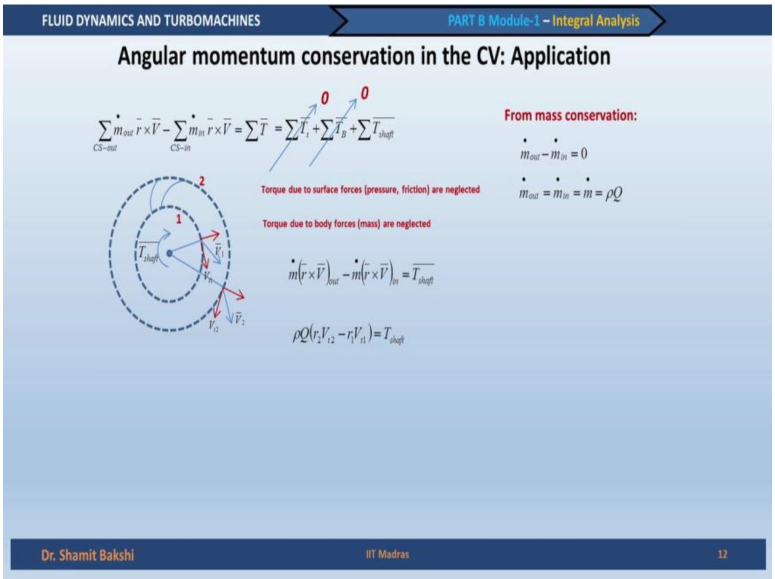

Mass flow rate is actually density into the volume flow rate. Now, if we write this, we can use this expression for M dot outdoor M dot in in this expression for angular momentum conservation. So, we do that, then we get M dot into multiplied by R Cross V out - M dot into R Cross V in is equal to shaft torque. Now, this R Cross V, let us look into this little further, we can resolve the V1 bar, that is the inlet velocity into 2 components, one is the radial component and another is the tangential component. The radial component is of, is not of importance to us because we are talking about the moment of the momentum or we are talking about a cross product, so what is important is the tangential component. So, similarly at the exiting point, exiting control surface we can resolve the velocity into 2 components, one radial, another tangential.

Now we can say, we can define the tangential velocity at the inlet as VT1 and the tangential velocity at the exit as exiting control surface, that is the control surface 2 is VT2. We are not naming the normal component because they do not produce any momentum. So, now we can then write from this, if we can replace this R bar Cross V with this radius multiplied by VT R bar Cross V out as this radius multiplied by VT 2 and above Cross V bar at the inlet as this radius, radius of the control surface 1 multiplied by VT 1. So, if we do that, we get a very simple expression and this expression is known as Euler Turbo machine equation.

This is a scalar form of the angular momentum conservation equation which is very useful for Turbo machine applications. You will be using this equation in the latter part in the Turbo machines part. So, we come to now as we discussed earlier, this shaft torque for the case of a pump or a compressor is positive because in that particular case, in the case of the pump and compressor, the shaft actually transmits angular momentum to the fluid. So the exiting fluid has higher angular momentum than the incoming fluid.

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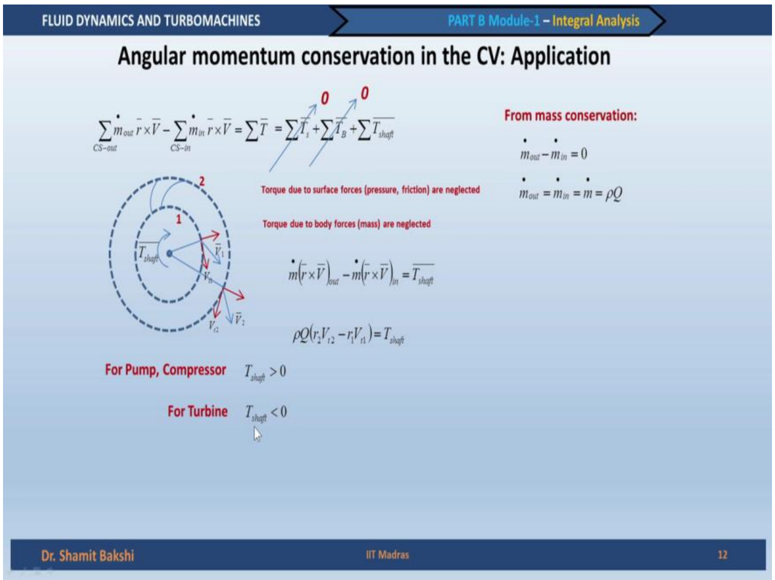

Whereas in the case of a turbine, the fluid loses its angular momentum to the turbine. It loses as it crosses that control volume and the same is transmitted as torque to the turbine. And the T shaft or the torque in the shaft is negative. So, this is a kind of convention and which is also consistent with this equation which we have derived here. Now the next part, this actually brings us to the end of the 3 applications of the other 3 applications of the Reynolds transport theorem, the first one was mass, the 2nd was momentum and the 3rd was angular momentum. Now, we will before coming to the end of this particular chapter on integral analysis what we want to do is also look at the momentum conservation equation for a control volume with linear acceleration.

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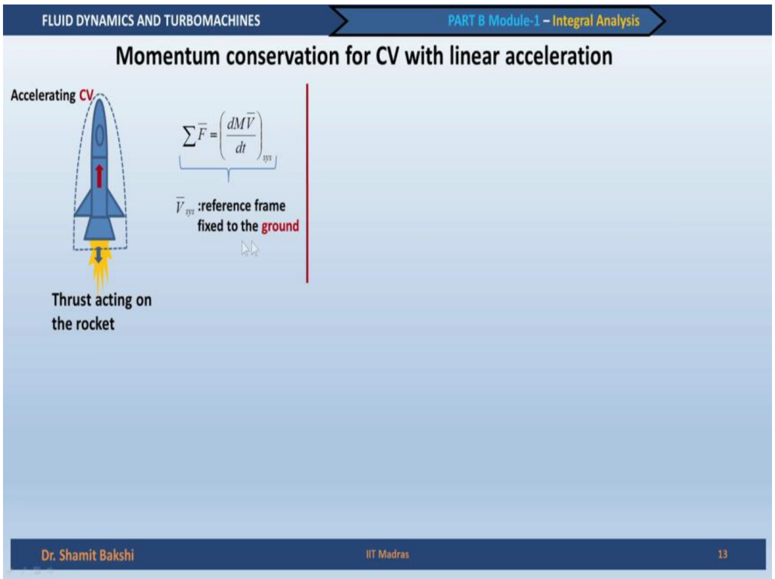

So, when we derive the momentum conservation, it was actually either for a control volume for a control volume moving at constant velocity. So, that form of momentum conservation equation is not applicable for an accelerating control volume. But in practical applications we often encounter situations where we have to use a accelerating control volume. In the derivation of Reynolds transport theorem we have assumed that the that our coordinate system for defining the fluid velocity is fixed to the control volume. If the control volume is accelerating, it means that the reference frame is also accelerating and such a reference frame is known as a non-inertial frame of reference. We will look at a situation where we have a linear acceleration, not a general acceleration but a linear acceleration of the control volume. We will see how the equation changes for that.

So, one such application is when we try to find using integral analysis the thrust acting on the rocket. While we try to do that, we actually put a control volume around the rocket and try to find the force but a rocket moves, it accelerates and the control volume attached to the rocket also have to accelerate. So, now we will see how to change our equations to accommodate this acceleration of the reference frame which is attached to the control volume. So, this is the case of an accelerating control volume. Before going into that, let us understand the reason for why we need to use a different equation. If we look at the, again look at the linear momentum equation, we see it as basically sum of all the forces acting on the system is equal to the rate of change of momentum.

Now this velocity of the system is basically the velocity with respect to the with respect to the control volume with or with respect to the reference frame with which we want to find the force. We will come into, come to that a little later. So, as you can see the velocity of the system is reference frame fixed to the ground. In most of the cases this reference frame is fixed to the ground because we want to find out what is the force respect to a ground fixed coordinate system. For example in this case of the rocket, we have to send the rocket out from the ground and we want to find out what is the force required to actually or what is the thrust required to actually accelerate the rocket away from the ground.

Now we can actually describe it in little more details that as to why the force acting depends on the coordinate system, the choice of the coordinate system. For example, if you take this rocket, then suppose for an observer who is sitting inside the rocket, he throws a ball upwards, so when he does that, if force is required to throw the ball upwards and that force is different than if a observer who is sitting on the ground has to throw a ball at the same speed but considering the acceleration of the rocket. So, if you at the ball which is thrown by the observer sitting inside the rocket from a ground fixed coordinate system the ball has an acceleration imparted by the observer inside the rocket as well as the acceleration of the rocket itself.

So, both these 2 will be required to consider to find the force required to lift the ball with respect to the ground. But for an observer inside the rocket, the force required to throw the ball is same if he sits in the rocket and throws the ball or if he sits in the ground and throws the ball. So, the force is actually dependent on the choice of the coordinate system. In most of the cases, most of our applications like I mentioned before, we want to find out the force with respect to a ground fixed coordinate system, so the velocity is also with respect to the ground fixed coordinate system.

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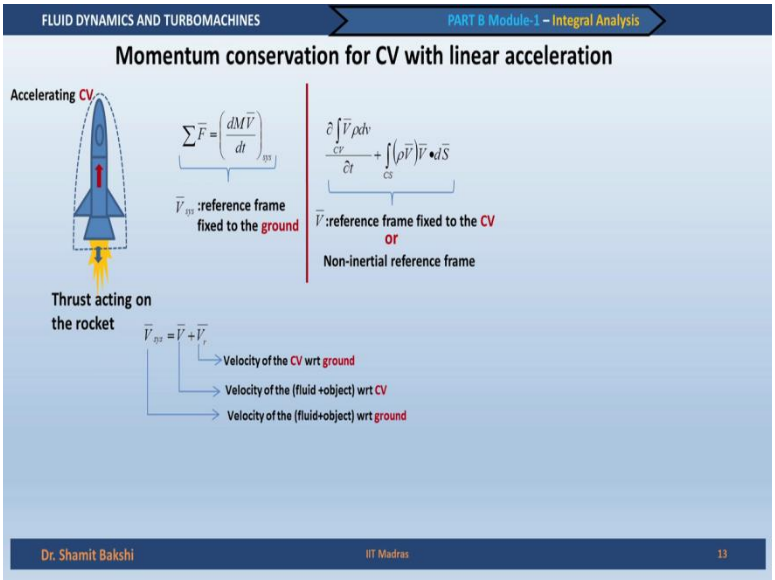

On the other hand if we look at the expressions in the Reynolds transport theorem, we see that the velocity which appears in the Reynolds transport theorem or in the expressions for the Reynolds transport theorem, the velocity is actually not with respect to the ground fixed coordinate system, it is a reference frame fixed to the control volume. So, for example in this rocket, the CV is actually moving under reference frame for defining this velocity, the fluid velocity is fixed to this control volume. If such is the case, then than these 2 terms now cannot be equated because they have velocity is defined on the left inside as the velocity with a ground fixed coordinate system on the right-hand side we have velocity is defined with respect to a accelerating reference frame which is fixed to the control volume. Or we can say it is a non-inertial frame of reference, the reference frame is accelerating with respect to the ground.

Now what is the relationship between these 2? They cannot be equated but there has to be some relationship between these 2, what is that relationship? If we can find that, then we can find out the modified form of momentum conservation for the control volume for the case of linear acceleration. So, this is V system if we look at can be written as the, so let us look at this, so this is velocity of the fluid and object with respect to the ground, so this V system is basically the velocity of this entire system with respect to the ground, this is the velocity of the fluid or object with respect to the CV. Okay. So, these 2 are related with the relative velocity. So, the velocity, this VR is actually the velocity of the CV with respect to the ground. So, it is like velocities, so this is, this relationship is true for any relative velocity expression that velocity of fluid + object with respect to the ground is equal to the velocity fluid + object with respect to the CV + velocity of the CV with respect to the ground.

So this is the relationship between the velocities. Okay. But here we can notice that although for the CV we write that it is the velocity of the fluid + object with respect to the CV but the object of velocity with respect to the CV is zero because the object is moving along with the CV with the same velocity as the control volume, only the fluid is moving at a different velocity than the control volume. How can you say that because if you sit on that control volume, then you can observe the fluid moving, so that says that the fluid is actually moving with respect to the control, accelerating control volume itself.

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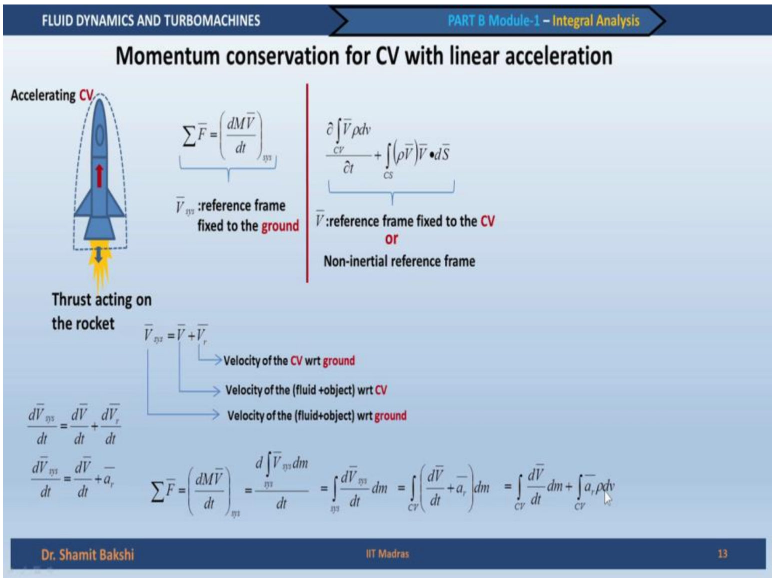

Now once we have this expression, we can, as you are dealing with derivatives here, let us take a derivative of this in both the sides. If we do that we get the expression that the derivative of the velocity with respect to the system is equal to the sum, the derivative of the sum of these 2 velocities. Now we can easily see here that if this velocity is constant, this relative velocity is constant, that means the CV is moving at a past that velocity with respect to the ground, then this term disappears, this dV R by dt is equal to 0. So, these 2 velocities, although V system and V bar are not same but these 2 derivatives are same. So, the same equation can be used only with the difference that this V bar is now the velocity with respect to the control volume.

We have demonstrated this before when we saw the example where the a plate was stationary and the plate was moving with a constant velocity, the value of the force changes. Now this situation is not like that, this situation is like that the VR is changing with time, it is not constant, that is the control volume is accelerating or decelerating, it applies to both. So, we write this further as the acceleration, this AR bar is basically the acceleration of the control volume with respect to the ground, so we write it in this form. Now let us look at this expression in little more detail that the force can be written as d M V bar rate of change of momentum or we can write this MV bar as integral of this V bar dM. So, what it means is dM is a elemental mass within the system and V system, V bar system is the velocity of that mass.

So the product of that if we take an integrate it over the system, it will give me the momentum of the entire system. So, it is written in that form, then we can actually take this derivative inside as the system is not changing with time, so just, if we take this derivative, then we can write it as dV system by dt dm, we did this so that we can utilise this expression, utilise this equation and bring in the velocity of the fluid + object or mainly the fluid with respect to the control volume, not with respect to the system. So, we can write dV system by dt as this, we can use this equation there, here and we can write in this form.

Of course once we write it in this form, we can actually take it as the CV because if you remember in our derivation of Reynolds transport theorem, at any point of time T where we are considering the CV and the system actually overlaps with each other. So, we write it in this form, we can further write it as dV bar by dt dm, this is basically rate of change of momentum of the fluid + object within the control volume with respect to the written in terms of the control volume and this is basically the acceleration of the control volume. So, and multiplied with the mass of the control volume, integral of the momentum of the entire momentum of the control volume.

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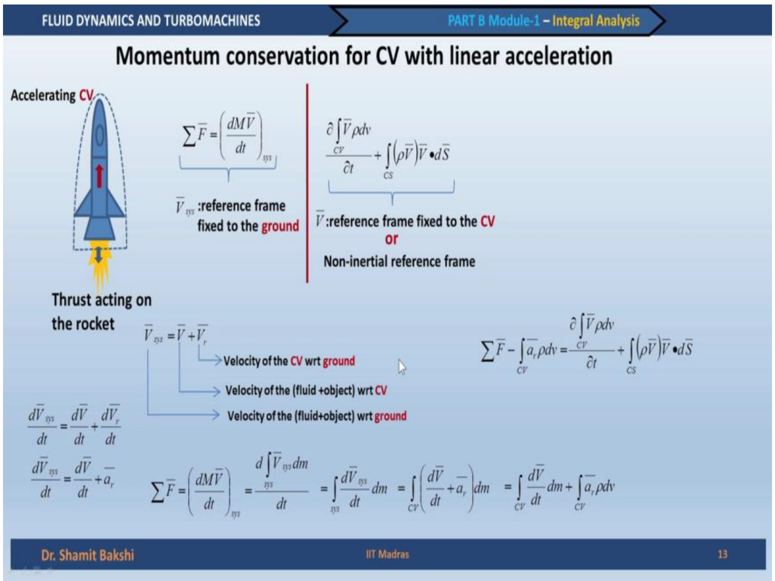

So finally we can write this F bar - this acceleration of the mass of the control volume as this expression. So, this is the modified form of the momentum conservation for a CV with linear acceleration. So, we do not say F bar is equal to this, we say F bar - this inertia force terms, this is actually an inertia for, if you see this is mass of the entire control volume, rho dV over the entire control volume multiplied with the acceleration. So, this is basically the inertia force coming in, so the net force acting - the inertia force, now is equal to the same expression. And these velocities are with respect to the control volume, reference frame fixed to the control volume. We will demonstrate this application of this equation further while we do a problem during our tutorial session.

So, this brings us to the end of the 3rd lecture which is the last lecture of the 2nd week of this course and we have in this lecture looked at the application of the Reynolds transport theorem for angular momentum conservation equation and we have also looked at how the angular momentum conservation can be simplified for the case of a turbo machine and we have looked that the Euler turbo machine equation. At the end of this lecture we have looked at the momentum conservation, the linear momentum conservation for a non-inertial, for a control volume which is more accelerating, that means for in the case of non-inertial frame of reference. Thank you.