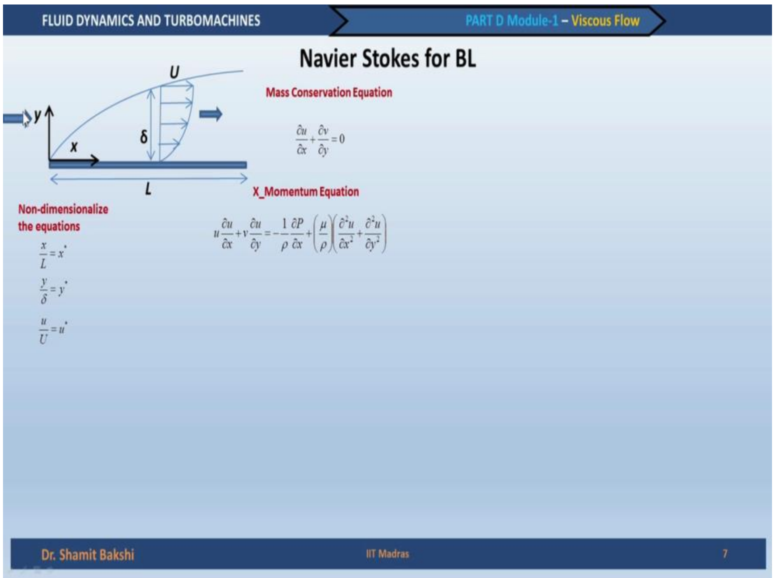
Fluid Dynamics And Turbo Machines.
Professor Dr Shamit Bakshi.
Department Of Mechanical Engineering.
Indian Institute Of Technology Madras.
Part D.
Module-1.
Lecture-2.
Viscous Flow.

Good morning and welcome to the 2nd lecture for the 4th week of this course on fluid dynamics and Turbo machines, we are looking at viscous flows. In the last lecture we had looked that the flow over a flat plate and defined different boundary layer thickness and we also looked at the momentum integral expression for a flow over a flat plate, derived the expression for the boundary layer thickness, the disturbance thickness in terms of the Reynolds number. In this lecture we are going to look at the same problem, the flow over a flat plate but starting from the differential analysis. In the last lecture we have looked at the problem more from the integral analysis point of view, in this lecture we will look at it in the differential analysis point of view.

(Refer Slide Time: 1:26)



So this also demonstrates how we can apply the different approaches of analysing fluid flows like the integral and differential approaches to a given problem. So today we will look at how to apply the differential approach to the flow over a flat plate. So basically when we say that we are going to apply the differential approaches, differential approach to this flow over a flat plate, it means that we are going to apply the Naviar-Stokes equation for boundary layer. So

BL stands for boundary layer. Whenever we use during discussion within this chapter this acronym it means boundary layer. So the mass conservation equation is Dell U by Dell X plus Delta V by Dell Y is equal to 0, this is well-known to us.

And of course this is a two-dimensional incompressible flow. X momentum equation is given as this and we have again taken two-dimensional incompressible and a steady flow, the unsteady flow, unsteady term is also removed from here because this is essentially what we are dealing with is a steady state boundary layer. You can have a unsteady boundary layer also but we are not dealing with that in this discussion. So this is a steady state boundary layer, that is a steady flow on the surface of the plate and what is the, how does the boundary layer forms.

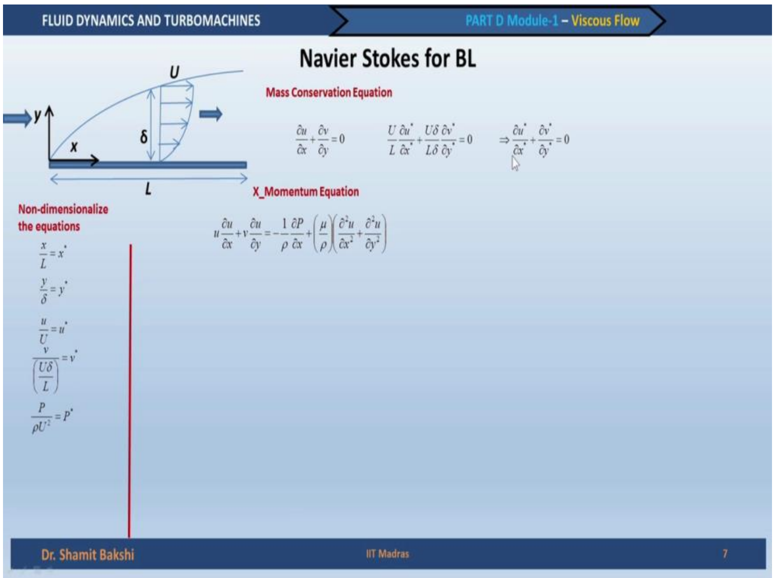
So basically this is the equation for that, these first 2 part is the convective part of the acceleration and what we have done, we have taken the rho from here and inserted it here and returned, this is the pressure gradient coming from the pressure forces and this is the viscous term. So basically this is our X momentum equation. Now what we do is in the next slide we will see how we can apply the Y momentum equation for the boundary layer flows. Let us see how we can reduce this X momentum equation for the case of a boundary layer flows.

In that context it is very useful actually to write this equation not in the form of a dimensional variable like velocity, each quantity here has a dimension, that means U velocity meter per seconds, pressure as Newton per metre square and so on so forth. So all the quantities which we are going to solve has a dimension, we want to express this as a non-dimensional quantity. And we will see how that will help us to reduce this equation to a more simplified form. So this is another way, I think you will be introduced to non-dimensional, non-dimensionalization process or dimensional analysis in the next part of this course, in the Turbo machines part of this course where it is more applicable.

But this is an application of that dimensional analysis to deal with the differential equations. So to non-dimensionalize the equation, what we need to do is basically we have to divide each of these terms which we are going to solve. So velocity has to be divided by a velocity to make it non-dimensional, it is clear. Similarly length, X or Y has to be divided by another length, so what is that length? See one when we do non-dimensionalizing, we should basically try to choose an appropriate quantity to non-dimensionalize.

So when we talk about U velocity, so we will come for the U velocity R, let us see the length scale for the X length, quantity which can be used to non-dimensionalize the X length is capital L because that is the length along X direction. So we and we name this as X star, the non-dimensional quantity as X star. Similarly we can define a non-dimensional Y star. But for that the length should not be this L because if you go in Y direction what you see, the significant length is Delta, the disturbance thickness. So we should define, we should divide that with Delta. That is the trick which you have to use while non-dimensionalizing an equation and you have to select the the appropriate length scale and velocity scale.

(Refer Slide Time: 5:25)



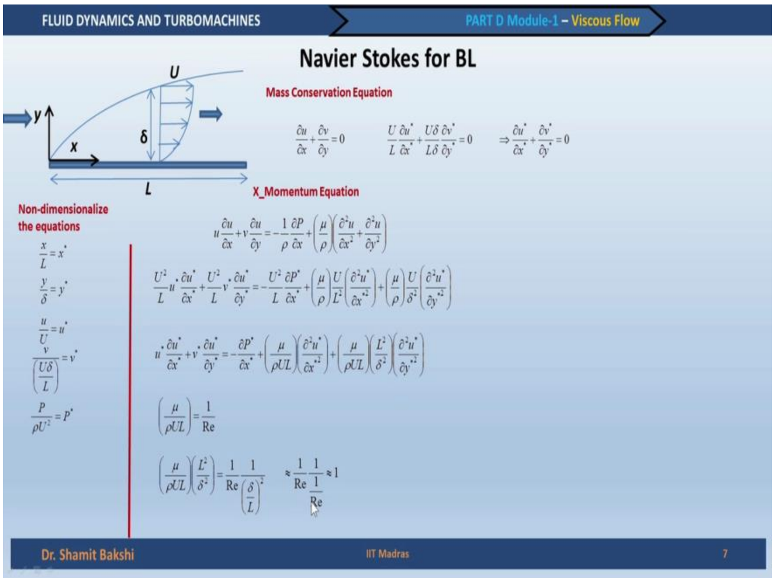
So this is done, so we have to replace Y in this equation with Y star, we will see how to do that but we will let us see other scales also. So for U velocity, again it is straightforward because the uniform velocity which is approaching the plate is capital U, so we non-dimensionalise non-dimensionalise it with capital U. And name this velocity, nondimensional velocity as U star. V is little tricky and we define this quantity U Delta by L, how is this arrived at?

You can see if we can do a scaling of this equation, the mass conservation equation, you see this is capital L, you replace this small u this with capital U and this with capital L and this with let us say some V and this with Delta, then the velocity scale which you will be left out with is capital U multiplied by Delta divided by capital L. So that is what **we** has been used to non-dimensionalise V velocity, U Delta by capital L obtained from the dimensional consistency of this equation. So we see this as V Star, now I think we are all set to write this equation in terms of this nondimensional variable now, that means X star, Y star, U star and V star.

Of course, okay we have not addressed P, so P has to be non-dimensionalised with rho U square with the severe it because explicitly there is no pressure term appearing here, so just use the same dimensional quantity rho U square to non-dimensionalise P and name this as P star. When you non-dimensionalise an equation, every quantity has to be nondimensional, so that is important. Now let us replace the quantities like velocity, pressure, etc. and the length scale in this equation with this nondimensional equation variable. For doing that what we should do is, we should write X here as X star into capital L.

Capital L will not do anything to this equation because they are all constants, similar, similarly all other variables. So if we do that now, now we can write these equations in this way, so before going here we have actually non-dimensionalise the mass conservation equation, although it was not necessary because the velocity scale was arrived at by using this equation. So because of that you will see the quantity equation remains in terms of non-dimensional variable is also U star, X star, V star and Y star remains in the same form as with the dimensional quantities because of the selection of the velocity scale like this.

(Refer Slide Time: 8:01)



Okay so let us come to the X momentum equation now. So this expression is what you get but we will go term by term for this equation and try to understand how it is written. So for small u, you can replace it with U star into capital U. So U is replaced with U star into capital U, similarly U star into capital U here, so one U comes from this, another from this, it becomes U square and X, instead of X, we write X star multiplied by L. So L can be taken outside, so L comes outside here, so you get a coefficient U square by L here.

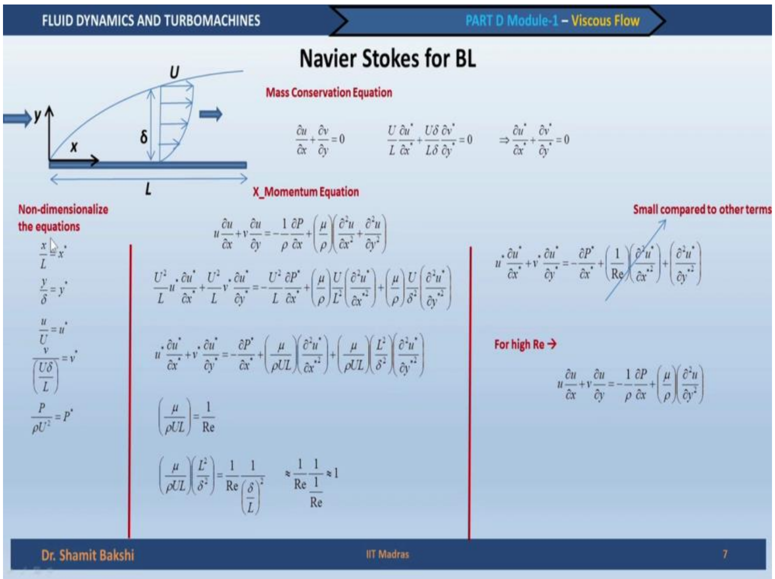
Similarly if you do for the V velocity you will get a U square by L coefficient here and same for your pressure gradient term. Here also you can apply the same technique, you replace U with small u star multiplied by capital U, capital U comes outside, U remains here and X, with X star multiplied by capital L and then this comes outside because this is X star square, so it is L square here, so you can now club these variables. So this is how, this equation is arrived at.

Now club these variables, see one thing you can do before going ahead, if you look at this equation carefully you have U square by L here, U square by L here as coefficient and also here as coefficient. So you can actually divide this, the left-hand side and the right-hand side of this equation by U square by L. So if you do that then what do you get? You get very similar for as the original Naviar-Stokes equation or the X momentum equation up to this up to this point. So this is very similar, here you get little different terms. So with the Dell 2 U star, Dell X star square, you get mu by rho U L, here you get mu by rho U L multiplied by L square by Delta square.

So this is just by organising all the terms together you get this kind of expression, and see it is nicely organised actually. Why is that, because see rho UL by mu, it is basically Reynolds number. For this flow over a flat plate, the Reynolds number at this edge is basically rho UL by mu. So we get that expression here. So this can be replaced by mu by rho UL can be replaced by one by RE, this also can be replaced by one by RE, so what we are left out with is this L square by Delta square. So let us see if we can write something in terms of Reynolds number for this quantity.

So before going into that we, okay, so we have taken this entire term here and put it here, that is mu by rho UL multiplied by L square by Delta square, we put the entire thing here, this part is one by RE like here, so this is one by RE, this part is one by Delta by L square. What is Delta by L square? If you remember our last lecture, we derived an expression for Delta in terms of X. So Delta by X is basically proportional to 1 by square root of RE. So Delta square by X square is one by RE or Delta square by L square is actually one by RE where RE is defined with respect to the length scale L.

(Refer Slide Time: 11:28)



So we can use that here. So this is one by RE and Delta square by L square is actually proportional to 1 by RE. So if you use Von Carmen relationship, then it is 5.48 by RE, okay. So what it actually tells us, it gives a order of magnitude value for this expression, for this entire expression. Order of magnitude value, what does it mean, it means whether it is 10 to the power zero or 10 to the power one or 10 to the power 2, like that. So if we look at this term, because you have only 5, you will even in the Von Carmen expression you will have 5.48 by RE here, so this will be of the order of zero.

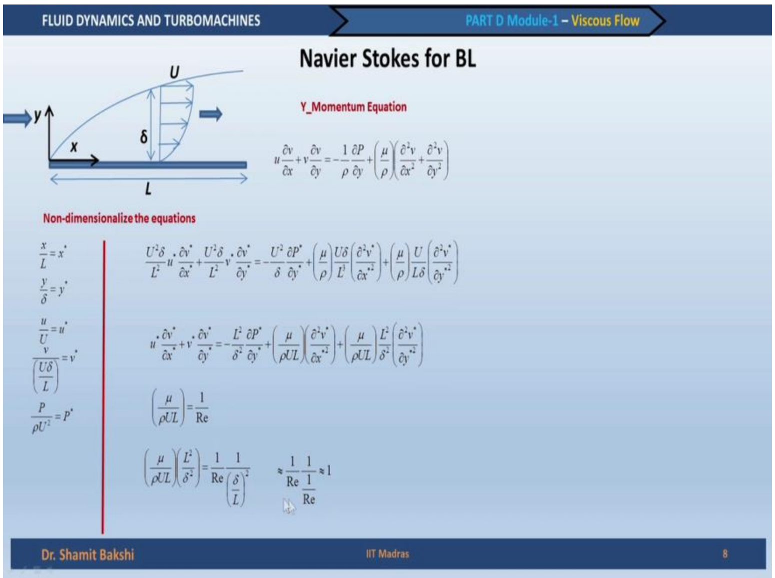
So the magnitude will be of the order of 1. So we can use these 2 information and rewrite this equation. So now if you rewrite this equation, what we get, this is multiplied Dell 2 U star Dell X star square is multiplied by 1 by RE and this is the coefficient here is of the order of 1, the value is of the order of 1, the order of magnitude is zero. Okay, so now let us see what it means for high Reynolds number. So generally the Reynolds number will be high, so if you remember the value of Reynolds number at which transition occurs is 300,000, 3 into 10 to the power 5, that is very nice.

So even if you leave that, if at least even for a your laminar flow we will be dealing with Reynolds number of the order of 100 and this Reynolds number which appears here is not the Reynolds number at a particular station, it is the Reynolds number with the length scale of the length of the plate. So this is basically Reynolds number of rho U capital L by mu. So for any, if it is a very small plate, it becomes like a small object, for that of course this equation does not apply but for any finite size plate, you will see the Reynolds number will come out to be 100 or so.

If that is the case, this is, this particular term is multiplied by 1 by 100 whereas the other terms as it is not multiplied, are multiplied by 1, so this term naturally is negligible because it is 100 times as it appears here of the other term. So this is not zero but it can be, it is small compared to the other term because of the multiplier of one by RE with this particular term. So this can be removed and we can write this equation. So this is a modified, so by doing this analysis what we get is a reduced form of this differential equation.

See know we are trying to apply the differential approach to the boundary layer problem and for doing that we need to deduce the equation in a form that we can solve it easily. So this is that effort actually, with that effort we try to non-dimensionalize this equation and arrive at the X momentum equation of this form. So now let us look at the Y momentum equation.

(Refer Slide Time: 14:30)



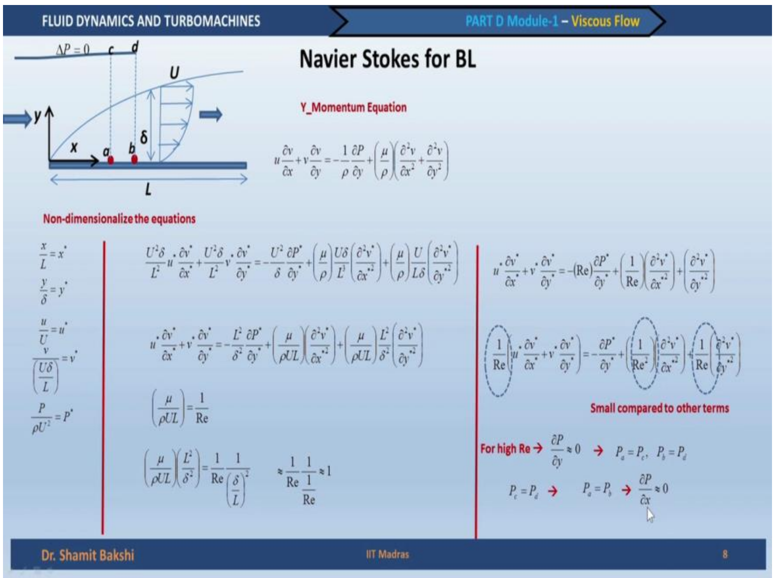
So Y momentum equation in its full form for a 2-D steady incompressible flow is like this. Okay unsteady term is not there, of course it is a steady boundary layer and incompressible flow. So now this is the full-fledged equation for a 2-D unsteady incompressible flow. So now non-dimensionalize the same equation as we did before, the same approach we adopt, X with L and name it as X star, Y with Delta and name it Y star, small u with capital U, V with U Delta by L and P with rho U square. So again we can use these expressions to rewrite the Y momentum equation in terms of the nondimensional parameter, namely U star, V star, X star, Y star, P star, so let us do that.

We already introduced the methodology for that that what we we do in this equation, we can write small u as capital U into U star and small v as this multiplied by V star, so U Delta by L multiplied by V star, so if we do that, what do we get, we get an expression like this. So just to look at the first term if you see we have replaced U with U star multiplied by capital U, so one U comes from here and the V star V here inside the differential with is replaced with V star into U Delta by L. So one U came from U star, another U Delta by L comes from the V star.

And this X star, X is replaced by X star into capital L, so another L comes from there. So that is how it becomes U square Delta by L square, so this is the coefficient of the first term. The 2nd term will also have the same coefficient if you follow the method which was introduced. So if you continue these are, like this, these are the expressions which you will get, again we divide the left-hand side and right-hand side with U square Delta by L square, we get this equation. So once we get this equation now, again we see that these terms can be organised in this form.

So the coefficient of Dell 2V star square Dell X star square is mu rho UL and here it appears like this, very similar to what we got before, except for the fact that the pressure gradient term also have this expression now, L square by Delta square. So this can be resolved in the same way as we did before, mu by rho UL is one by RE, okay, and then mu by rho UL into L square by Delta square which appears as a coefficient here is of the order of 1 as similar to before and L square by Delta square is of the order of RE because L square but Delta square if you see, L square by Delta square has appeared here, it can be written, it can be written as one by Delta YL square that is 1 by 1 by RE, that is RE essentially.

(Refer Slide Time: 17:45)



So L square by Delta square is of the order of RE, so now let us replace these expressions with this particular expressions which we got here. We can rewrite this equation in this form, the RE is the multiplier here and the 1 by RE is the multiplier here. So what we do is, we divide the entire equation with RE again, so if we do that we get this. So now see for high Reynolds number, again this term is multiplied, this entire term is multiplied with one by RE, this term is multiplied with one by RE square and this is multiplied with one by RE. See if every term was multiplied by one by RE, then we could not have done anything.

But fortunately we see that only that these returns are multiplied with, 2 terms with one by RE and one with 1 by RE square. So compared to this term, naturally these terms will be negligible because even for Reynolds number of 100, 2 of these terms are multiplied with 10 to the power -2, one of the term is multiplied with 10 to the power -4. So all of them can be neglected compared to the term which is not multiplied with any RE factor, so that is the dominant term in the equation.

So that is basically the advantage of using this dimensional analysis or writing the equation in a nondimensional form. If you do that, then you can see which terms will play more important role in an equation and you can reduce the form into more simple simplified form and you can solve it easily. So for example in this particular case you will see it becomes very simple. Now you are only left out, we began with this entire equation. This is not solvable, this is very difficult to solve but just by doing this nondimensional, just by non-dimensionalising this equation what we got is that for high RE this is the equation.

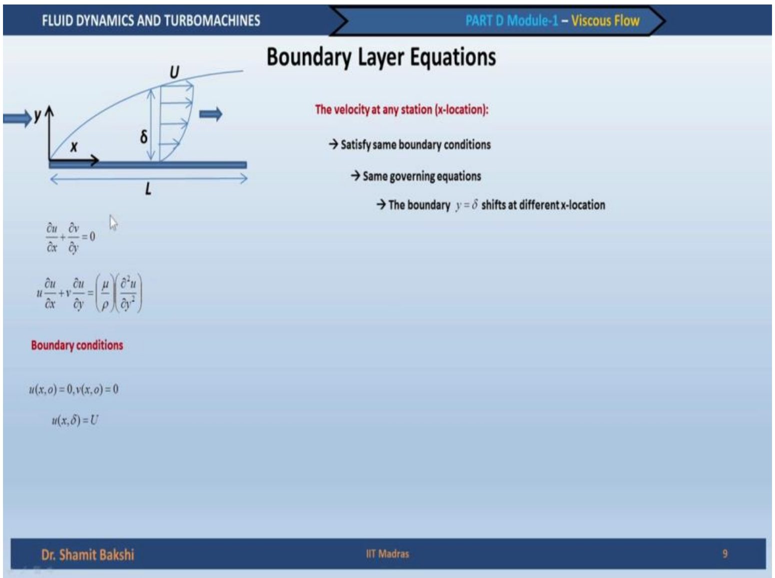
Dell P by Dell Y is equal to 0 is equation, this is the equation which you have to solve, which is very easy to solve of course. And now we will come to a very important conclusion from here. So that is Y momentum equation gives us a very important conclusion about the boundary layer characteristics and what is that, that is for explaining that we take 2 points on the top of the plate. So you have this point A and point B on top of the plate. What we know from this equation is Dell P by Dell Y is equal to 0 within the boundary layer.

So if we use this, then and draw a streamline outside, okay we draw a streamline outside and extend these points to the outer streamline, to the point C, A to C and B to D, now within the boundary layer Dell P by Dell Y is zero, outside the boundary layer also it is zero, so that what does it mean? It means that if we do this, then PA is equal to PC, pressure at A is equal to pressure at C and PB is equal to PD, pressure at B is equal to pressure at D. Now we already saw for the case of this flow over a flat plate, okay, Delta P along the streamline is zero because Delta U is zero.

And using Euler equation for an inviscid flow, of course this flow outside the boundary layer is inviscid flow, so you can use Euler equation and if Delta U is zero, Delta P will be zero. So that means PC is equal to PD, so that means PC is equal to PD. So if that is the case and you can use these equations now, what you get is basically PA is equal to PB. And you can do this for any set of 2 points drawn within the boundary layer. So what it essentially means is within the boundary layer Dell P by Dell X is 0 for the flow over a flat rate.

So but this is quite general, Dell P by Dell Y is zero, it can be extended to flow over curve surfaces also but we will not go into that, we will see here is simply that whatever we actually assumed in the last lecture that Dell P by, the same pressure gradient outside the boundary layer is imposed in inside the boundary layer is actually correct, it can be mathematically proved like this. So Dell P by Dell X is 0 for a flow over a flat plate and this has a very important application which we will see in our next lecture.

(Refer Slide Time: 22:04)



That what what it actually means, that pressure gradient has a very important role to play on particularly with relation to viscous flows. So we will see that how that happens, now see what we got is the, we got the equations, reduced form of the equations, so we got the finally of course this is our mass conservation equation, continuity equation, this is our X momentum equation and that is our Y momentum equation just give us Dell P by Dell Y is equal to 0 and we use that for also showing that Dell P by Dell X is equal to 0 and that is how you see in the X momentum equation here now Dell P by Dell X we have removed because that is zero.

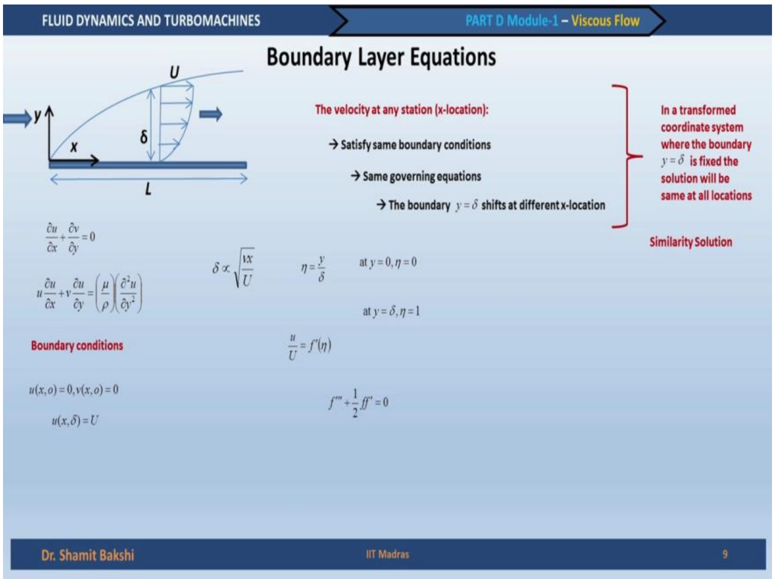
So this is basically the reduced form of this equation. And let us see if we can solve this equation, these 2 equations together, U and V are the 2 unknowns and these 2 are the equations, so can we solve it. Still you see that it looks difficult to solve being a partial differential equation but if we make some observations, it becomes easy to solve these equations. So we will see what observations we can make.

Before going into that, let us see these are the boundary conditions that U at X and Y is equal to 0 will be zero, that means if you move along this X at any point U is zero, same is V. That basically no-slip condition applied on the wall and no velocity perpendicular to the or the same velocity as the wall perpendicular to the wall. The wall is stationary here, so this is zero. So this is one boundary condition at one edge and the other edge is UX, Delta is equal to U, free stream condition. By applying this condition we can actually solve this equation.

But let us see what more simplification we can do to this particular set of equation. That can be done if we make some few observations about this equation. So let us consider a velocity at any station, so by seeing station what we mean is any X location. You have any X location, what can you say about the velocity. So what we see, it is it comes from, of course we can see this velocity comes by solving these 2 sets of equations, mathematically it is that that is what it is.

So at each point it actually we solve the same equations, the same set of equations. So it seems and it satisfies the same boundary condition, at each point the boundary condition is also same. So governing equation is same, for the velocity at any station, the governing equation is same, this, these 2 are the governing equations, the boundary conditions are also same. So same boundary conditions, same governing equations, so solutions should also be same right, but it is not. Why it is not? Because the boundary Y is equal to Delta shifts at different X location, the Delta is actually shifting.

(Refer Slide Time: 25:05)



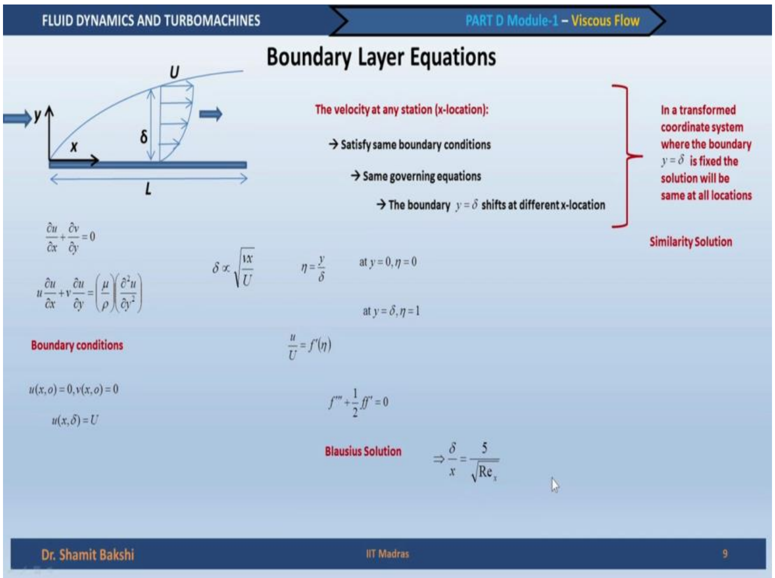
If you move along the X, the Delta is different, so that is basically the problem. The governing equations are same, boundary condition is same but the boundary is shifting, so how to solve that? We can solve it very easily is we can make a small observation. What is that? So if we look at a transform coordinate system. How do we transform, that means we do not have to write this equation in terms of Y, we write it in terms, in a different coordinate system where this Y is equal to Delta, this boundary becomes fixed, it does not become, it does not shift, it does not shift to a new position for each location, if it becomes fixed, then all the points will have the same solution because there is the same governing equation, same boundary condition.

Only boundary has to be fixed. So this approach is called, is seeking a similarity solution, the profile is actually similar but it is not appearing similar because it is not written in the correct form. If you write it in terms of correct variables, it will be similar. So it will be same actually but because it is not same it is called similar. So this is basically the similarity solution, so how to see that kind of a approach. The only task now remaining before us is to write this equation in a transformed coordinate system in such a way that this boundary becomes a fixed boundary.

So we define, so Delta of course we know, basically it is a function of X, so it is changing at every X, so we define a new variable called Eta. What is this Eta? Eta is defined as Y by Delta, so see this is defined in an very, in a way that it satisfies this criteria that at Y is equal to Delta, the boundary becomes fixed, because at Y is equal to 0 Eta is zero, if you just plug-in Y is equal to 0 here, at Y is equal to Delta, you plug-in Y is equal to Delta here, Eta becomes one and this is true for all the Deltas, all the X location.

So this writing the equation in terms of the new variable Eta actually transforms this boundary which was changing, that means that U X is equal to Delta is U to a fixed boundary. So now what you can get, you can write the velocity profile in terms of a single variable Eta, you do not have to bring in X now and this equation becomes, if you have single variable for a differential equation, single independent variable for a differential equation, it becomes an ordinary differential equation. And this equation with this substitution becomes a 3rd order ordinary differential equation given as this.

(Refer Slide Time: 27:52)



And this is known as, this approach was developed by Blaussius and this solution is called a Blaussius solution. Using this Blaussius solution, now of course given this equation and the boundary condition has to be written in terms of this variable F which is not written here but it can be written, we just want to introduce the approach. So now this, if you solve you cannot get a analytical solution to this equation also. You can get a numerical solution to this ordinary differential equation and if you find a value of Delta by X using this, you get 5 by square root of Rex. Using Von Carmen approach we got a value of 5.48 by square root of Rex which is very close to what we got.

The difference is only because of the fact that the velocity profile is not quadratic which was assumed to be quadratic by Von Carmen. So this brings us to the end of the 2nd lecture of the 4th week of this course. In this lecture we actually looked at how to deal with the flow over a flat plate using a differential approach. That means starting from the Naviar-Stokes equation and how to reduce the Naviar-Stokes equation using a nondimensional, using nondimensional variable or by non-dimensionalising the equation.

So if you write that, you actually drop lot of terms and then you can write equations in a very in a more simplified way and we also demonstrated that how to approach towards a solution of this kind of a reduced equation using the Blaussius, using the similarity solution approach introduced by Blaussius. So this brings us to the end of the 2nd lecture, in the next lecture we will look at flow, so far we have looked at flows where the pressure gradient outside the boundary layer was zero, that is a essentially a flow over a flat plate.

That also means like we introduced here that the pressure gradient inside the boundary layer is also zero. In the next lecture we will look at flows where the pressure gradient outside the boundary layer and so and that is why inside the boundary layer also is non-zero and what happens there. Thank you.