Fundamentals of Gas Dynamics

Dr. A. Sameen
Department of Aerospace Engineering
Indian Institute of Technology, Madras

Week – 11
Lecture – 44
Prandtl-Meyer flow

In this lesion, we are going to learn Prandtl-Meyer flow, which is essentially a turning that we are going to make isentropically. So, in the oblique shock, we have seen we the flow can be turned,, but there is also an entropy loss, entropy increase. Now what we are going to do is, is there a possibility of turning a flow without an entropy change and such flows are called Prandtl-Meyer flow.

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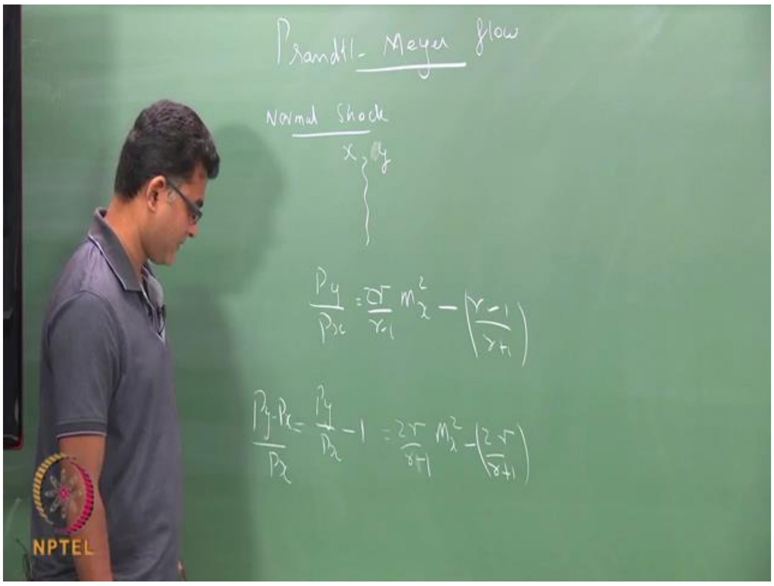

And we are going to discuss some equations related to that. So, our aim is to turn a flow without an entropy difference. So, I am going to rewrite as some of the equations that we have done for normal shock. So, whatever that I am going to write now we have already done the derivation, I will just write down the equations.

So, if I have a shock, after the shock is y, before the shock is x, then I have P y by P x 2 gamma by gamma minus 1 M x square minus gamma minus 1 by gamma plus 1, which I can rewrite in the following way I subtract minus 1 from both sides. So I get minus 2 gamma by gamma plus 1. This is nothing but P y minus P x divided by P x, so this is the delta pressure rise due to your shock.

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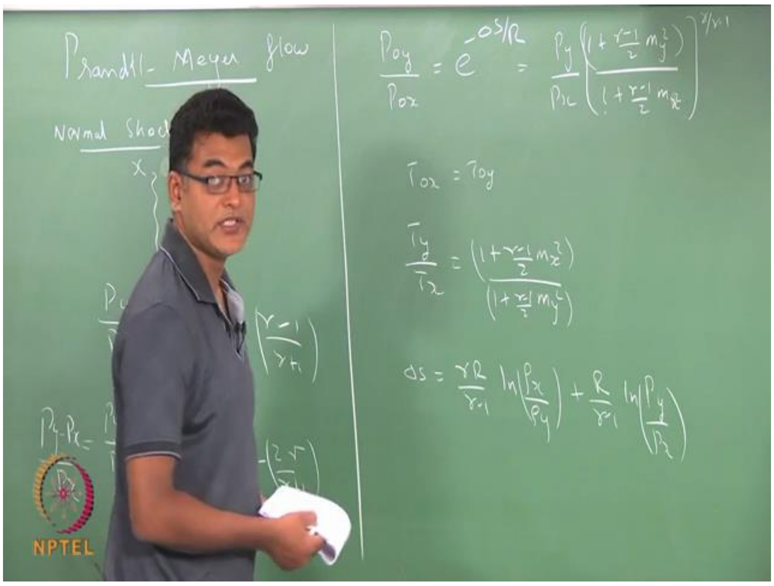

And we have also seen for an flow with entropy change, you can write this in this particular form. And I can replace P 0 y in terms of P y and mach number at that particular equation x square into gamma by gamma minus 1. I also know that across the shock, so I can also write T y by T x to be 1 plus gamma minus 1 by 2 M x square divided by 1 plus gamma minus 1 by 2 M y square. And some Gibbs relation delta S is gamma R by gamma minus 1 into ln rho x by rho y plus 1 into ln P y by P x. So, these are some of the relations that we had derived a very earlier in our course. And I am going to use that to prove that when we have small turn my entropy change is going to be 0.

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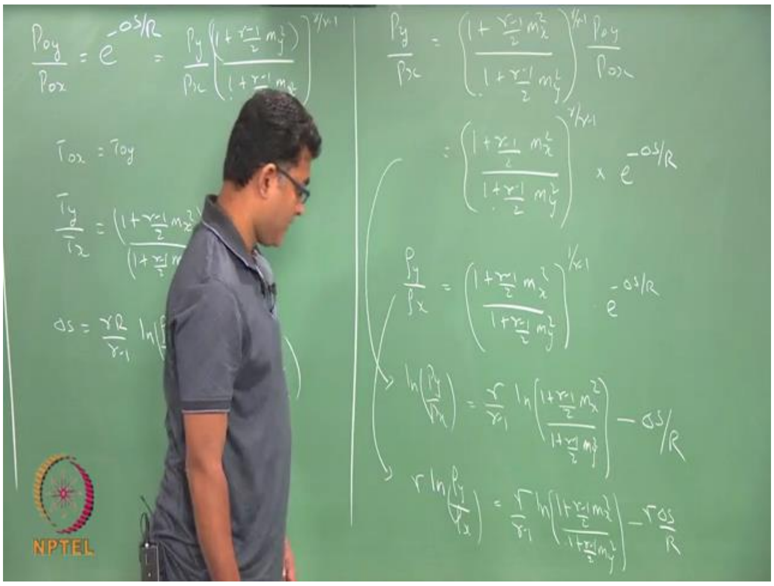

So, I rewrite my P y by P x as 1 plus gamma minus 1 by 2 M x square divided by 1 plus gamma minus 2 by M y square into P 0 y by P 0 x P 0 y by P 0 x is nothing but I have an exponent there so e power delta S minus delta S by R. So, I rewrite this equation minus 1 into e power minus delta S by R. Likewise I have also derived rho y by rho x which is 1 plus gamma minus 1 by 2 M x square divided by 1 plus gamma minus 1 by 2 M y square to the power 1 by gamma minus 1 into e power minus delta S by R. Now, I am going to use this to find delta S by R. So, if I take the logarithm here natural logarithm here, I can rewrite the first equation as ln P y by P x equals gamma by gamma minus 1 into ln of this term which is 1 plus gamma minus 1 by 2 M x square divided by 1 plus gamma minus 1 by 2 M y square minus delta S by R.

Likewise take the natural logarithm here, and multiply by R. So I multiply by gamma, gamma into ln rho y by rho x, so this would come here taking the logarithm here and here and multiplying by gamma. This would get me gamma by gamma minus 1 ln 1 plus gamma minus 1 by 2 M x square divided by 1 plus gamma minus 1 by 2 M y square minus gamma delta S by R. So, I have taken the natural algorithm here and multiplied by gamma on every term. Now, I subtract these two equations, so this equation 1, equation 2.

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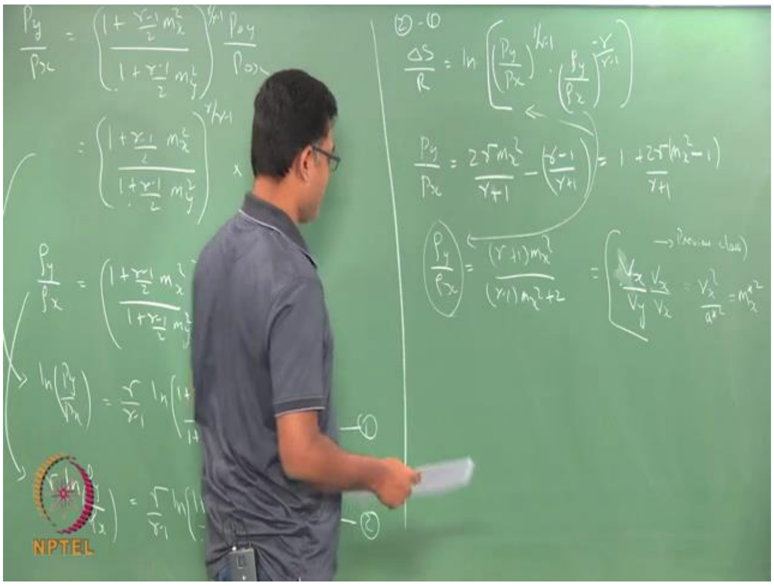

So, 2 minus 1, I would get my delta S by R as ln delta S by R as ln P y by P x to the power 1 by gamma minus 1 into rho y by rho x to the power minus gamma by gamma minus 1. So, I have subtracted this equation from this equation, and so these term these term cancel out. So, what is left is this term and this term and this term and this term. And I can reduce it to in particular form. And from here, I substitute our relation that we have written before my P x by P y is 2 gamma M x square divided by gamma minus 1.

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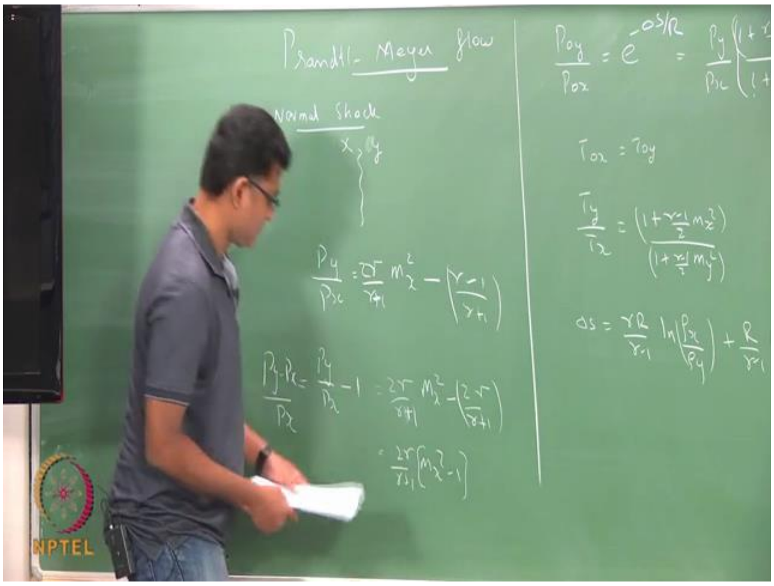

So, there is a correction here, this is gamma plus 1. What we have derived few classes early, this is gamma plus 1. So, this would be gamma plus 1 which I can rewrite this as gamma R by gamma plus 1 into M x square minus 1, so this is gamma plus 1 so this correction is gamma plus 1. So, what I have here is gamma plus 1 minus gamma minus 1 divided by gamma plus 1 which I can again modify as 1 plus 2 gamma M x square minus 1 divided by gamma plus 1. So, I substitute this here.

And rho y by rho x as gamma plus 1 into M x square divided by gamma minus 1 into M x square plus 2, this you have derived from V x by V y which we multiplied it by V x and V x. So, this is V x square by a star square so this is my M x star square which is what you have written here. So, this is something which we have derived it in the previous class. All this relation we have derived it in previous class, but I am now rewriting everything to get a delta S in the particular form. So, I will substitute rho y rho x here and P y by P x here.

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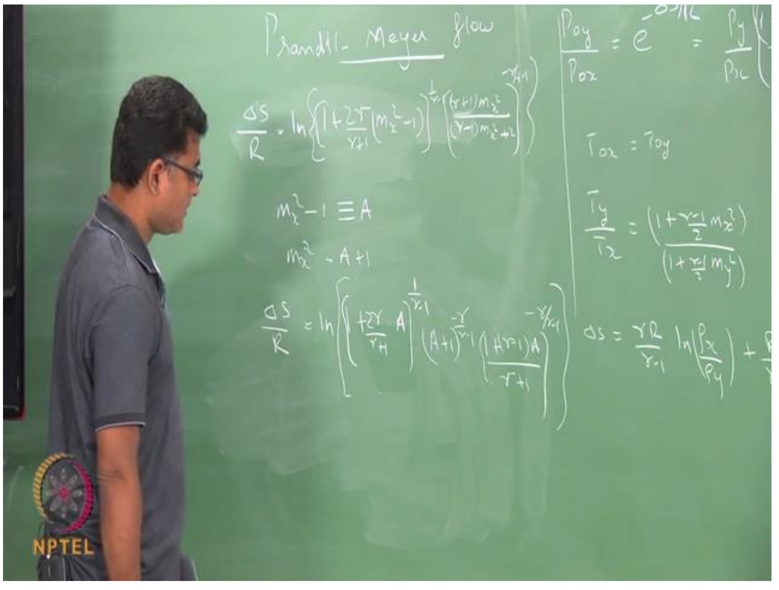

So I end up with relation on the following form ln 1 plus 2 gamma by gamma plus 1 M x square minus 1 to the power 1 by gamma minus 1 into gamma plus 1 by M x square divided by gamma minus 1 M x square plus 2 to the power minus gamma by gamma minus 1. Now I substitute M x square minus 1 as a quantity A, so M x square is just A plus 1, I rewrite M x.

And this thing for convenience in terms of A, so I would get delta S by R equals l n 1 plus gamma 2 gamma by gamma plus 1 into A to the power 1 by gamma minus 1 into A plus 1 to the power minus gamma by gamma minus 1 into 1 plus gamma minus 1 into A divided by gamma plus 1 to the power minus gamma by gamma minus 1 whole inside the natural logarithm.

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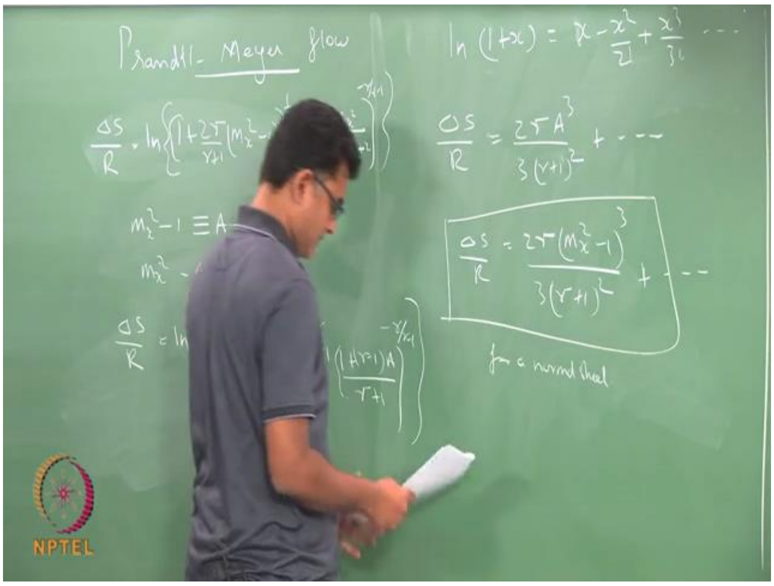

Now I use the binomial expansion of natural logarithm ln 1 plus x that is 1 x minus x square by 2 factorial plus x cube by 3 and so on. So, if I substitute that expansion here for delta S, my delta S by R would be 2 gamma A cube divided by 3 gamma plus 1 the whole square plus the other terms. We will worry about only the first term because that itself is M square minus 1, so which is nothing but 2 gamma M x square minus 1 the whole cube by 3 gamma plus 1 the whole square plus the other terms. So, this is for a normal shock.

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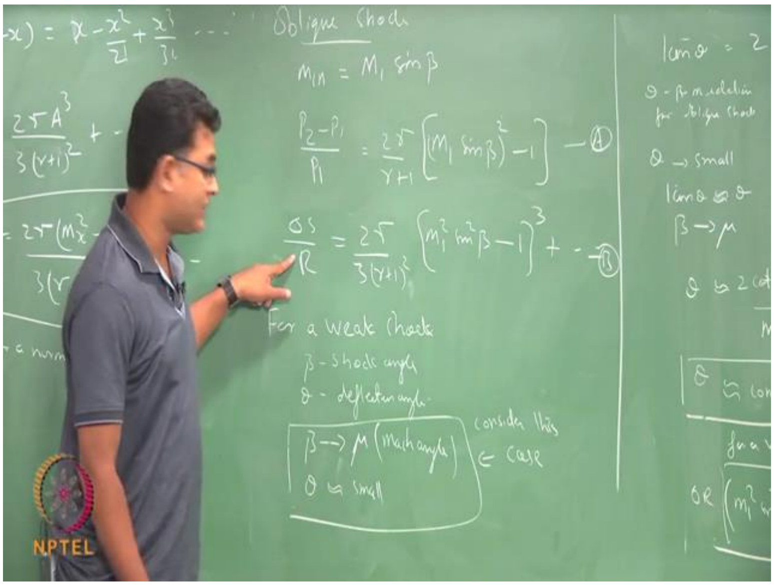

Now, let us turn to the oblique shock. So, my delta S is proportional to this particular term or the first order of change in my entropy is this. Oblique shock, so we are used instead of x and y, we are used 1 and 2, so it is M 1 sin beta. And our relation P 2 minus P 1 by P 1 which is now modified to this particular form, instead of M x square minus 1 now we have M 1 sin beta square minus 1. So, my delta S by R is also modified 2 gamma by 3 gamma plus 1 the whole square instead of M x square now I have M 1 square sin square beta minus 1 the whole cube plus other terms.

Now, we are going to discuss our Prandtl-Meyer relation. Now if we have derived this equation from whatever we had learnt, so it is just the rearrangement of whatever algebra that we had done in the last few weeks we get this particular relation for an oblique shock. Now for a weak shock, weak shock, so we have defined our beta to be our shock angle and theta to be our deflection angle. So, I am going to have a very weak shock. My beta approaches the mach angle which is mu which is something we have already seen. So, the limit of beta is 90, which are your normal shock and the mach angle mu.

So, now, I am going to discuss a case where your shock oblique shock is approaching the mach angle. And if my deflection angle is very, very small, so I am going to consider a scenario this. So, I am going to use my oblique shock relations in this particular limit where my beta approaches the mach angle and deflection angle is very, very small.

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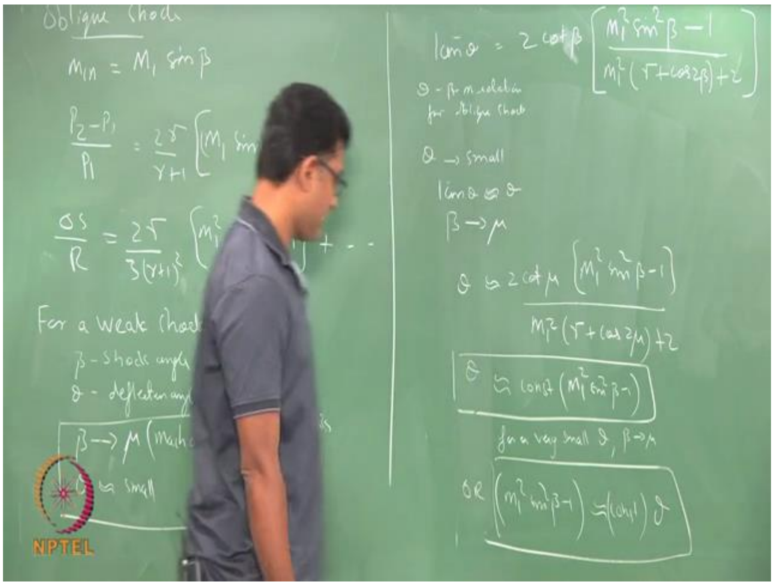

So, in that case we have written tan theta to be 2 cot beta into M 1 square sin square beta minus 1 divided by M 1 square into gamma plus cot 2 beta plus 2. So, this is our so called theta beta M relation for an oblique shock. So, theta is very, very small as the case we are going to consider here, theta very small, my tan theta also is approximately theta, and my beta approaches mu. So, what I want to consider here is that if I look at my pressure change and the entropy change, I have a term sin square M square sin square beta minus 1 how does this term relate to our deflection angle is what I am going to consider, within the limit of a small very small deflection angle.

So, when this happens my theta is approximately 2 cot mu, so I will leave this quantity as it is because this is what we need to compare with our deflection angle divided by M 1 square into gamma plus cos 2 mu plus 2, or this is a constant into M square sin square beta minus 1. So, when I have a small deflection angle it is just a multiplication of some constant into this particular factor or my M 1 square sin square beta minus 1 is a constant into theta. So, this quantity M 1 square minus M 1 square sin square beta minus 1 is just the constant into theta if theta is very, very small and beta is approaching your mach angle.

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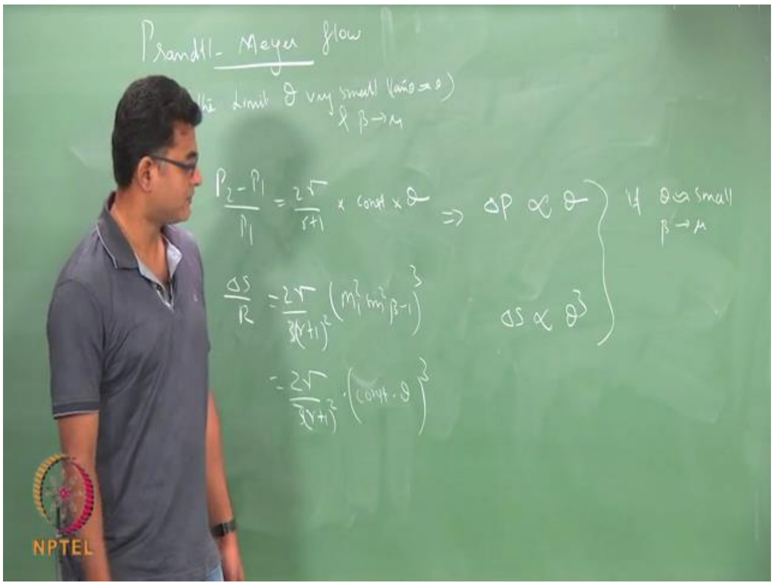

Now, I am going to substitute that here in these two equations - equation a and equation b that is my pressure change and my entropy change. So, what I do is I take that and I substitute that here. So, this is within the limit theta is very, very small where I can approximate tan theta to be theta and beta approaching mu, I can write P 2 minus P 1 by P 1 equals 2 gamma by gamma plus 1 into M 1 square sin square beta minus 1 as constant into theta.

And my delta S by R is now replaced whole cube is now replaced as 2 gamma by 3 gamma plus 1 the whole square into some constant into theta the whole cube. So, this tells me my delta P is proportional to theta here this tells me my delta S is proportional to theta cube both valid only if theta is small and beta approaches mu if theta is small beta approaches mu. So, this is an estimate of your change in your pressure and entropy.

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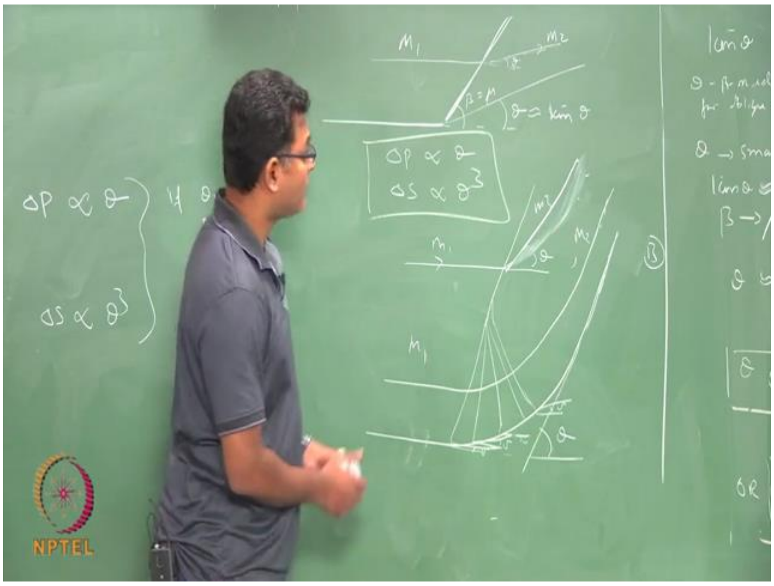

So, what did I do let us draw the scenario and see. I have a deflection theta; theta is very small, such that theta is same as my tan theta. I have a shock - oblique shock, this is very weak and the beta is same as my mach angle mu. So, this stream line that is come here is deflected by theta degrees which are very small. So, I have M 1, I have M 2 so this deflection is very small. So, my beta approaches mu mach angle and my theta is approximately same as tan theta and that is what we have done. In this scenario, my delta P is proportional to theta and delta S is proportional to theta cube. So, again my entropy change is proportional to theta cube.

Now we are going to discuss Prandtl-Meyer expansion. So, instead of this one single deflection, suppose I have a series of deflection, so this is a straight line so my total this angle is theta, but there are several weak oblique shocks here, all these oblique shocks turn by a small degree.

So, let us take delta, this also turns by small degree delta, this also turns by a small degree delta. And every shock turns by a small degree theta is delta such that this is valid such that this small condition is valid this weak shock condition is valid. So, my stream lines come and deflect like this for a angle theta, but whereas in this region instead of one single oblique shock, I have several oblique shock and each of that will turn by small degree such that there is a smooth curve that is so these are parallel lines.

So, I have M 1, M 2, M 2 and M 1. So, M 1 comes and turns by a small degree and then it goes like this. Let us evaluate the entropy change when you have such a scenario. So, I have a small shock as I have described here. Now, here in this scenario, I have a series of such weak shocks which makes very small deflection say n number of times and then you get the total deflection to be of all added of added all those small deflections.

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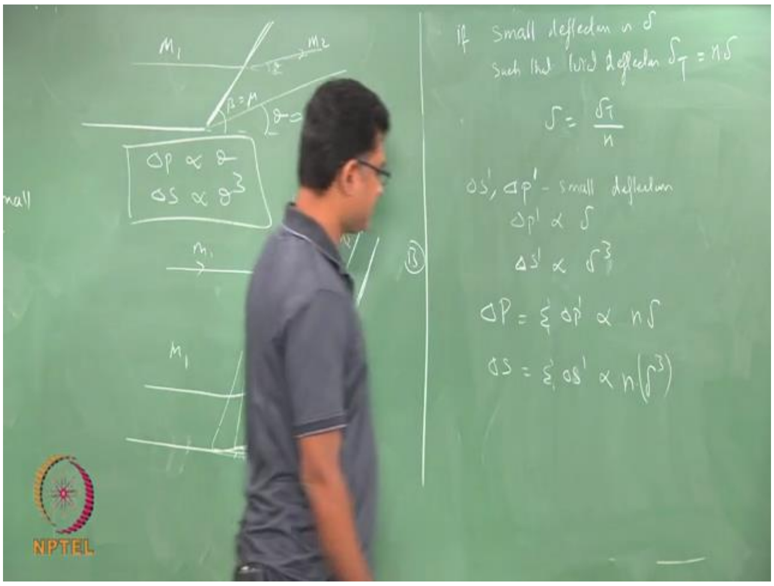

So, if my small deflection is delta such that the total deflection delta T is composed of n such small deflections. So, I have delta 1, delta 2, delta 3, delta 4 everything and achieving all those deflections are same, and there are n such deflections I have n into delta or my delta is nothing but delta total by your n, so this is my delta total delta total is the total deflection that it takes.

Now, my delta P dash associated with delta pressure associated with small deflections is proportional to the deflection. So, delta P dash delta S dash associated with the small deflection when delta P dash is proportional to the small deflection and the entropy small entropy change is associated with my delta cube. So, the change in pressure due to each of the small deflection is delta and the entropy change associated with this small deflection is delta cube. So, the total change in pressure is the sum of all this which is proportional to n into delta; and delta S is sum of all those small changes in entropy which is proportional to n into delta cube.

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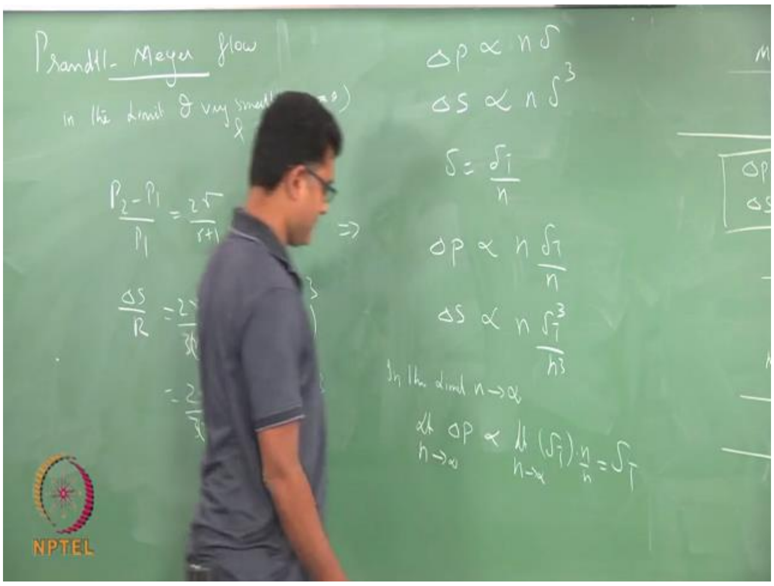

Now in such a scenario proportional to n delta proportional to n into delta cube, where delta is delta T by n delta total deflection by the number of ways. So, my delta P is proportional to n into delta T by n delta S is n into delta T cube by n cube. Now in the limit, n tending to infinity my delta P is proportional to limit n tending to infinity delta T into n by n, which is nothing but delta T.

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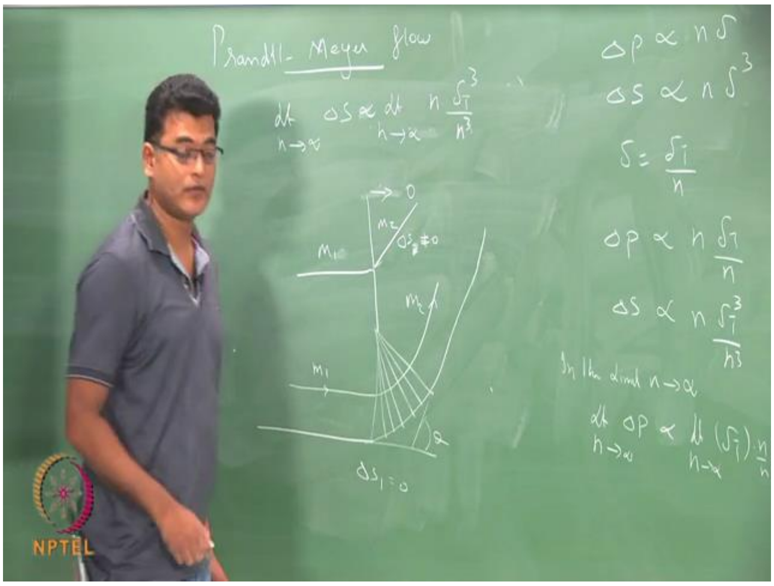

Whereas my entropy limit n tending to infinity delta S is proportional to limit n tending to infinity n into delta T cube by n cube as approaching 0. So, if my n is infinity my delta

S is approaching 0 which means this very simple case. So, if I have a small turn, so if I turn an angle theta with infinite number of very small deflections, the entropy change here in this turn is 0.

So it is this is delta S 1 is 0. Whereas, if I have an oblique shock to change the same Mach number and the same pressure, delta S 2 is not equal to 0, so this is some positive value. So, this oblique single oblique shock is now assume to composed of infinite number of ways infinite number of Mach waves and that entropy change associated with this is 0.

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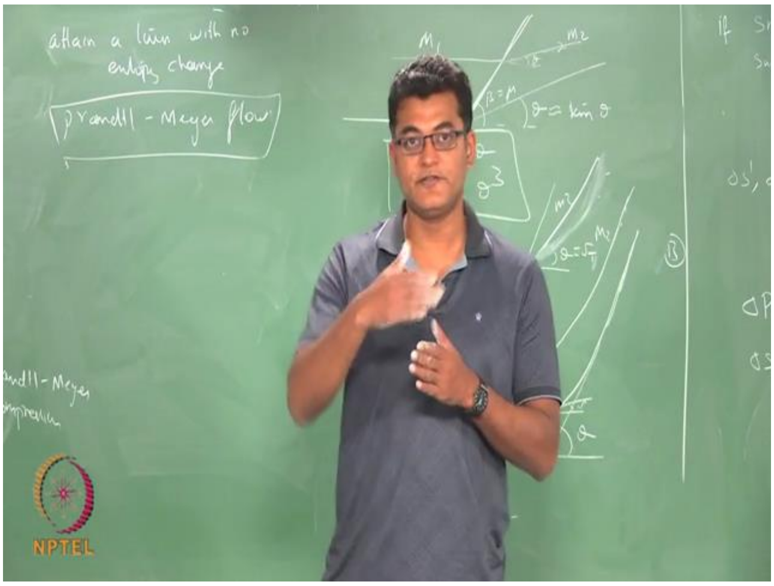

So that such flows are called Prandtl-Meyer flows where you attain a turn with no entropy change that is your Prandtl-Meyer flow what we have described now is a Prandtl-Meyer compression. So, you have attained a delta P change which is proportional to your delta total change in your theta, so you have attained a compression because of the shock ways, but the entropy is as you we have seen from the equation the entropy is 0. The change in entropy is 0; there is no change in entropy.

Now this scenario where you make a turn using a series of weak oblique shocks so that you gain some pressure, but you do not change your entropy such cases are called your compression Prandtl-Meyer compression. Next class, we are going to derive the other way, we are going to make a turn without change in entropy,, but you are going to lose your pressure or you are reducing your pressure which is an expansion. So, we will see in the next class the Prandtl-Meyer expansion. Now the point here is there is no entropy change in a Prandtl-Meyer flow that we had derived recursively and shown that. Next class will see Prandtl-Meyer expansion.

Thank you.