Fundamentals of Gas Dynamics

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Week – 10
Lecture – 40
Shock Polar diagram and Prandtl-Meyer relation for Oblique shocks

This class we will discuss something call shock polar related to oblique shocks and we will draw the velocity of triangles again.

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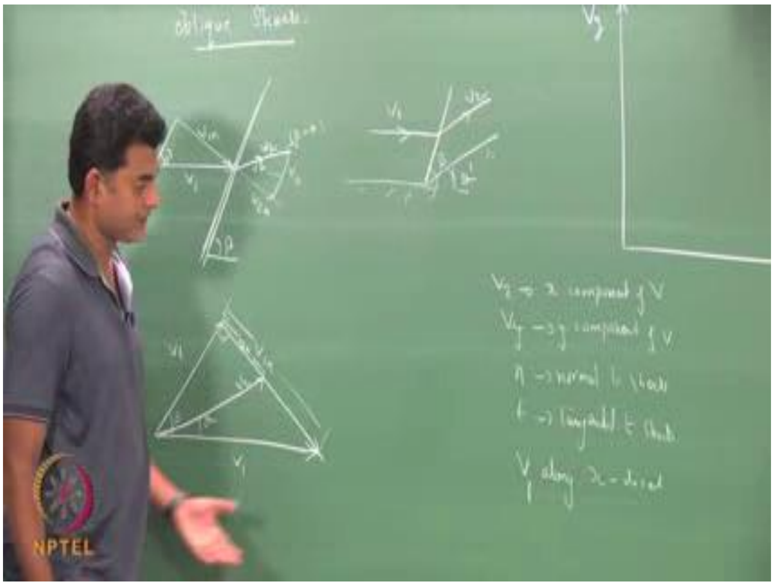

Oblique shocks; let us start with the velocity triangle this is a shock, I have V 1 approaching the shock, there is a tangent then there is a velocity component normal to the shock then after the shock we have the velocity V 2 n deflected V 1 is deflected by some angle and the tangent V t. We have seen to be same as V 1 t. So, V 2 t is same as V 1 t. So, I use V t as the notation for V tangent, this is my V 2, this is theta, this is beta minus theta and this is the shock angle beta. So, this is for a scenario where we have something like this. So, I have a shock and this is theta and this is beta. So, V 1 comes deflected to V 2. Now, I am going to draw this velocity triangle in the following way.

So, I have V t, V 1 and normal to V t is my V 1 M. So, I have V 1, V 1 M and V tangent. So, this is my beta. So, this V 1 is deflected at theta degrees somewhere here. So, this would be my V 2 and this distance is my V 2 n. So, all I have done is I have drawn these 2 plots together in this particular case. So, this would be my theta. This is my V 1 n, this my V 2 n. Now, what I am going to draw is I am going to plot this plot these components in x y plane where my x-axis represent V x and y-axis represent V y. So, V y is x component of velocity, V y is the y component of velocity, whether it is 1 or 2 it is different.

Now, suffix n is normal to the shock, t is tangential to the shock that is a nomenclature we have used for the oblique shock. Now, I will take V 1 along the x direction. So, I am going to plot this thing with the direction V 1 along the x direction. So, I have V 1 here which is also your V 1 x, then I have a component here this is my V tangent and V 1 n and this would be my beta.

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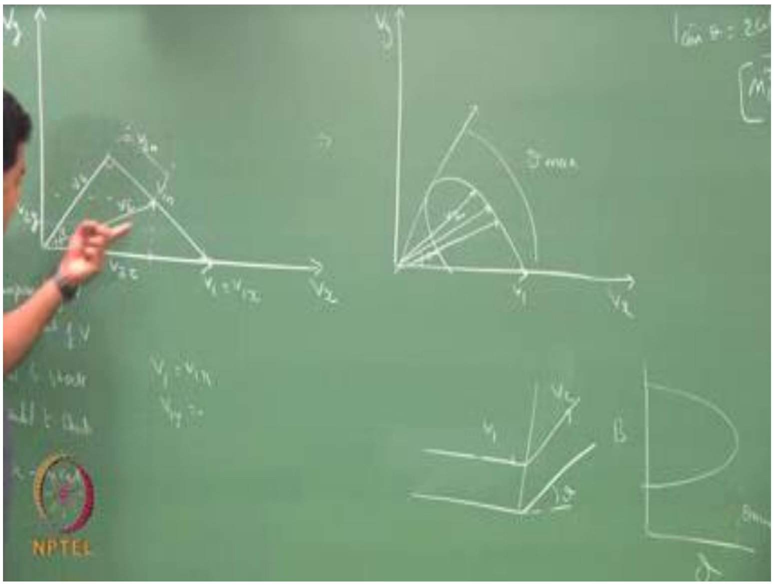

So, for V 1 is V 1 x which means V 1 y is 0, there is no component of V 1 along the y direction, Now, because of the oblique shock there would a deflection of your velocity which is V 2, which produce normal component V 2 n which are the same tangential component; normal and tangential to the shock. So, this V 2 has a component V 2 x and component V 2 y.

Now, I am going to draw the same plot without those details just V 1 and V 2. So, I have V 1 and a V 2 all these components are assumed to be present there to. So, without these detail just this vector and this vector is what I have drawn here. So, this is for a given beta and theta for some mach number for a given theta this is what I do. So, what does it mean I have a plot like this, I have ramp like this.

With some value of theta there is a V 1 coming it hits the oblique shock then take a turn is V 2. Now, if I keep changing my theta this point increase or decrease as I increase or decrease my theta and it we also have seen there is a theta max associated with this for a given mach number, for a given M 1 there is a theta max. So, there is a limit to which you can increase or decrease. So, that you have a solution from our theta beta relation which is your tan theta is 2 cot beta M 1 square sin square beta minus 1 divided by M 1 square gamma plus cos 2 beta plus 2.

So, I am going increase my theta and I would get a different value of this point. If I decrease it I would get a different value here and if I connect all these dots this would become something like this where this is my theta max. So, all I have done is I have changed my theta to get this particular point and for different values of theta I would get a locus something similar to this. So, that I would draw it with a red line, this is the so called hodograph which is nothing, but the locus of points related to the velocity that is before and after the shock. So, if I know the hodograph something like this and draw a smaller curve.

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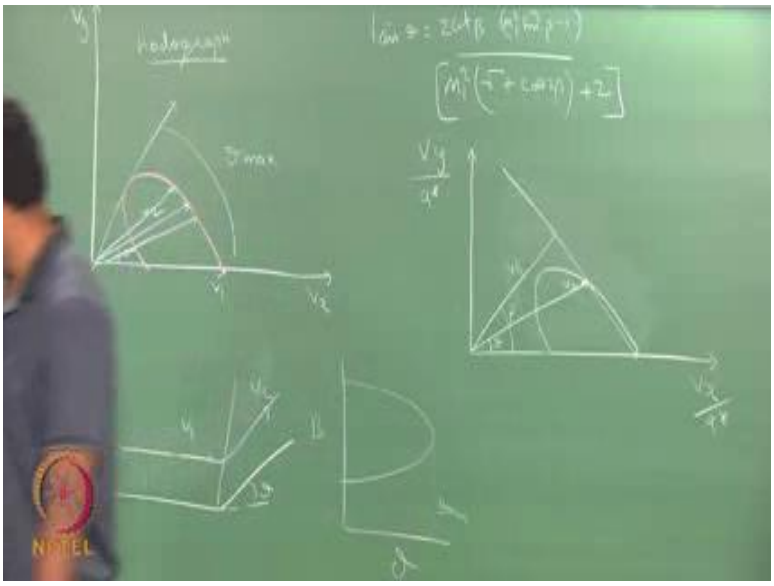

So, I have with some point I have for some value of theta I have V 2 from the V 1 what is kept constant. So, for this particular value of theta I have a different value of V 2. So, if I connect this line and the perpendicular to this to the axis. So, if I get a perpendicular to this particular line to the axis that would give me the shock angle. So, compare this with the velocity triangle that we have drawn here this is what you would see there. So, the point connecting V 1 and V 2 and extend that the normal from there to the origin it should give me the V tangential and the angle between that and V t should be your beta, which is what you would see here.

Now, what I am going to do is I would divide. So, this is my V y, I am going to divide this by A star and this by A star. So, all I am going to take I will take this particular curve and divide it by A star and A star. So, I will rub these plots, remember this is not after the shock, this is the V x component of the velocity and the star associated.

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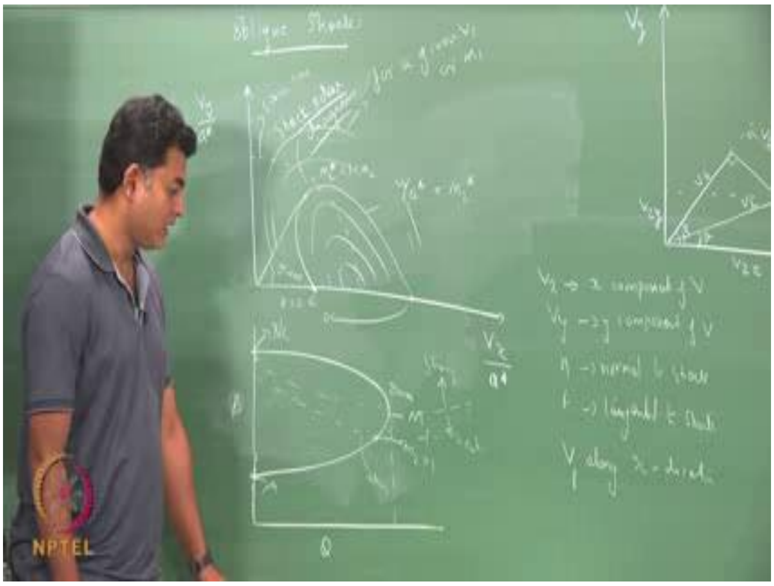

So, it is not M x star as we see it in the normal shock and this is not the M y star after the shock, this is the component of velocity in the x and y direction. So, probably I should I would leave it this as it is instead of using the mach number relation. So, if I get this plot, this is again the velocity divided by A star into divided by A star. So, that would be my M 2 star. So, this particular line is for a given V 1 or M 1 and the tangent would give me the theta max and somewhere here is the point where I would get my M star equals 1. So, M 2 star equals 1 which is also your M 2.

That is what you would see in your theta beta relation, beta theta for a given M 1, this is my theta max for some combination of theta and beta I would have my M 2 equals 1. So, this line would give me that particular demarcation which gives me strong shock and weak shock and these are called sonic line that we have already seen in theta beta M relation. So, this point is what I have written here as dot and this is the theta max plot.

So, I would put this red dot as my sonic line and this as the theta max. They are 2 different points as we have already seen in the theta beta M chart and remember this is for a given mach number, given M 1 and this is the so called shock polar diagram. So, if I know the mach number based on my velocity components x and y, I can draw this and find the solutions where I get theta max and M 2 the sonic condition. So, this is for a given M and for other M’s you would get plot similar to this and you can find the locus of all these points which gives M 2 equals 1. So, that is your sonic line corresponding to what you have seen here in the theta beta M relation and this point on the x-axis, these 2 point on the x-axis gives me theta equals 0, which is precisely these 2 points for some value of your M 1.

There is no deflection of your velocity vector V 1 if your theta is, if your beta is this or this which is this 2 values with the these normal shock pi by 2 or your mach angle mu. For these 2 values there is no deflection of your velocity vector which means theta equals 0 which has the points that is lying on your x-axis. So, for other values of mach number you would get something like this and this is what you call as the shock polar. Now, you could also get the equations for this you can derive the equations for the shock polar for that first we will derived the Prandtls relation for oblique shock and then we will proceed to this.

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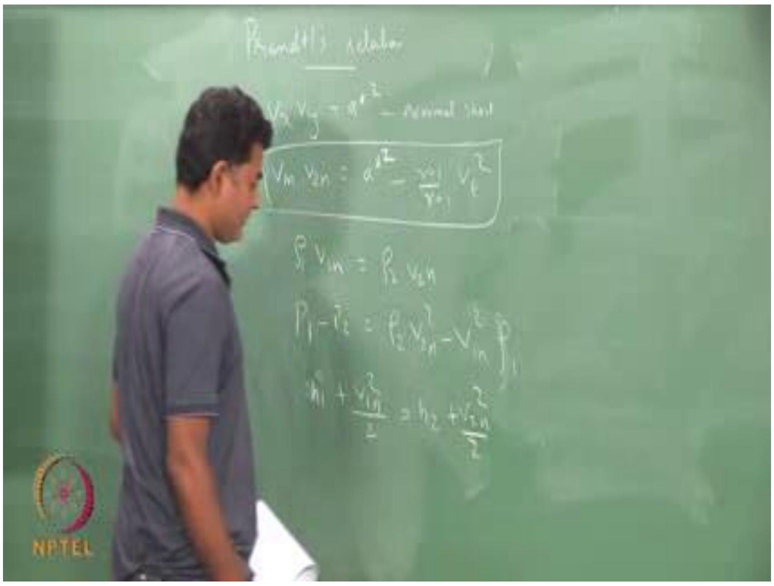

So, for a normal shock we have seen V x V y is a x star square now for an oblique shock, this would be modified to something which is the following. So, this is what we will derive and for which we will start from the continuity equation, rho 1 V 1 and equals rho 2 V 2 n and the momentum equation is this.

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A square; this can be re arranged as, Now, first we will write the energy equation h 1 plus V 1 square n by 2 equals h 2 plus V 2 n square by 2, which I will rewrite as V 1 square by 2 plus gamma by gamma minus 1 into P 1 rho 1 is V 2 square by 2 plus rho minus 1 P 2 by rho 2 which is for a normal shock or just simply the energy equation between 2 points V 1 one 1 and 2 which we can related to the stagnation condition which is c P t 0 into r t 0.

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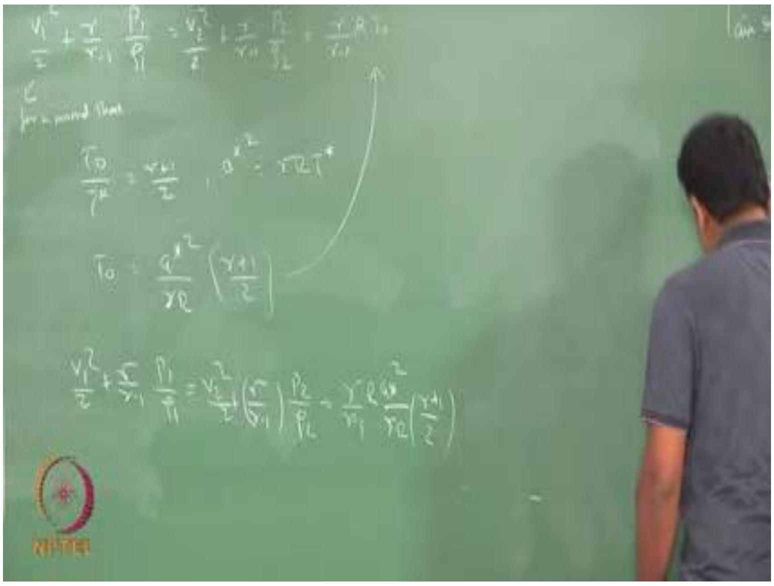

So, this is your stagnation, the total temperature that is given from the energy equation. So, which we also know t 0 by t star is gamma plus 1 by 2 and your, a star is nothing but gamma r t star a star square is gamma r t star. So, I am going to use these 2 relations to get t 0 as a star square by gamma r into gamma plus 1 by 2. I have just converted t star in terms of stagnation temperature that is all I have done here. So, across the oblique shock the temperature the stagnation temperature is constant. So, I can try like this. So, a star will also be constant across the oblique shock. I am going to substitute this here in this equation.

So, I will rewrite that as V 1 square by 2 plus gamma by gamma minus 1 into P 1 by rho 1 equals V 2 square by 2 into gamma by gamma minus 1 into P 2 by rho 2 plus equals gamma by gamma minus 1 into r instead of t 0 I would write this a star square divided by gamma r into gamma plus 1 by 2. So, this give me 2 relations P 1 by rho 1 equals gamma 1 gamma plus 1 by 2 gamma a star square minus gamma minus 1 by 2 into V 1 square by 2.

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Likewise P 2 by rho 2 is gamma plus 1 by 2 gamma a star square minus gamma minus 1 by 2 V square by 2 from this equation I have split this equals into those 2 equations. So, when it comes to the oblique shock relation all I need to substitute is instead of V 2, I would substitute as V 1 and V instead of V 1 and substitute V 1 n and V 2 is V 2 n.

So, for oblique shock I would replace V 1 n with V 2 n with the normal component and from momentum equation which we have written as P 1 minus P 2 as rho 2 into V 2 n square minus rho 1 into V 1 n square, I would rewrite this as P 1 n minus P 2 by rho 2 V 2 n equals V 2 n minus V 1 n. So, this is my equation 1 and these 2 equations I can get V 2 n minus V 1 n from this equation.

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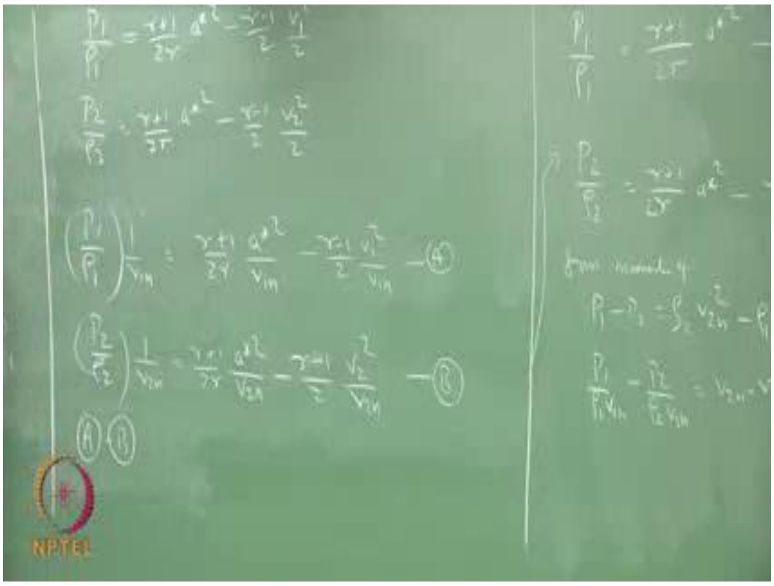

So, my equations are P 1 by rho 1 equals gamma plus 1 by 2 gamma a star square minus gamma minus 1 by 2 V 1 square by 2 and P 2 by rho 2 into gamma plus 1 by 2 gamma into a 1 star minus gamma minus 1 by 2 V 2 V 2 by V 2 square by 2. Now, I am going to multiply this equation with divide by V 1 n and the second equation by V 2 n. So, that I would get P 1 divided by 1 by V n a star square by 1 V 1 n minus gamma minus 1 by 2 V 1 square by V 1 n.

So, to get this particular form, P 1 by rho 1 I had divided by V 1 n. So, at 0 this equation I have divided with 1 by V n, likewise P 2 by rho 2 into 1 by V 2 n is gamma plus 1 by 2 gamma a star square by V 2 n minus gamma plus gamma minus 1 by 2 into V 2 square by V 2 n. So, this is my A and this is my B. So, I subtract A minus B. So, I would get gamma plus 1 by 2 gamma a star square by V 1 n minus gamma minus 1 by 2 into V 1 square by V 1 n divide by V 2 n.

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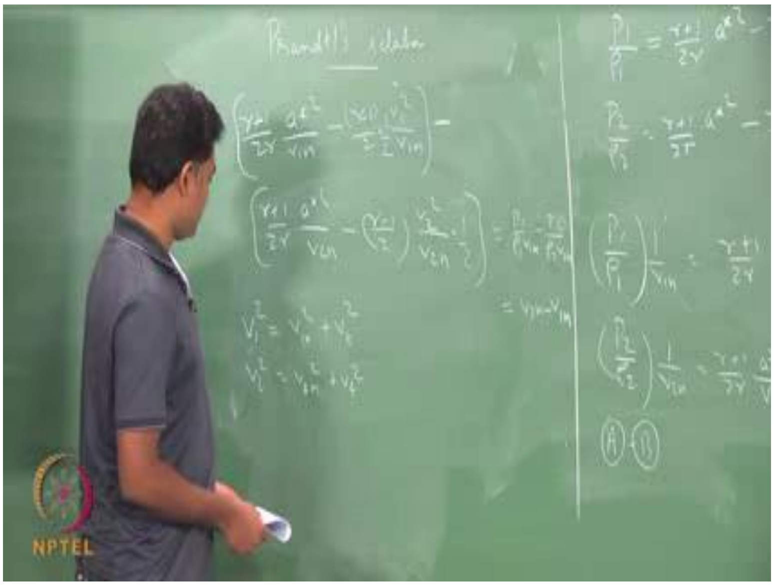

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Divide by V 2 n.

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So, there will be 1 by 2 here. So, there is a correction here there is a 1 by 2 here 1 by 2 here. So, this 1 by 2 is missing here. So, so there would be a 1 by 2 here and gamma plus 1 by 2 gamma a star square by V 2 n minus; we have subtracted this, which would be equals to P 1 by rho 1 V 1 and minus P 2 by rho 2 V 2 n, which we know from equation 1 as V 2 n minus V 1 n. So, we are substituting this value this and this from the equation just we had seen, V equal to V 2 n minus V 1 n which is what I have written here.

So, this can be modified if I do this t square and if I substitute this here and we also know V 1 t is same as V 2 t. So, I have just written V t here. So, if I substitute that and do some simplification I would end up with the relation that is that looks like the following gamma plus 1 by 2 gamma into a star square V 2 n minus V 1 n divided by V 1 n V 2 n plus gamma plus gamma minus 1 by 2 V 2 n minus V 1 n plus gamma minus 1 by 2 into V t square by 2 n minus V t square by V 1 n equals V 2 n minus V 1 n.

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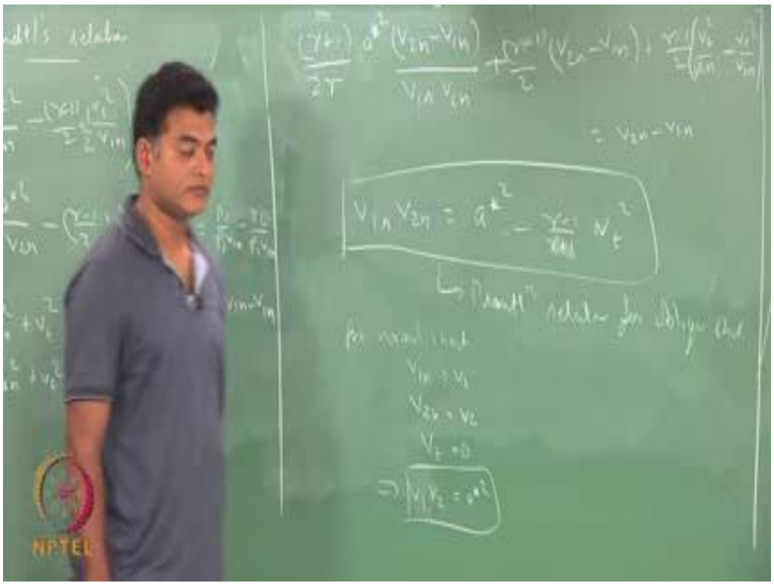

So, if I can take V 2 n outside from these quantities I would get V 1 n and V 2 n equals a star square minus gamma minus 1 by 2 into V P square. So, I divide this equation with V 2 n minus V 1 n. So, this cancels out, this cancels out then the numerator here cancels out. So, I will end up with equation that is here.

Student: (Refer Time: 33:12).

This is gamma minus 1 by 2 by gamma plus 1 correct; this is gamma minus 1 that is the Prandtl relation for oblique shock. So, for normal shock V t is 0. So, you would equation varies to the normal shock form V 1 n is V 1, V 2 n is V 2 and V t is 0. So, this would reduce to V 1 V 2 equals V 2 star. So, V 1 is velocity before the shock, V 2 is velocity after the shock is a star square. Now, we will use this relation to find the equation for the shock polar which is not very difficult, we will just draw that particular velocity diagram which we had just now drawn.

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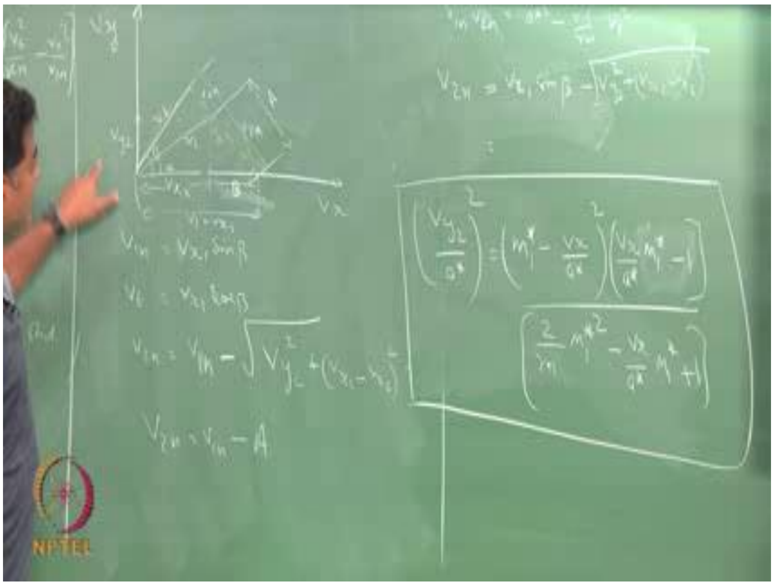

So, I have V 1 then I have a V 2, this is V 2, this is theta, this would be my beta, this is V t, this is V n 1, this is V 2 and this is my V 1. So, this is beta, this will also be beta and this component is your V x 2, V 1 is same as my V x 1 or V 1 x. So, I can write V 1 n is V x 1 sin beta, V tangential is V x 1 into cos beta and V 2 n is V 1 n minus root of V y 2 with this is V y 2 plus V x 1 minus V x 2 to the whole square. So, the V 2 n is this, you can look at the triangle and get this relation. So, V 2 n is V 1 n; V 1 n is this much minus this length.

So, V 2 n is V 1 n minus, let us say this as length A. So, that is minus A. So, this A is from this angle or from this triangle you can get this length and that length is what is written here this height is V 2 A, V y 2 and this is this whole length minus this. So, this is my V x 2 and this all length is V 1 which is V x 1. So, this length is my length b. So, from this triangle which the height is V y 2 and this length is B, I can compute this particular length which is my A and form if I know a V 1 minus a is my V 2 that is all I have done here which can be modified with the particular relation, which is the Prandtls relation V 1 V 2 n equals a star square minus gamma minus 1 into gamma plus 1 into V P square.

So, you can substitute that relation use that relation to get this equation which can be modified V 2 n can be modified as V 1 n s V x 1 sin beta minus V y 2 square plus this quantity root. So, if you substitute all this I would end of with a relation that is the following. V y 2 by V star square equals M 1 minus V x by a star the whole square into V x by V star into M 1 star minus 1. So, this is star; divided by 2 by gamma plus 1 M 1 star square minus V x by A x M 1 star plus 1.

Student: (Refer Time: 40:53) M square.

Plus 1.

Student: (Refer Time: 41:04).

No, it is plus 1. So, for every for a given M 1 star for every V x, we can get a V 2, the moment you get a V 2 you can extend that for change your M 1 and get this thing, in this plot. So, how do we plot this that particular equation.

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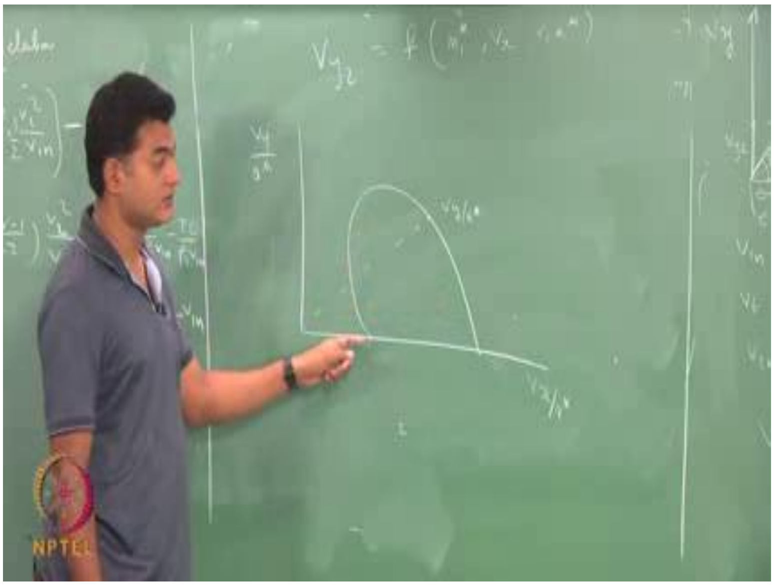

So, V y 2 is a function of M 1 star and V x and of course, gamma and a star. So, let us assume gamma and a star to be constant, this is a function of M 1 star and V x star. So, what we need is this particular plot. So, this is V x by a x star, V y by a star. So, this point somewhere here is my V y by a star. So, what we have is V y. So, if I change for a given V x, keep changing my M 1 or for given M 1 change these theta values, so that I would get the point here which is a locus which is what is given by this equation. So, this is your equation to your shock polar.

In this class, what we have done is the concept of shock polar and how we can plot the velocity components in the x and y direction and get some information about the flow directions and the shock waves, and if you have theta greater than theta max the solution does not exist for whatever we have done. So, it would be a detached shock or something or about shock as it is called. So, all those condition comes out of this particular relation between theta beta M.

Thank you.