Fundamentals of Gas Dynamics
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Week - 08
Lecture – 32
Rankine - Hugoniot equation

So, last class, we were discussing about normal shocks and the variation of properties across the normal shock. So, what we had seen is, if I have a shock, given my value of Mach number before the shock, I know an expression to find the Mach number after the shock.

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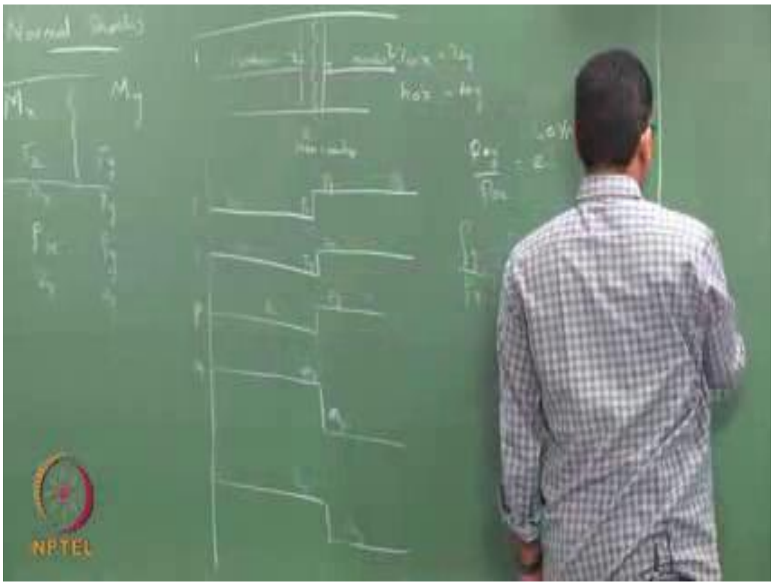

We also know how to find the ratio T x and T y, likewise, P x and P y, and rho x and rho y; again, v x and v y, and all such quantities, we can get, if we know Mach number before the shock. So, these ratios, we had seen as increasing with, with respect to a, with respect to the shock. So, once it crosses the shock, you have an increase in all these properties, except the stagnation pressure.

So, we have also seen the stagnation temperature to be same, because, there is no heat, and there is no shaft work. So, this is going to be true; but, it is not isentropic process. So, up to this, we had seen, from 1 to x, we have an isentropic flow; and then, suddenly, it encounters a shock, which is a non-isentropic flow; and, from shock to some other location 2, from y to 2, you have, again, isentropic solution.

So, if I know these properties, isentropic properties up to shock, then, I can find, across the shock, what happens, and from across the shock, after the shock, you again compute everything, using isentropic solution. So, if I plot my density, if this is my location of shock, the density would be some value up to the shock, then, suddenly, there is a jump, and then, increases my... So, this is my rho y, and this is my rho x. Likewise, your temperature also increases by some amount, which would be here, and here, assuming, assuming the thickness of the shock as very, very small, so, if there is a sudden discontinuity of density and temperature.

Then, you have the pressure also, does the same thing. The Mach number could be large. The moment it encounters the shock, it goes down to subsonic value. So, it goes down to subsonic value. So, I have an m x here, and m y here, which is lower than; this is rho; this is t; this is P; and this is m. And, my stagnation pressure also goes down; P 0 x, P 0 y, because it is an isentropic, non-isentropic process, your P 0 y by P 0 x is given by e power minus delta s by r, where delta s is the entropy change that is happening due to the irreversibility that are associated with your shock.

So, expression for this all, we have seen. We have also seen, how these ratios change with function of m x. So, you could actually write all these ratios in terms of Mach number before the shock. To consolidate what all we had said in the T S diagram?

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So, from 1 to x, 1 to x, 1 is here; x here, which is an isentropic process; from x to y is a non-isentropic process; but, there is an increase in my temperature. So, I would increase my temperature here. And, from x to 2 is again, y to 2 is again an isentropic process. So, I have 1, goes to x; then, suddenly it encounters a shock; there is an entropy change; and then, it goes to y after the shock; and, from y to 2, you have again an isentropic solution.

Both this have the, all this states have the same stagnation temperature, because, there is no shaft work; there is no heat transfer. We have also seen this, how this would appear in a p v diagram. So, I have 1, goes to x; then, there is a shock; because of the shock, there is a density increase, which means the volume would decrease. So, I have a point somewhere here, which also means, my pressure has increased. So, my y is on a line, isentropic line above this.

So, this would be my y, which again goes to 2, isentropic way. So, these are isentropic lines, and this is my shock. This would produce a delta s; delta s is here. This would be my delta rho. So, this is a pressure, some pressure here; some pressure here. So, if you look at the stagnation pressure here, this P 0 2 is now less than P 0 1, according to this relation. So, your P 0 y is always less than your P 0 x, if you want an increase in my entropy. Difference is, yes, 1 by delta v. So, this difference is delta v, which is 1 by delta rho. So, this is for the shock that is happening here; from 1 to x, it is isentropic; x to y is non isentropic; and, from y to 2 is again isentropic.

So when, if I have a shock in a, shock in a varying area duct, so, up to the shock, I can assume to be isentropic; if there are no heat, and no shaft work, I can assume that to be isentropic. So, we can use all the relation that we have been using for isentropic flow; then, after that, non-isentropic, and then, again, isentropic. So, in the, whether it is a converging nozzle, diverging nozzle, a diffuser, or a c d nozzle, wherever there is a shock, you can assume that, it is, up to that, it is isentropic, and then.

Now, we will try to derive an equation which relates P 1, P x and P y, in terms of rho x and rho y. This equation is called Rankine - Hugoniot equation. So, there is a jump in the density and pressure, and this jump, and density and pressure are related; and this expression that gives you the density jump in terms of pressure jump, or pressure jump in terms of density jump, is typically called as the Rankine - Hugoniot equation. So, we will derive that, and we will, after that, we will try to do some numerical problems to see how this works.

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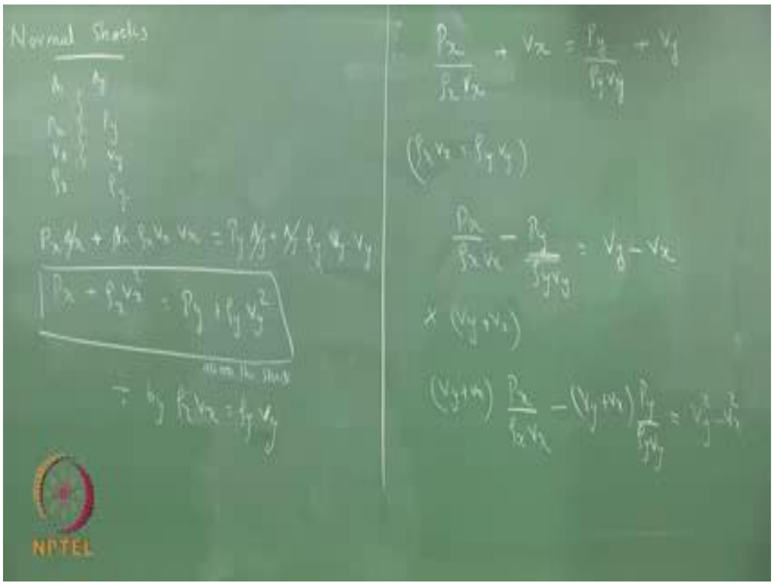

So, like before, I start with the shock here, which has a control volume that is very close by. So, my A x is same as my A y; and, I have a P x here, P y here; v x and v y; rho x and rho y. I apply the momentum equation. I would get P x A x, plus A x rho x v x into v x, which is equal to my P y A y plus A y rho y and v y, multiplied by v y.

So, this is your force due to pressure, and this is the kinetic energy. So, this is m dot into v x; this is m dot into v y. Since A x and A y are same, I can cancel out A y. So, I would end up with P x plus rho v x square, equals P y plus rho y v y square, across the shock.

So, now, our job is to eliminate v x and v y, and we end up with Rankine - Hugoniot equation; that is all that is need to be done here. So, for that, we will do some minor algebra. So, I am going to divide by rho x v x. So, this would be x that would divide by yes. So, I would divide by rho x v x, plus v x, equals P y by rho y v y, which is same as my rho x v x plus v y. So, my rho x v x is equal to rho y v y. So, what I have done here is, I have divided this equation by rho x v x, which is also equal to rho y v y.

So, I would end up with an equation. Now, I would multiply this equation with v y plus v x. So, I would get v y plus v x multiplied by P x by rho x v x, minus v y plus v x multiplied by P y by rho y v y, equals v y square minus v x square. Now, I expand that term, term on this side.

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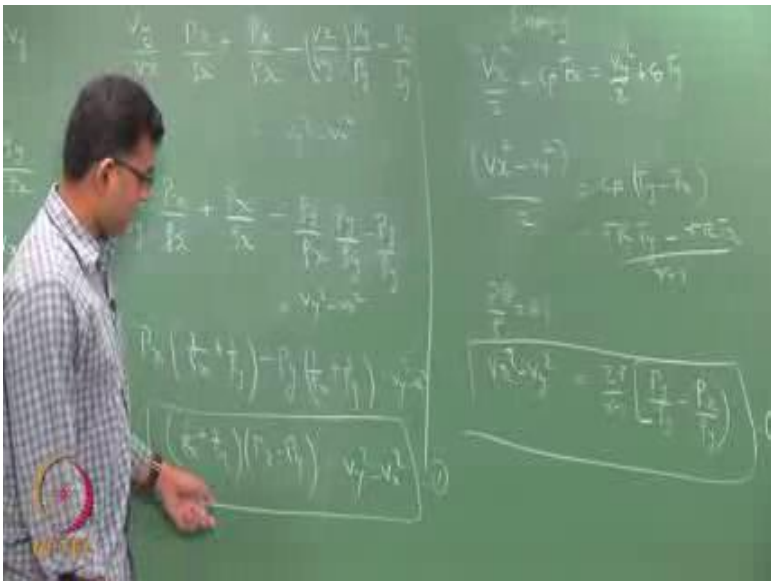

So, this would be v y by v x into P x by rho x, plus v y v x, v x v x cancels. So, it would be P x by rho x minus v x into v x by v y into P y by rho y minus P y rho y equals v y square minus v x square. Now, from the continuity equation, which I have written here, my v x by v y is also my rho y by rho x. So, wherever I see this ratio, I substitute that in terms of rho y and rho x.

So, my v y by v x is now rho x by rho y, into P x by rho x, plus P x by rho x, minus v x by v y is rho y by rho x, into P y by rho y, minus P y by rho y, equals v y square minus v x square. So, this and this cancels; this and this cancel. So, I can write in this particular form, where I write a P x into 1 plus rho x, plus 1 by rho y, minus P y into 1 plus rho x, plus 1 plus 1 by rho y, equals v y square minus v x square; or, 1 by rho x plus 1 by rho y equals P x minus P y, equals v y square minus v x square; yes.

So, this is from the momentum equation, and I have used continuity equation to substitute v y by v x as rho x by rho y. Now, we will use the energy equation. So, this is my equation 1. Now, from the energy equation, I can write v x square by 2 plus C p T x equals v y square by 2 plus C p T y, which is nothing, but h 0 x equals h 0 y, across the shock, from which I can write, v x square minus v y square by 2 equals C p T y minus T x, which is gamma R T y minus gamma R T x, divided by gamma minus 1. So, I had to change to C p to gamma R by gamma minus 1. So, R T I can write in terms of p v; p by rho equals R T. So, I use this particular relation to get, convert that T to pressure and density ratio. So, my v x square minus v y square equals 2 by gamma minus 1, into gamma; R T y I substitute as P y rho y, minus P x by rho y.

So, this is from my energy, which is equation 2. So, what I have done here is, I have taken the momentum equation, I eliminated my velocity in this particular form. Here, I have taken the energy equation, eliminated the temperature, and I have come across this particular form. So, velocity in terms of pressure and density; here again, velocity in terms of, now, equate these two that would give me the Rankine - Hugoniot equation.

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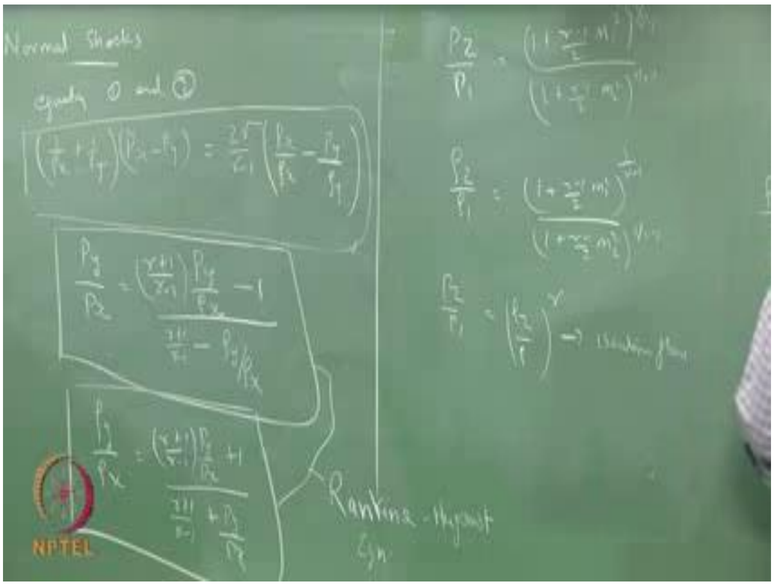

So, equating equation 1 and 2, which I have just now written, this would be 1 by rho x, plus 1 by rho y, into P x minus P y, is minus of the other one. So, it would be 2 gamma by gamma minus 1, into P x minus P x by rho x, minus P y by rho y. So, I have included the minus sign inside the bracket.

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So, this is P x by rho x; everywhere else its fine. So, I get this equation, which can be reduced to the following form. So, I can write this in terms of P y and P x, equals gamma plus 1 by gamma minus 1, rho y by rho x minus 1, the whole divided by gamma plus 1 by gamma minus 1, minus rho y by rho x. Or, rho y by rho x equals gamma plus 1 by gamma minus 1, into P y by P x, plus 1, divided by gamma plus 1 by gamma minus 1, plus P y by P x, these two equations are called Rankine.

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Rho y by rho x; Rankine - Hugoniot equation; this is pronounced as Hugoniot, Hugoniot equation. Now, here, in this process, we have not assumed anything that is irreversible; all that x 0 1, x 0 x equals x 0 y, is the only condition that we had taken, which is no heat, no heat, no shaft work. Then, that includes your T 0 x equals T 0 y. Then, it is the momentum equation, and continuity equation. So, energy equation and the perfect gas equation, and the momentum equation and the continuity equation is what we have used. So, if it is an isentropic flow, you know how P 1 and P 2 vary with respect to rho 1 and rho 2. So, P 1 and P 2 vary with respect to.

How does it vary? So, if it is an isentropic flow, in order to get P 1 and, P 1 by P 2 in terms of Mach number 1 plus gamma minus 1 by 2, m 1 square to the power gamma by gamma minus 1, divided by 1 plus gamma minus 1 by 2 m 2 square to the power gamma by gamma minus 1. Likewise, your rho 2 by rho 1 is 1 plus gamma minus 1 by 2 m 1 square to the power 1 by gamma minus 1, right; divided by 1 gamma minus 1 by 2 m 2 square to the power 1 by gamma minus 1; or, your P 2 by P 1 is rho 2 by rho 1 to the power, this to the power gamma is, to the power gamma.

So, this is your isentropic flow; because of non-isentropic, you have this relation across the shock. So, if I plot P 1 by P 2 versus rho 1 by rho 2.

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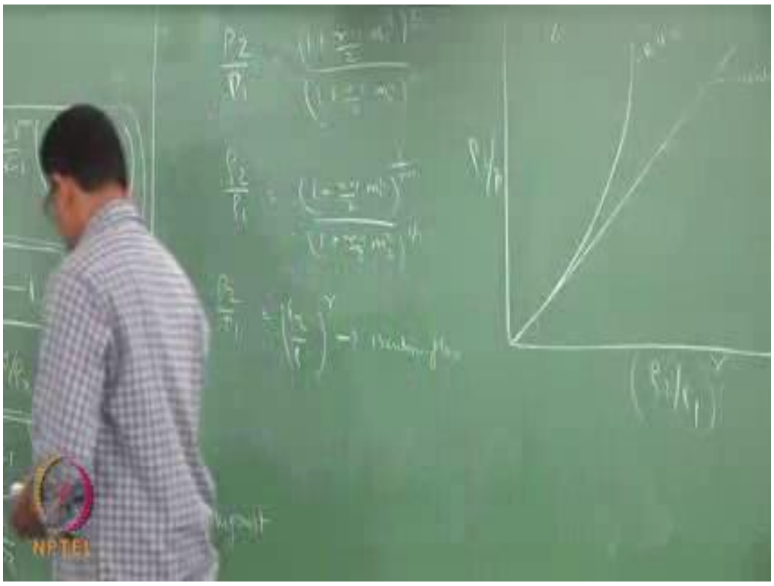

So, let us take rho 2 by rho 1, P 2 by P 1. If I, instead of y and x, I put 2 and 1, then, I would get something like this, for Rankine – Hugoniot, something like this for isentropic flow; this is Rankine - Hugoniot equation. So, there is a difference between these two ratios, because of the shock; that is what it.

Student: Just in case a straight line.

This will not be a straight line, because, there is an exponent here. So, this line is some curve with, you know, slope changing at every location. Or, what you can do is, you can plot this; and this, if I plot this versus this, this would be a straight line with 45 degrees, which means that, for every P 2 by P 1, which is equal to this quantity; whereas, in Rankine- Hugoniot, it will be something else. Now, we will see what happens to the scenario, when we have a pitot tube kept in a supersonic flow.

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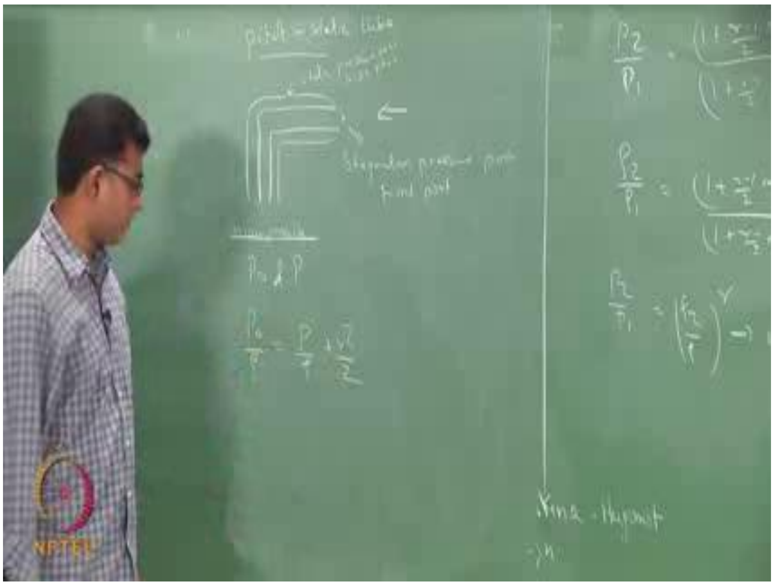

So, pitot tube is something that measures your, a stagnation pressure, at this point, if the flow is in this direction, flow comes, and then, takes, the streamlines take deviation like this. So, there is a stagnation point here. The velocity is about 0 isentropically; hence, you measure your stagnation pressure here. And, somewhere here, if you have a port, that will pressure your static, our static pressure, which is also, which is something different called the pitot static tube; we will discuss that, may be at the end of the class. But now, for the time being, we are going to measure only the stagnation pressure.

So, in an incompressible flow, or, let us discuss pitot static tube right away. So, this, I will modify. So, I have a tube that is going to measure my stagnation pressure. It is also going to measure a static pressure at this surface. So, this is my static pressure port, or, simply, side port. This is my stagnation pressure port, or simply, the front port. So, if I say front port, it essentially measures stagnation pressure, and side port of the pitot static tube measures the static pressure. So, the flow come and goes to a 0 value at this particular point, because, that is a stagnation point there, as you would have discussed in your fluid mechanics course.

So, there is a stagnation point, and there is a static port here. So, I am not going to discuss the pitot, working of the pitot tube, but just these 2 points will be stressed. So, I measure P 0 and P, which is my static. If it is an incompressible flow, I use Bernoulli's equation to find my velocity. So, I pressure my, I measure my stagnation pressure, and the static pressure, I get my velocity of the fluid.

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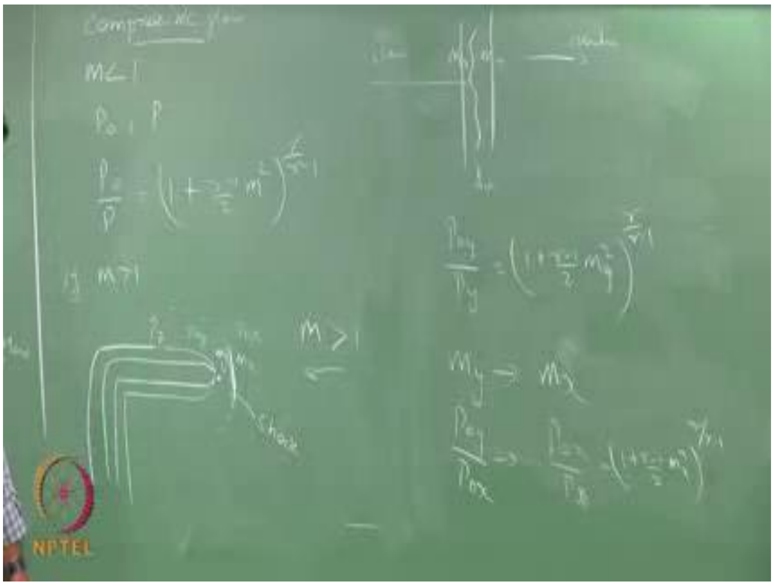

If the flow is compressible, with the case subsonic flow, m less than 1, then, that particular port will measure your P 0 and P, and you know, the P 0 by P is related to your Mach number, which is 1 plus gamma minus 1 by 2 into m square to the power gamma by gamma minus 1, from which you can find your Mach number. So, you know, if you measure your stagnation pressure, you measure your static pressure, then, you get your Mach number, which can be converted to your velocity, if you know the temperature.

Now, if m is greater than 1, then, we have a problem, which is the following. So, I have this tube, then, I also measure my static pressure. So, it measures something here; but, the problem here is, I have a flow which is m greater than 1, and it suddenly sees an obstruction. So, what does it do? There is a shock that is generated. So, there will be a shock that is generated. It would be something called a shock; for the time being, let us assume this to be a normal shock.

So, there will be a shock that is generated in front of your measuring device, which is your pitot static tube. Now, if you go back to your description of a shock, to the schematic which I had described sometime back, I have a shock here, m x, m y; and, this is isentropic; this is isentropic; only the shock is non-isentropic. Same thing happens here; the flow comes, and upto this point is isentropic, and after this also is isentropic; only the shock is non - isentropic.

So, you can relate all the properties up to the shock using isentropic relation, and after the shock, and further downstream, using your isentropic relation; this jump, you have to use shock relation. So, I have my m x coming here, and, I have an m y here; I have a P 0 x here; I have a P 0 y here. This P 0 y is measured at your pitot tube. So, this point which measures some pressure is your P 0 y; not P 0 x.

So, now, if you know P 0 y and your P y, if I know my P 0 y and P y, I can get my Mach number, m y square to the power gamma by gamma minus 1. So, I know my P 0 y; I know my P y, from which I find my m y first. Now, if I know my m y, I can get my, I can get my m x. If I know my m x, I can also get P 0 y by P 0 x. I repeat; I get, I measure P y and P 0 y, because, there is a shock sitting here. So, whatever I measure in the static, pitot static tube is P 0 y and P y. The moment I get this, I can get my Mach number after the shock.

The moment I get Mach number after the shock, I have a relation which will tell me what is the Mach number before the shock, I can get a Mach number before the shock. So, I get a Mach number before the shock. If I get a Mach number before the shock, then, I can also get the pressure ratios, stagnation pressure ratios, from which I can find P 0 x. If I know P 0 x, then, I can use P 0 x by P x relation in terms of Mach number to find my pressure before the shock. So, the static pressure before the shock is also obtained, once I have this. And, once you have the Mach number before the shock, you can find, eventually, find all the quantities, if you know the temperature at that location; the velocity can be easily found out.

So, the point here is, in supersonic flow, there will be a shock sitting, which would actually make the measurement not as direct as this, but, there will be one more step involved in finding the Mach number.

So, what we have seen is the relation in normal shock, the properties, how the properties change across the normal shock, and how is it useful, when it comes to something like a measurement here, or in a, say, for example, the inner diverging section in the c d nozzle, if there is a shock standing somewhere in the diverging section, the properties across the shock is going to be different, which actually takes it to a, some other Mach number than the designed Mach number. So, all those conditions, we will understand this better by doing some numerical problems.