Introduction to Boundary Layers
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Module - 02
Lecture - 06
Concepts of BL displacement thickness and BL momentum thickness

Hi! So, welcome back as we did talk about boundary layer displacement thickness. Well, so with this, something else to be done next, which is say, boundary layer momentum thickness, let me write that down. We denote it by variable theta.

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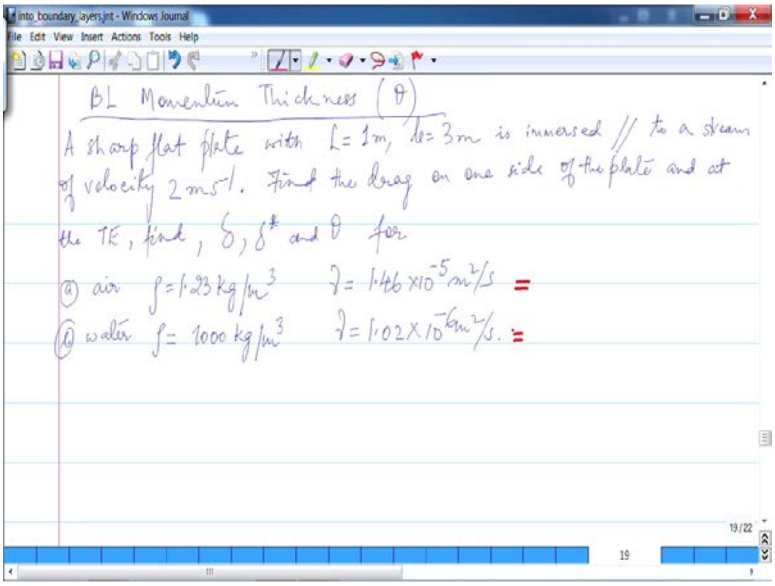

So, boundary layer momentum thickness, which we denote by this theta. So, again we will try and understand what that means and we will do the math to understand that. Now before we do that, now let us sort of look at the problem that what sort of questions could be asked, right, what sort of answers are we looking for. One, we are looking at, you know, boundary layer momentum thickness or displacement thickness and so on and so forth.

So for example, this is a problem we could have. A sharp flat plate; a sharp flat plate with dimensions L is equal to 1 meters, b is equal to; so, length 1 meters and breadth 3 meters is immersed parallel to a stream of velocity, to a stream of velocity 2 meters per second. The question is; so, you have a physical problem like this. You have a flat plate, which has certain dimensions and you just immerse it and there is a flow which is happening across it. So, now, what should you be concerned about?

What you should be concerned about or what the questions that which you could have is find the drag on one side of the plate. So, find the drag on one side of the plate and at the trailing edge, which is the far end of the plate. At the trailing edge, find delta, delta star and theta. So, what exactly is this theta now? For; there are some cases given to you. Now a; so, case a is one is for air, when rho is 1.23 kg per meter cube, nu is 1.46 into 10 to the power minus 5 meter square per second, b is water, rho is 1000 kilogram per meter cube and nu is 1.02 into 10 to the power minus 6 meter square per second. This is very interesting. So, hopefully we will get some very good insights into this, once we do this.

So, to answer this question we should be able to find out delta and delta star by now. But, we do not know what this theta is and which is the momentum thickness, which is boundary layer momentum thickness and we also do not know how to find out the drag. That is how something we need to find out. So, let us see how we will go about this problem.

Now, what exactly is this momentum thickness? Now; and how do we sort of go about this? Now, so what we did in order to calculate the boundary layer displacement thickness. So, we took the; we took the mass flow. We basically exercised the fundamental equation of conservation of mass.

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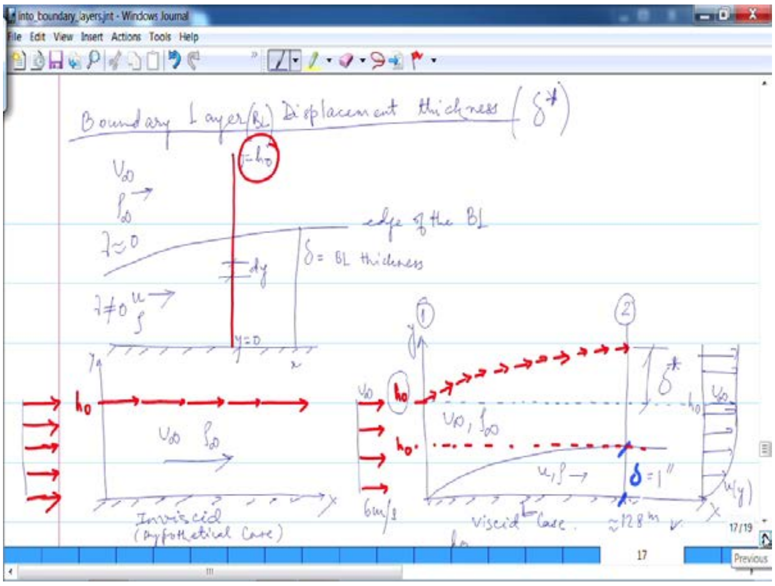

If I where to go back to this picture that we had drawn last time, so this is the picture which we had drawn. If you look at this, so basically what we said? Is that whatever mass enters, you know, at section 1, we will need to leave section 2 because that is the fundamental law of conservation of mass. Mass cannot be created or destroyed. Now, having said that, so when it enters the space, it is entering at a distance h naught but for that entire mass to exit, it cannot exit only through h naught.

It needs more distance; more height. It needs more space, it is big because in the scenario of existence of the boundary layer, it displaces or pushes the external flow away from itself. So, this is actually acting like almost a solid in obstruction. Solid obstruction and so, it pushes away the flow. So, therefore you can see the stream line. It gets displaced and therefore, you need a height which is h naught plus delta star to allow that entire mass, incoming mass, to exit and, so that is what we did.

Now, of course in this particular case, you have a mass which is coming in with velocity and we have a mass which is going out for the velocity. So, clearly there is also a change in momentum when that happens because you can see the velocity is changing. The velocity is not same.

Now, so what is that mean? In terms of physically, what is that mean? So there, when I do that, what kind of; what is that boiled on to? That is I need a height of delta star over and above my h naught for my mass conservation to be valid. Then, what sort of a height should we need for my momentum conservation to be valid because momentum also needs to be conserved. So, it is basically the same principle again that we going to exercise or implement the law; the fundamental law of conservation of momentum.

So, if I do that again. So, I can; am not sort of repeat the flow; the diagram which we did. So, if you look at this picture here, so we are looking at this height h naught, which is extending outside of the edge of the boundary layer. So, if I do that and I have a small little element of d y. Say if I do that.

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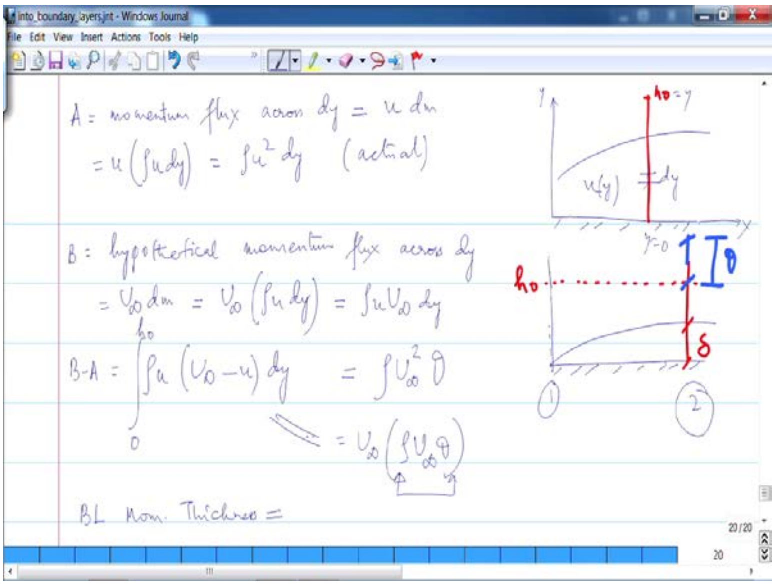

Let us go to the next page. So, A is essentially; this is momentum flux. So, momentum flux; momentum flux across d y. d y is a small (Refer Time: 09:15). Let me just draw the little picture here for, you know, for reference. So this is, this is the plates. This is y and we do have that. So, what we are going to do is we consider some height, which is this height is say h naught and what we are doing is considering a small element and this is d y. So, here Y is 0 and here Y is h naught. So, what we saying is momentum flux across this little d y is nothing but u into d m because here the velocity. It has the velocity profile u y, essentially. So, this is; so when I say u d m, basically I mean u y d m. So, that is why. So, we will have to integrate it. So, u d m, so now, then we are going to use some math; u. So, what is the mass? What is the mass of this d y?

So, density and velocity which goes through that d y, so this is essentially the mass flux. Then, what this becomes is rho u square d y. So, this is what it; what actually goes through. Now, B is the hypothetical momentum flux across d y. So, hypothetical momentum flux across d y basically means this is equal to because it is for inviscid case. Inviscid case of velocity profile will be U infinity, U into d y. So sorry, not across d y, it should be into d m. So, is nothing but d m, this is total mass. So, then this is essentially this. Therefore, this; now, this d m remains the same.

Now, what we basically say? This, we have to be very careful when you do this. So, U infinity d m is still this. Does it make sense? So, then this is essentially rho u U infinity d y. So, this is essentially due to the; this is the momentum flux across; momentum flux into free stream. Now, what you basically say is we have a little mass. We have a little mass that mass is not changing and we say that this mass here is a rho u d y. This small u is basically in the presence of the boundary layer, the velocity is u, u y; so, u y d y. So, this is the mass corresponding to the velocity u and height d y, for unit length, so, now; for unit depth.

So, now the thing is the momentum flux. So, what would be the momentum? With which this small mass will be actually moved? How will it be moved in the sense that when the flow comes in, right, it is going to move this mass. How will it move? It will move it with velocity U. So, therefore, what is the momentum; corresponding momentum? So, which is U into d m, so U into this. However, the same mass, if its existing here but if the what came in, what came in was not the small u, was the hypothetical velocity which is U infinity.

Then, what would be momentum would be? Momentum would be this; U infinity. So, U infinity is essentially the momentum, which is basically for the inviscid case. So, again obviously B is greater than A. So, the loss or the decrement in the momentum is B minus A, which is equal to rho u U infinity minus u d y and if we integrate that, the total decrement, so that we shall integrate from 0 to h naught that would be my total decrement in the momentum.

Now, what we did last time? In terms of the mass, we said that this is the total decrement in the mass. So, the mass needs a separate height to move, it cannot. The entire mass cannot exit through height h naught it needs an extra height, which is delta star. In this case, you need an extra height for this momentum to exit. So, let that height be theta, so that, so when I do that, so then I call this; how do I therefore equate it? Then I say rho U infinity square, which is theta rho U. So, this is the total momentum flux, through that extra theta.

So, what basically I am saying is that you got this delta, this plate and you got this; say, boundary layer and this height is delta, and this height is h naught. Then, I need another height, which is like this; this bit and that is theta. So, now I can say that when I have, so, in this case the way to look at this. The way to look at this, that if I have this flow which is entering here. So, the way to look at this, what I am trying to say is that if the flow enters here at this location, at the height h naught it has a certain momentum.

Now, for that entire momentum to be conserved I need to see the entire momentum go out at the exit, as well. That is not going to be possible with just the height h naught. One would need an extra height. And that extra height, corresponding height is theta. So, which is similar to what we were saying about the displacement thickness as well, so that in this. So, rho infinity square this. So, if this one I can also write this I can write as U infinity rho U infinity theta.

So, basically what I am saying is that this entire decrement in the momentum flux is equal to; equal to what? If you look at this, this is essentially the mass which is corresponding to this height theta. So, what you mean is that the mass; what is the momentum generated by the mass corresponding to height theta. So, in other words what you can say is that what does the boundary layer momentum thickness. So, we said boundary layer momentum thickness that essentially would mean the momentum, caused by the free stream across a mass, which has a height theta and this theta is; therefore this is the height. So, this is the momentum which is caused due to the free stream over a mass corresponding to the boundary layer momentum thickness and this is equal. This and this, entire momentum is equal to the momentum decrement from section 1 to section 2 and this, whatever height h naught you take, the h naught could be anywhere it could be delta as well.

So, that is what? So, this basically, so all or another way of looking on it is the delta is the height of the fluid, the mass, the height of the fluid. So, the mass of which corresponds to the momentum caused by the free stream, which accounts for the momentum decrement as the fluid moves from section 1 to section 2 over a boundary layer. So, that is essentially our momentum thickness. Well, having said that; so now again we have all these equations here. So, if I were to equate this, this is our equation that we have. This is our equation.

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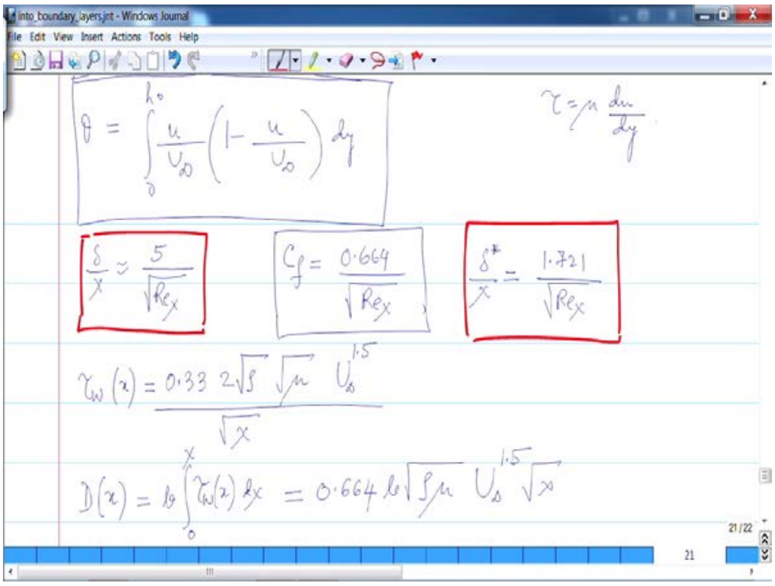

So, if I would equate this, what I would get is this 0 to h naught u by U infinity 1 minus u by U infinity. Again, what you increasingly see here is that, if you can find an expression for the velocity profile for this, then you should be able to find an expression for displacement thickness, momentum thickness and so on and so forth.

So, now having come across that, now let us sort of go ahead and so we should be able to get some; let us see. Let us see how we are going to do this now. So, there are few things have to go about this solving the problem. Of course, you can insert the values of u and so on and so forth and do that.

Now, one of the important things; one of the reasons of studying the boundary layer is because of this whole drag, a thing. It is a viscous effect. So this question here, find the drag on one side of the plate; that is important. That is the very important thing. So, we found out various expressions and all that. So now, let us just go ahead and see some formula and we will kind of develop that and then do this problem before going into the details of it. Some details, is to how Prandtl came upon his formula and place his; was able to give. Who is Prandtl? He is a student by the way. In 1908, he was able to propose solutions for this. Now, these are things that we will discuss as we go further in the course.

So, as of now what we know, what we? Let us just look at some formulas. This is basically coming from this boundary layer theory. Now, this is something that we know. Let us go further down. So, what we know so far is delta by X, which is 5.5. So, I am going to just write that. This is one of the things that I know. Now, C f is coefficient of friction actually, this is given as this, let us see. These are things we tend to use right in our study and this is an expression that; this is for displacement thickness. So, that is, I think we wrote something about 1.83, so similar.

Now, wall shear stress, which also depends on x of course. Now, this is an expression for wall shear stress. So, you know what wall shear stress is. So, shear stress is nothing but, I think we talked about this in detail, in the first couple of modules. So, sort of remind it as per that. So, this is; these are some formulas which we were kind of trying to do. So now, this is an expression for wall shear stress.

Now, essentially that this should be sufficient because when you are talking about drag, what we get from the boundary layer is essentially an expression or understanding of the shear stress. We talked about the shear stress at the beginning. So, shear stress d. So, you can see d u d y. So, which is how the, how is a velocity varying as we move along Y and this is obviously the maximum near the wall. So because it is 0 near the wall, just above it, it is little more. And, so that is how we get the velocity profile. So therefore, the wall, the shear stress of the wall is maximum, obviously. So, as we go, it, you know decreases. So now, so therefore this is the shear stress. So, this is the shear, which is being caused by the fluid on the surface; on a solid surface. So, therefore, it causes a drag. So, all we are going to talk about the drag is integrate this shear force, sorry, it is the shear stress. So, when we integrate it over the area and that should give us the drag.

So, if I would calculate the drag which again will depend on the length. The x that I am considering because as you can see here; if you see here, the shear stress, wall shear stress is inversely proportional to the root of x, so this is no like linear correlation, d x. So, if you see this, we are looking at length x, tau w x d x and you multiply that by the breadth of the plate. So, d x is basically b tau x d x. So, if I use the above formula, what I get is this; 0.664 b under root rho mu U infinity 1.5 under root x. So, you can cross check, if I did. So, what I did is I integrated this tau x into d x. Now, multiply that by b. What I got here is 0.664 b and the b comes here, under root rho mu this U infinity is at, and yes, this is what we get; x. So, this is; cross check if this is ok. So, this is what we get.

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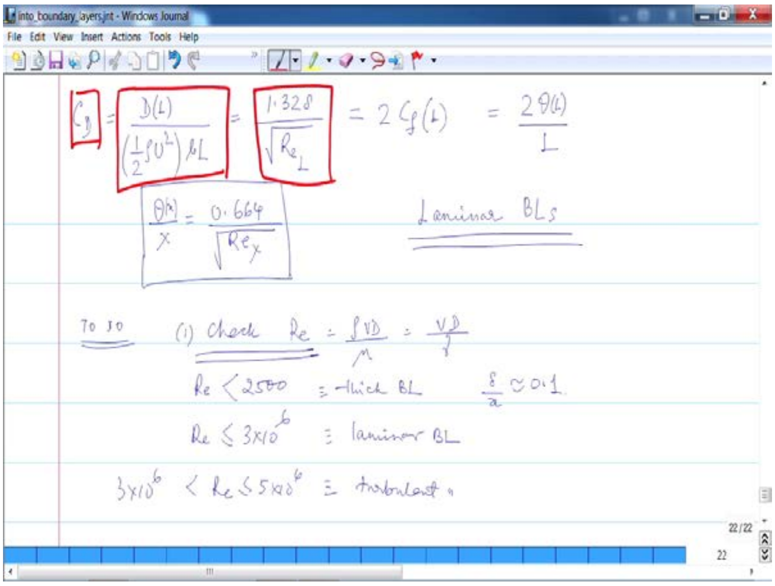

So, then coefficient of drag, that is, that comes from that is basically how we call C D. I mean, how do we define C D? So, this basically comes from the formula. So, if you do that, this is basically drag; over the length L. You could say x. Yes, it is basically half rho U square. This is a dynamic pressure into the area. So, this is a standard relationship, in the sense that when in cases of wings coefficient and lift coefficient drag. So, this is how we define it.

So, this is the total drag by dynamic pressure into the area. So, if I do that in this particular case, what I get is 1.328 by R e, whatever here, length and that is equal to; if you see two times; that is basically, if you see two times coefficient of friction because as if you see I wrote this at the first time itself. So, when I calculated this and finally I come to this, I saw C D to be this, this is equal to nothing but two times C f L. So, you can sort of expand that little bit. So, you do this.

Now, did I just say that theta by x is point; so, I have not given you an expression for theta. Or, did I? Let us go back. No, I do not think I have. So, let us go back. So, let us go and see if you can find something regarding that. Now, momentum thickness, this C D thing, we can write this as 2 theta. Of course, that depends on L as well. So, did we say something about theta by x somewhere? I do not think we did it. So, we did not say anything about theta by x. So, basically this is what we get.

So, now this C D is equal to 2 into theta by L. So, then, therefore if I do that, now if I equate all of this. So, therefore what I get is theta by x is nothing but 0.664 by R e x. How? Because if you see if theta by L is nothing but C f L. What is C f L? C f L is nothing but 0.664 by R e x. So, 0.664 by R e x, this L, I have just replaced that by x. So, this is nothing but theta x. So, this is my expression for the displacement; the momentum thickness, actually.

Now if I do that, if I do that when; how do I go about this problem? Well, let us sort of; now before I delve into all these things, the relationships that we have drawn here. You know, that I have just derived or talked about here, they are for laminar boundary layers. This is very important to understand. These are for laminar boundary layers. They are turbulent boundary layers, as well. And then, we have to take different; slight different things on a consideration. So, then the formula will be slightly different. That is very important.

So the first thing, one thing that when you do the problem that we just discussed. Let me; what I will do here, before I close this module is that I will just give you some hints, you go ahead and do it yourself and we will discuss as soon in the next module. This is like little homework for you.

So, what the first thing is that you do. So, let us do this. This is your to do list. So, the first thing you do is a check. Now, check the Reynolds number. Now, the Reynolds number is what? It is rho V D by mu or this is V D by mu, this is what it is. Now, then I will give you these; these matrix. So, if R e is less than 2500, then we have a thick boundary layer and delta by x is usually; a delta by x is around 0.1. So, R e is less than equal to 10 to power 3 million then it is laminar boundary layer and if R e is greater than 3 million and less than 5 million, then it is turbulent boundary layer. Now, these are the checks.

Now, what you need to do is go ahead and first check the R e and make sure that you can actually use this formula that we have just discussed. So, once it is added, what you can see here is that, when we did this, so look into the formula here. So, this is your C D and we use this, which is we relate it to the Reynolds number. So, here if you find out the Reynolds number corresponding to the length L and you should be able to find out the C D and if you find out the C D, then you can find out the drag. How? Because the C D is connected, you get directly from the drag. So, then you can find out the drag and then, if you find out the drag, then you need to find out the delta, delta star and delta theta. So, you need to find out the drag. So, you should able to find out from there.

So, then again we have this expression for delta. So, hopefully you can find that out. This is the boundary layer thickness and then you have this, which is the displacement thickness, laminar flat plate and there is you get that. So, I will sort of leave you there and you could also find out then theta. You know, you could find out theta from here as well and there is something called a shift factor. We will get it to that later on; you basically have two cases to deal with here. So, get some practice. So, you have got two cases; one is for air and one is for water. So, it will be very interesting to see that for the same plate L is 1 meters and b is 3 meters and the same velocity of 2 meters per second. If the fluid is air, then what is the drag, delta, delta star and theta? Instead of if it is water, which is almost thousand times heavier; 1000 kilograms per meter cube. Then, how does the drag, delta, delta star and theta changed? So, this is a very interesting problem. So, you should sort of engage yourself quite a little bit. So, as you just; if you go ahead and try to look into this problem by yourself and we will come back and look at this in the next module. So, that should close it.

Thanks.