Introduction to Boundary Layers

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Module - 04
Lecture - 04
Concept of a Boundary Layer (BL)-II

So, let us therefore now begin, like we said in the previous module that we will be going to find that relationship between Boundary Layer thickness etcetera. Let us first start by doing this. This is basically, what we are going to do is, we going to do a control volume analysis of the drag force on a flat plate which has been caused by the boundary layer because the flow is being retarded by the flat plate, so therefore that is going to cause a drag on the plate.

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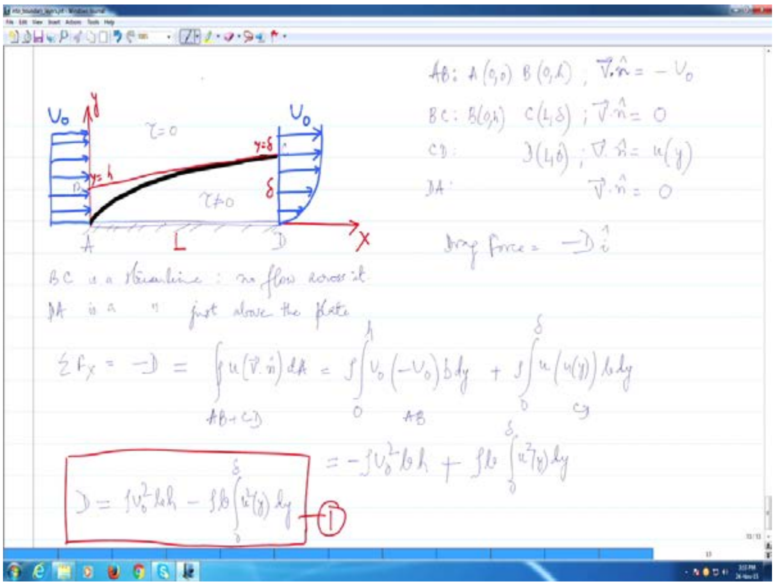

So, let us see what that mean. What we going to start doing is, let me draw, this is my flat plate, alright. This is x and you have in a flow coming in so let me draw. This is the flow coming in, so I would have, this my velocity profile right at the entrance and we will look at this entire plate, at the outer edge. At the other end, this is my velocity profile, what you get at this side is this is kind of a velocity profile. Now, this I am going to call this is U naught and this is same as U infinity if you will, so this is it.

Of course, we have the boundary layer, so say the boundary layer is something like this: so boundary layer is something like that. Essentially, this please do not get confused so therefore, this here itself is also U naught, this velocity is also U naught, meaning these two arrows, so now if I go even further up this all will be U naught same as this. Alright, this is my boundary layer. Now, what I need? What I am going do here is? I am going to draw a control volume and the length of the plate is L, this is the lengths of the plate which is L. Now, let us call this height, the height of the boundary layer at this edge let us call that as delta.

And, what we are going to do is we draw a control volume in this way. So we are going to draw a tangent to the boundary layer at delta and we are going to pull it down. If we do that then basically here y is equal to delta and this height is essentially edge, we are just calling in that so y is equal to edge. So if I do that. I am going to label this, so I am going to call this as A, this bit as B, this as C and this as D. So, this is what we know.

Again shear stress is not 0 here and shear stress is 0 here Now, what I did here essentially this one, this line I am trying to draw control volume here which is A, B, C, D. This is nothing but a streamline. This line that I have drawn is nothing but a streamline which come which is moving. Basically, you do not have a flow across the streamline and you cannot have a flow across the streamline. That is from your basic knowledge maybe you should wash it up a little bit then what exactly is streamline. So the tangent to the streamline is going to give you the velocity, but there is no flow across the streamline.

The way I am trying to draw the control volume in such a way that is in-composes this parts of the flow where the viscosity effects are present which means in our case in existence of the boundary layer. So I am cutting it up exactly there. I am using a streamline which best defines that. Now, once I do that, and please remember that is this is like a square plate maybe so there the plate does have a width. As of now what we are assuming is that this picture does not change as we move along in to the plane of the paper. Therefore, this is my control volume A, B, C, D.

Now similar to what we did the last time let us find out the direction of the velocity. So, we have got, now look at this wall here so look at this side. So, A, I am going to write the coordinates of A are 0, 0 and B is 0, h. So this is essentially an inlet and here V dot n, V dot n is what? Like we did yesterday, this is like a little area, so the area vectors. This is minus U naught; this is from what we did yesterday, from a previous introductory class. Then we got B C. Let us write on the coordinates, so B again we have written. So, B is 0, h and C is L delta. Now here is what? V dot n, now like I said this has to be 0 because B C is a streamline. So, B C is a streamline and there is no flow across it. I might as well write that here B C is a streamline, no flow across it.

Then again we have C D. So, C is here and the coordinates of D are L, 0. Here, V dot n is equal to this. If I write that, so that is if I write U 1, it is not wrong but then U naught is variable, if you see it is not constant is it not. So, unlike at the entrance it is a constant velocity which is the free stream, in this case denoted by U naught. But on the exit, C D here, U naught is not wrong but it is equal to U naught only at y is equal to delta at D. Therefore, what we will do here is best to write this as U and that it is a function of y.

And then again we have finally D A, so we got the coordinates here. Now, D A is again the streamline which is right above the flat plate, well again there is no flow across the streamline but in any case here the velocity is 0, because it is new at the flat plate and there is new to the viscosity. So, D A is a streamline just above the plate. Here to the velocity is 0. Of course, the fact that is streamline and there is no flow across it, that holes. But in this particular case because it is a boundary layer and you are considering viscosity and this streamline is right next to the flat plate anyway the velocity is 0. Therefore, we have this. Now what we are going to basically do? Therefore, there is a certain retardation being caused due to the effect of viscosity on this boundary layer and that is causing a drag force, which is what we are trying the consider and evaluate and put a quantitative to it and qualitative analysis.

And here, we are going to sum up all this shear forces which is going to cause the drag force and the drag force is of course in direction opposite to the flow, so we are going to write it like that. So my drag force will be, and this is the unit factor in the direction of x; positive x and of course, pressure is uniform. Pressure acts uniformly in all the surfaces, forces due to pressure is 0, uniform pressures. Therefore, let us to try to sum this up. Therefore, sigma F x which is equal to D, so when I say F x I am talking about two forces in the x direction and that like we said is this.

What are we going to do here, how do we calculate force? In this particular case, forces given by rate of change of momentum, this is something that we did in the review so shall go back to it. So what we will do here is integrate, this actually surface integral you could write a double integral there I am not doing that. So, A B, so U V dot n D A integral. So, V dot n as you can see is 0 at B C and D A. So, all we have to do is A B and C D, this whole thing is A B and C D. These are the two entrance exit. So that is all we have to do. What we have here? So V dot n into D A into row is essentially the mass flux right. So, mass into velocity and this is a mass flux, so rate of change of mass into velocity that is momentum, so rate of change of momentum. Therefore, this rate of change of momentum is equal to force. What I would urge to do is? Do write down the units and make sure that this boils down to the units of force, so then both the sides will be same.

Let me just repeat that. So this bit, this and this is mass flux right; meaning it is mass per second and this is m into U is what momentum, and this per second means rate of change of momentum. Hence, this thing is rate of change of momentum which boils down to force. So, let us write this down for say A B. So, for A B now Rho is constants so we are going to take that out we will write it for A B, so A B the surface goes from 0, h. So, U is nothing but this thing and V dot n we have already written it is minus that, area is nothing but B into D y. So, B is the width, width into the plane of a paper. This plus again row and this is going from 0 to delta.

So, we are looking at C D, this one is for A B and what we are doing here is for C D. So, C D we go from 0 to delta, U will remain as U, we cannot do anything and V dot n is nothing but U y and d A is again B into D y. Again if I do this then what do I get? So what I get here is minus Rho U naught square B y, so which is h, I am missing that step here, plus Rho B 0 to delta u square y d y. Therefore, the drag force, so I could write the drag force see is the negative sign there, is nothing but Rho u naught square b h minus Rho b integral 0 to delta d y. If I do that, so this let us call this as, we have got some expression for the drag here.

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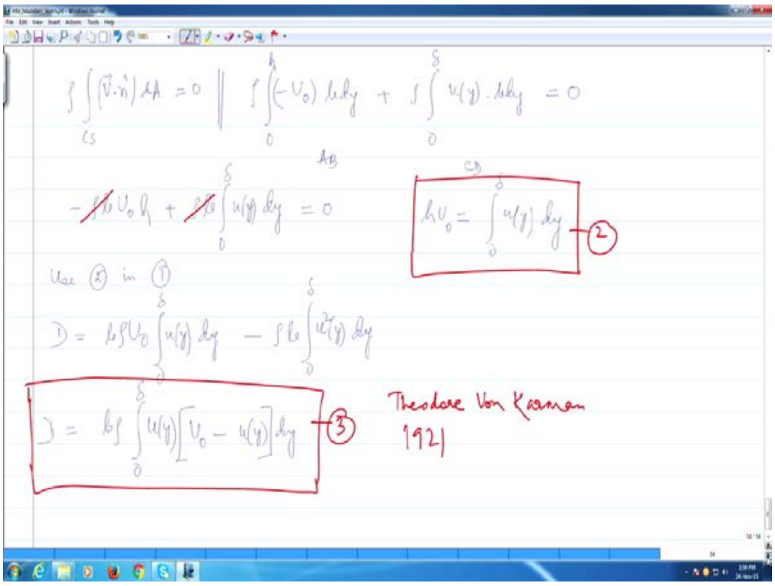

So if I choose that, now from the mass conservation equation, we know that Rho control surface V dot n d A is equal to 0. This is from mass conservation. Again will at the two boundaries, so A B and C D. So, this if we apply to the boundary so what we get is 0. So 0 to h, again this is for A B. V dot n is minus and b d y plus Rho 0 to h u y into b d y. This is again for C D and this whole thing is equal to 0. So, if I do the math here, I will not miss the step here. So, what will get is minus Rho b u naught h plus Rho b, sorry this is not h right this has to be delta correct. So 0 to delta u y d y that is equal to 0, so Rho and b right, so Rho and b canceled out you get that. What we get here is h u naught is equal to 0 to delta u y d y. This is what we get here and let me do this.

Therefore, we needed an expression for this right here, so, h into U naught, so let see if we will insert what we get there into this. We are going to insert or rather use two in one. If I choose that what we get is that a drag is equal to b Rho U naught, so I am going to just replace h naught by this integral so it is by delta u y d y minus Rho b 0 to delta d y, or I will write d to be equal to b Rho integral 0 to delta u y into U naught minus u y d y.

This was actually first proposed by Theodore Von Karman in 1921, this was first proposed by him and it is momentum interval theory for boundary layers. So basically, we used to control volume approach and we do an integral of the shear forces in order to get an expression for drag. So now, if you look at this we should be able to find out this drag provided we have some idea of the behavior of u y. If we knew u y then we could easily get this. Now, the main thing is how does the velocity so what is the velocity profile? There we come down to that. So, various things have been proposed.

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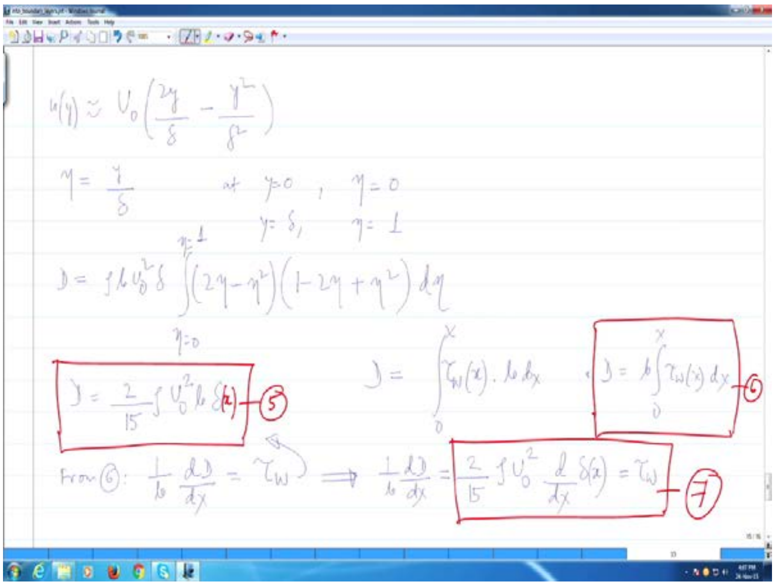

For example, if we use this approximate value. So, approximate say we use this value this expression 2 y by delta minus y square by delta square. This is the expression for u y. This is between 0 and delta. So, if I use this for an expression for u y and then we substitute it into this expression for drag given by the equation 3 then, of course, I mean I will let you do the math yourself what I will give me some hints, so what you should do is you could get another variable call it eta is equal to y by delta. Y by delta so that at y is equal to 0, eta is equal to 0 at y is equal to delta, eta is equal to 1

So, therefore, what happens is you could take. Therefore, the drag, so just replace it, I am going to replace that. Then you get all this b U naught square delta. The integral then becomes eta is equal to 0 eta is equal to 1 and inside 2 eta minus eta square 1 minus 2 eta plus eta square d eta. If I chose this and then finally, this boils down to D is equal to 2 by 15 Rho U naught square b of delta. What you see here? Is that for a given flat plate if you know is width it, if you know the free streaming velocity which is coming in, if you know is density of fluid it did not be water it is could be anything else or anything else honey then this delta at a particular point, this delta is the height of the boundary layer at a certain point and then we can calculate drags. So, drag basically depends on the height of the boundary layer. It is very important to mention here that this delta will be different with x. As we move along x, delta is going to change right so we kind of know that, so let us call this that.

Therefore, this is from our expression here. Now, Karman of course, is we are integrating the shear a along the plate. Now, in our case of course, the drag is essentially the shear stress which the fluid is exerting on the plate by means of its viscosity. Let us say the shear stress is tau w and we multiply that by the area to give us a shear force. So, how do we do that? So therefore we write the drag, it is the wall, it depends on the x and the b is the width. So, this is a tau x. So we integrate that over the length x. A certain length x will be responsible if the issue sum up all the shear forces, the entire shear force which is exerted on the length x because the shear stress will depend on the x, so that should give us the drag. Therefore, if I do that so then I also or I can write or I will take the b out, so b 0 to x tau w x d x.

If I do that, I am going to call this as 6. Now that I have that, so now from 6 I can write from 6 what I can get, this is what I get from 6. What I get from 6 is? 1 by b, if I differentiate the drag force with respect to x, I get. This is yet in another way of looking at it right that essentially the drag per unit x will give you the shear stress. The drag force divided by the area should give you the shear stress at the wall. So this is what we get and then what we will do is, with this expression we will insert this into 5. If we insert this into 5 what do we get? What we get then is this, that 1 by b d D d x is equal to 2 by 15 Rho u square d d x of delta x is not that fine. We bring the b here and d d d x, we basically differentiating this drag with respect to x so we get something like this. If I do that, and this is basically equal to tau w. Therefore, this is equal to tau w. So, then this is my tau w is it not. Now there is something else also.

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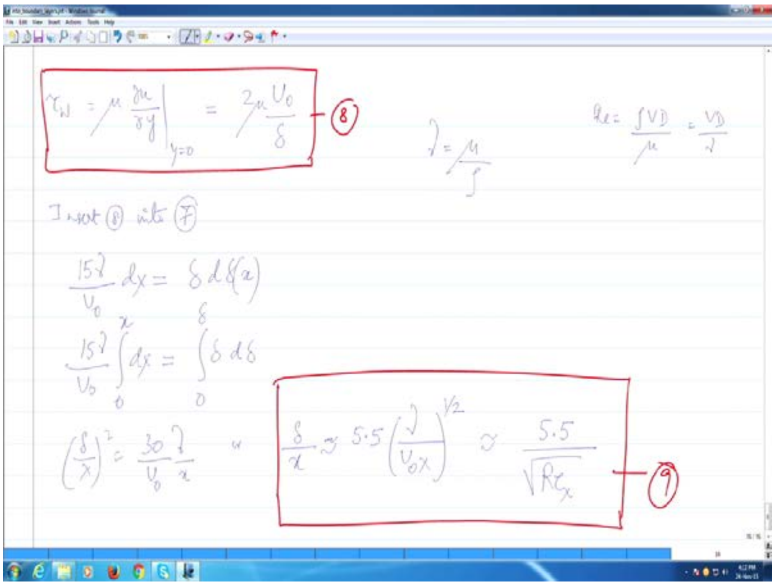

Now, there is also the fact that tau w is equal to mu del u del y, that is from expression and del u del y, if I do it at y is equal to 0, so shear stress is nothing but which we did in the previous modules. Shear stress if I did not have the say this it is nothing but coefficient of viscosity into the gradient of the velocity. Now, shear stress at the wall is nothing but this gradient at y is equal to 0. What is that? Therefore, in this case it is equal to 2 mu U naught by delta.

So, we are going to box enough and say this is 8. So, 8 and this one the previous one what we had done here, this one is 7. So, now that we have an expression for tau wall, we are going to insert that into 7. This 8 is basically tau wall, so this is the expression 2 mu u naught by a delta, we are going to insert 8 into 7. If I am going to do that, so we insert 8 into 7. If I do that what we get is something like this, 15 d x. I am going to leave you to and then I am going to integrate this. If I integrate on integration, so 15 nu what is mu? This is nothing but this by Rho. So, U naught d x, x is going from 0 to x is equal to delta goes from 0 to delta, so delta d delta. So if I do that what I essentially get is delta by x is equal to 30 by U naught nu by x or essentially delta by x, it is very interesting nu u naught x to the power of half.

So this is ultimately what we arrive, we started out calculating the drag force and so on and so forth. What you seem to have found out is that, this x is basically the distance along the flat plate. If we see so, if we do all this then we get this. Now this U naught x by nu, this is nothing but reynolds numbers. And this here reynolds number and when I say x because reynolds number is basically what, the expression for reynolds number is Rho V D by mu this is what you know right, this is nothing but V D by nu. Now this D is the distance. Now, this distance in this case is along x which is in the direction of the movement of the fluid. So therefore this x, the x is important because there is also displacement in the y direction it is not r e y, the direction is y. So this is not r e into s this is r e subscript x denoting that this. Therefore, what you see is that, if you move a certain distance x. You can find out the height of the boundary layer for a certain delta. So, delta by x is around 5.5 by r e x.

These are etcetera formulas, then of course I am not getting in to more issues now you know laminar turbulent and all of that. Therefore, now for the problem that we have set up that for a given plates, what is the distance at which the height of the boundary layer is 1 inch? So I think you have sufficient information here to solve that problem. So what I will do is I will go ahead and stop here and meet you again for the next lecture, and let me not give you the answer to the other problem. You do it and do this yourself and then come back and check it. I think you have enough sufficient information available here to find out that. So, I think we will stop here and pick it up next time.

Thank you.