Introduction to Boundary Layers
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Module – 02
Lecture - 18
Similarity Solutions to the BL
Equations Applied to a Flat Plate-II

Hi; so, welcome back. So, what we (Refer Time: 00:15) basically the position that is that, we were looking at the boundary layer and we said that the boundary layer profile at two x locations are similar to each other.

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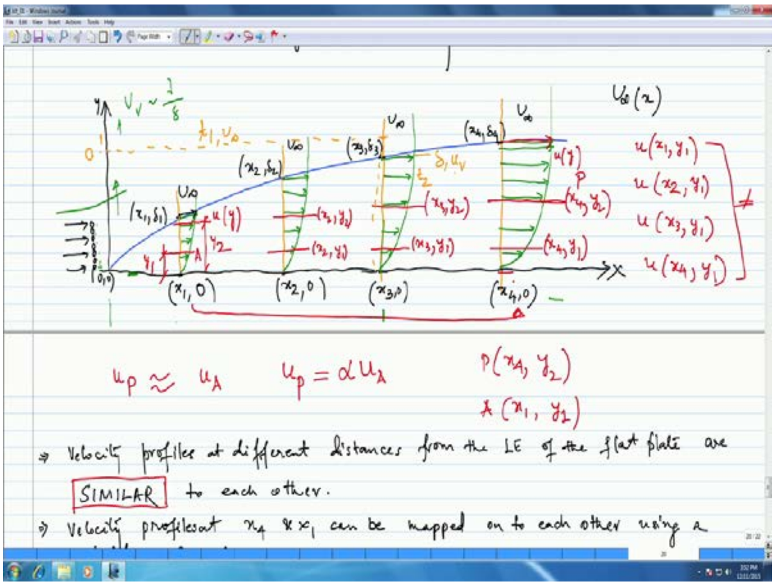

And, what do we mean by that is that, basically we can superimpose the velocity profiles on one on each other. And therefore, for example, if this – if we look at this diagram here; so, the velocity profile at x 4, I can get this by just multiplying the entire velocity profile at x 1 by a suitable scaling factor. If I do that, I should be able to get. So, basically just to give you a very crude example, say if I say multiply the velocity profile at x 1 by say 2; then, I should be able to get the velocity profile at x 2; maybe if I multiply it by 3.5, I should get at x 3 and something like that. So, that is what I am trying to mean.

Now, what we were just saying is that, there is a certain momentum transport in the y direction. Now, why are we talking about this right now? Because we are trying to understand – see these equations and these numbers and lots of thing that we are writing also comes from the factor – we are trying to physically understand the flow. So, we are trying to mix both; we are going to use, trying to understand the physical process as it is happening and then use quantitative mathematics to develop equations and solve them and all of that. Now, look at this for (Refer Time: 01:57) So, we said there is a certain momentum transport in the y direction. And, we said the velocity of that; we are going to denote that as u v – u v. And, this is the function of kinematic viscosity and delta. So, we do not know exactly what the function should be; what this scaling factor should be, but dimensionally correct. So, therefore, this is some – it is a possibility.

Now, what we are going to do is – now, what is – how is this happening? Like I said, there are these fluid particles, which I say – coming in. Now, what happens – I mean look at this – say at location x 3. Now, what exactly is happening when these particles come here. Now, as these particles sort of – there – now, say the particles are passing x 3; now, the question is that, what is the time taken for the particle to cross this position at x 3. And, while it is doing that, during this time that it takes to go from 0 to x 3, some particles have also moved away from the world, because of the momentum transport – because the viscosity transports some momentum away from itself in the y direction. So, these two things were happening at the same time. I think that make sense; so that and because in innovative, look at this is that, it is being sort of pulled along x-axis as well as pulled away along the y-axis. So, therefore, if there are – there are certain – there are the particles – I mean the time that these particles stay to go from the first loca – from the edge or leading edge of the boundary layer to leading edge of the flat plate to say x 3 – to x 3. And, during that time, there are particles which are being transported away into the y-axis. So, is the same time.

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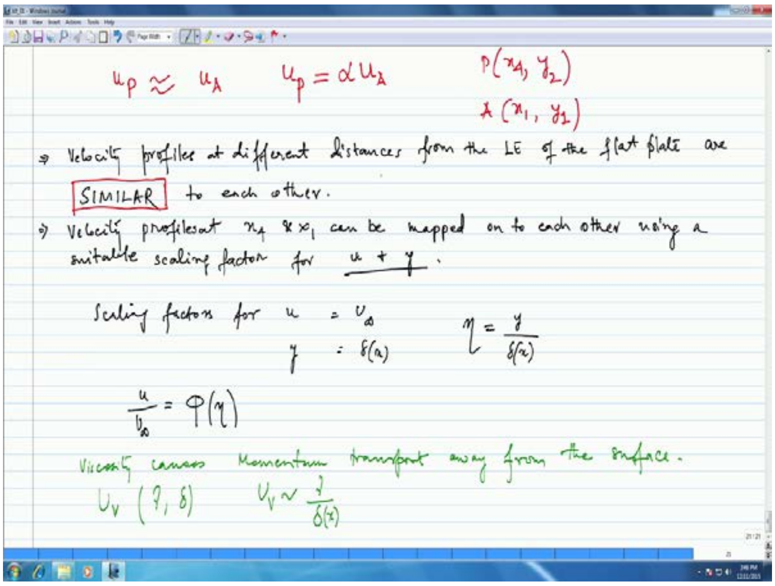

Now, let us see if I can put a – do a little bit of a quantitative analysis for that. So, let me sort of write that down. And hopefully, this makes sense to you.

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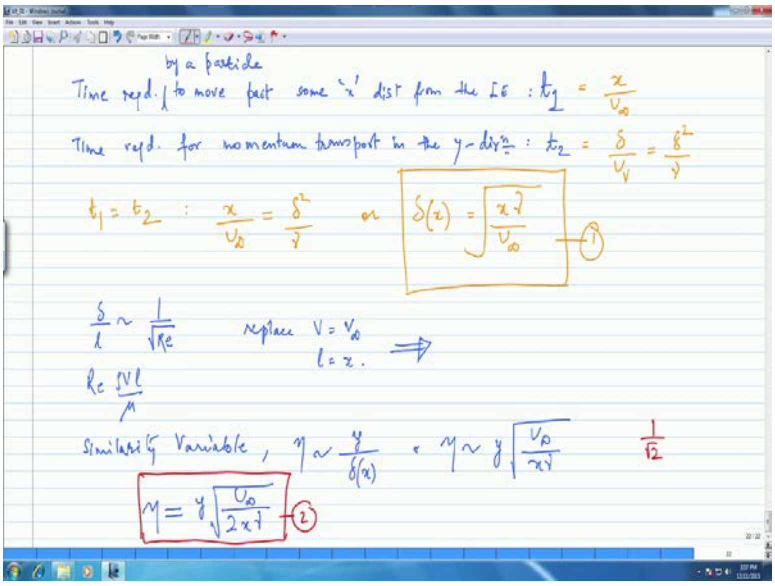

Now, let us say – let us look at this – the time requires; so, what I will be looking at is time required to move past say some – past some x distance from the leading edge by a particle. Time required by a particle to move past some x distance from the leading edge – I am going to call that as t 1. And, the time required for momentum transport in the y direction is t 2. Now, so, what should be… So, the time required to move past some x distance from the leading edge; what should that be? So, if I said x 3; so, this is the x 3. And, what is the velocity with which – So, this is the particle. In the normal case, what is the velocity, which is required? What is the velocity with which it will require to move from x direction. So, it – this is basically – this is the total distance, is x and it is moving with free stream velocity. And, that is the total time which will be required.

Now, what is the time required from momentum transport in the y direction? So, we said that the distance – there again the time required. So, the distance is delta. And, the corresponding velocity is u v. And then – So, again this is distance by velocity and this is again distance by velocity and that gives the time. Now, when I say that – now, again u v is something that we said we could write like this. So, then – So, therefore, this ultimately becomes delta square by nu. So, therefore, if I take – So, basically, what I am saying is – So, if I take a particle; so, if it is – the total time that will be required for the particle from going to here to here is basically free stream. And, if it is – and during that time, there is also manual transport in the y direction, which we are looking at like this. So, therefore, what we are saying is if you look at this; so, when we say the distance is delta; so, we are looking just this test in distance – delta basically means I am looking at a particle just about at the boundary layer edge. That is why – so, we have this particle, which is moving in – so, in the free stream; and, this is the particle, which moves up by delta. It is like these two are kind of meeting each other.

So, I have a particle – free stream, which moves – which moves along here – which moves along here and it – at the time when it is at x 3; it takes time t 1. And, this is moving with free stream velocity. It is at the distance say x 3. At the same time, there is this particle, which is being transported from y is equal to 0 to y is equal to say delta with a velocity u v. So… And, this time is basically t 2. Now, what we are saying is the t 1 is equal to t 2. So, when I say that – so, when I – so, that is what we get. And so, then we said that. And, we say that t 1 is equal to t 2. If I do that, what do I get? So, I get x by u infinity is equal to delta square by nu – nu or delta, which is a function of x is equal to x nu by u infinity – this.

Now, what is interesting is now we looked at this. Now, if we have not already noticed, now what we have done earlier is that, delta by l, is a function of 1 by under root of R e. Now, in this particular case, you replace – here you replace v, because R e is v – rho v l by mu. So, replace v by v infinity and l by x. So, I would like you to do this yourself; just cross check – just cross check. Do this and cross check and see this is something that we had pulled up earlier. What we saw – and this – we got this from another point of view or another way of looking at this boundary layer. So, tell yourself what this will come about. So, this is something that you had looked at earlier; fine; so, looked at that.

So, now, let us go back and say that we had this similarity variable, which is y by delta x. So, we had this. So, we are going to call this. So, we said eta. So, we are going to call this eta as a similarity variable because that is what we kind of using to scale up the velocity profiles at different x locations. So, what we are going to say is that, similarity variable – similarity variable, that is, eta, which is equal to – So, we said basically this is y by delta (Refer Time: 12:31) I mean we do not know what – it is the function of that – something like that. So, now, if you look at delta – if you look at delta from one; so, we have got this. Now, we are going to use this expression. We are going to replace this delta x by this expression here. So, if I do that, what I will get is that or eta is a function of y by – delta is under root x of eta. So, we will write this as this is under root – this is u infinity x of – or, what am I doing wrong? No, y by delta x – that is fine. So, y by this; so, y – u infinity; correct. So, it is fine.

Now, what we are going to do is introduce – introduce some sort of a factor, so that we get an equality here. And, what we are going to do is multiply this whole thing by 1 by under root of 2; that is, say that our eta becomes – eta is equal to y under root of u infinity 2 x and nu – 2 x and nu. And now, what is – why am I using this root of 2? What – from where did I sort of come up with this – I multiply this basically by a factor 1 by root 2. The only reason for this at this point of time is that, it helps in simplifying the differential equation, because this has been tried and tested. So, we kind of know from that – know from experience, so that 1 by root 2 – the factor with which we multiply; it is – helps in simplifying the differential equation; all right. So, then we get a similarity variable, which is as you can see, it is a function. Therefore, of x and y; and you got the kinematic viscosity and the free stream right. So, that seems like a good deal actually.

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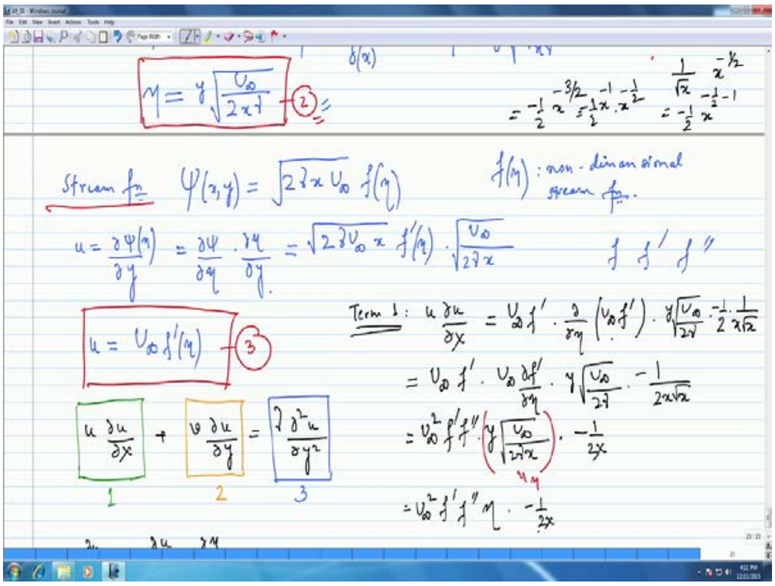

So, now, let us also define a stream function. So, we what we will do is we will define a stream function. And, that is stream function is x and y, which is equal to 2 nu x u infinity and a function of eta. Now, this f of eta right – this f of eta is essentially the non-dimension – non-dimensional free stream function for – I mean stream function. So, f of eta is the non-dimensional – non-dimensional or basically dimensionless – non-dimensional stream function, that is, f of eta. So, we have – So, we have now go this – stream function is this. So, what is the usefulness of these stream functions? Well, stream function is basically we can get the velocity. So, this in a 2D case right. So, stream function is basically one function. And, using the same function, you can – by differentiating the same function, you can get the two velocities. So, numerically, what happens is when you are basically solving from one function; that is the usefulness of it.

So, basically, what I am saying is that, u is del psi del x and – no, not del psi del x; it is del psi del y. So, v is basically minus del psi del x. So, that is how we get. So, in this particular case, as you can see, the psi is basically a function of x eta, which again by itself is a function of x and y. And of course, co efficient of – I mean kinematic viscosity in free stream. So, if I want to do this here; so, then this becomes del psi del eta – del – del eta del y, which is equal to – which is equal to del psi del eta is equal to x and f of dash eta; and, del eta del y – del eta del y; u infinity 2 nu x while if you were to look at this. So, it is basically calculus; and, this is basically calculus. So, del psi del eta. So, now, psi is basically a function of eta. So, here – So, psi is basically a function of eta. So, when I say del psi – del; so, first I – what I need is del psi del y. Now, since psi is a function of eta – if I were to write this psi is a function of eta; so, what I do here is – so, del psi del eta and del eta del y. So, now, del psi del eta is nothing but this. So, this bit. So, this is the components. So, this one is 2 – this thing – f dash of eta – and, del eta del y. So, you can use this equation 2. So, del eta del y has basically just u infinity to this. Let us just pause down to this under root thing.

So, if I do that; So, now, please remember that, when I – I will just write stuff like f, f dash, f double dash. So, all these differentials are basically with respect to eta. Now so, if I have to do this here if you see; so, 2 nu x – So, 2 nu x. So, therefore, what I get is that u is equal to – So, then u infinity and u infinity – So, that stays. So, u infinity f dash of eta. This is a very important relationship. It is an extremely important relationship. So, what do you understand from here? Let us take a step back. So, what we wrote out here – we just came up with this similarity variable, which is given here in 2. And then, we said we will define a stream function in this fashion. So, we defined the stream – And, before I do that; so, when I wrote down the eta here, I introduced this factor 1 by root 2 and I said this helps in the differential equation. So, I do that. And, accordingly, I come back and then I do right an expression for stream function, which is a function of x and y and I get this. I have this. And, from the stream function – we use the stream function in order to – basically, we resolve the stream function. So, then we get the velocity u, which is del – del psi del y and we do the differentiation. And so, of course, as you can see the way I have chosen the stream function with a root 2 here and the way I have chosen eta with the root 2 here kind of cancels that and it helps. So, that is what we get. And so, then finally, what we get is that, u is nothing but u infinity f dash eta.

So, now, if I can solve for f of eta, which is the non-dimensional steam function; if I can solve for f of eta, I should be able to get u. So, all I need here is f of eta. Then, I should be able to get the velocity components. So, now – So, therefore, let us go and look at the boundary layer equations. Now here – so, let us look at the first equation, which is u del u del x plus v del u del y is equal to nu del 2 u del y 2. Now, let us call these; this is the first term; this is the first term; this is the second term; and, this is the third term. Now, look at the first term. So, we are going to look at term 1. So, u del u del x. So, what is u then? So, what is u? So, u is nothing but choosing 3 – it is u infinity f dash. Is that fine? Then, it is del del x of u; it is del del x of u. So, which means that it is nothing but – so, del del x of u. So, I am going to write that. So, del u del x is basically equal to – because as you can see now that, the u is a function of eta. So, then I can write del u del eta and del eta del x. So, I can write that. So, therefore, I can – I will write this as del del eta of u; u again is u infinity and f dash. And, del eta del x – del eta del x – look at eta value. So, this is the value of y – eta. So, this is del eta del x. What will that be? Del eta del x; so, x to the power minus half – x to the power minus half. So, basically, then this becomes y u infinity 2 nu into minus half – minus half into – x of under root x; is not it? So, if I am going to write it this way; so, then what do we get here? So, therefore, if I do this; so, then what I get is (Refer Time: 26:07). So, what we get here is so; then, that is equal to – So, this u infinity here in this here. So, this can come out. So, then this becomes u infinity square into f dash, so that I get del del eta. So, I get del del eta from here; I will write this. So, it is basically del del eta.

Let me write this better; I will write this in clean way. So, then I get u infinity f dash into u infinity del f dash del eta into y under root u infinity 2 nu minus 1 by 2 x under root x – x to the power minus – yes, because basically what I am writing here is it is x – see it is 1; we have taken the derivative with basically taking the derivative of 1 by root x or it is the derivative of x to the power minus half, which is equal to minus half and x to the power minus half minus 1; which is basically then again equal to minus half x to the power minus 3 by 2; which is nothing but minus half x to the power minus 1 into x to the power minus half. So, therefore, I wrote it like this; it is just easy; that is all. So, minus half 1 by x and this.

So, then if I do this; so, then what I will get is – So, I will multiply these two. So, u infinity square f dash; and the del f dash del eta is nothing but f double dash – f double dash. Then, we have a y; we have a y. And, let me sort of introduce this thing. So, I got u infinity 2 nu and I am going to introduce this x over here into minus 1 by 2 x. Now, y into u infinity by 2 eta x is nothing but eta. This whole thing is eta. This whole thing is eta. So, therefore, what we get is – this is equal to u infinity square f dash f double dash into eta into minus 1 by 2 x – minus 1 by 2 x. So, basically, what we are saying is therefore, term 1.

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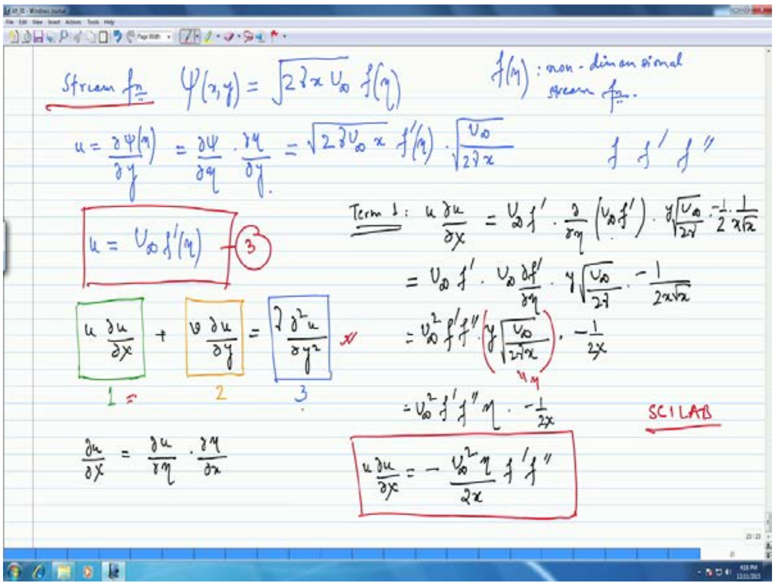

So, therefore, ultimately, u del u del x is equal to minus u infinity square eta by 2 x into f dash f double dash. So, what basically you – what I am doing here is that, we have this set of equation; I have my equation. So, this is my equation that I am trying to solve here. This is the experimental boundary layer equation. So, what I did here is I transformed this first term here – this first term here in terms of the new variables – the non-dimensional stream function and eta and the similarity variable that we have come up with. So, using that; so, therefore, I am able to write that as eta and f dash and f double dash. So, similarly, what we will have to do here is do the same thing for term 2 and term 3. So, once we do that; so, basically, we get expressions and then we will also transform the boundary layer equation, I mean, the boundary conditions. And then, we will get – we will put all that together and see what equation we get; and, is it simple or is it difficult to solve for what is it. So, I will do that. So, I have a little piece of a little code, which I have written and I will show you the output from that. And hopefully, you should be able to write that up yourself and be able to do it; it is quite fun actually; and, all these things that we are discussing. You begin to get those as plots. And, I think that is very exiting when you – when you see that.

So, what I will do is I will stop here and we will finish this in the next two modules. And hopefully, I will have time in the next two modules itself to show you the actual plot from the (Refer Time: 31:39) code. So, which by the way I am using scilab. If you are not aware of, I am using this because this is free and I do not have matlab on my maths. So, you can basically right this whatever; you can use matlab if you have. Scilab is a free software; it is similar to matlab. But, then the syntax are quite different. So, if you have a little bit of coding and you should be able to do it though. And it is not too complicated; say it is simple. So, the equation that we have is all these differential equations. And, what is nice is that, we kind of use this understanding of similar – similarity. So, basically, what we are saying is that, since I can actually – if I. So, basically what I am saying is that, if I can solve for the u and v at location x 1, I can just multiply those by a certain scaling factor and I will get the velocities at x 4; hence, this similarity principle. So, therefore, what I am using is a simple one variable. So, I am using – So, basically, what I am saying is I can use one variable to define the flow at x 4 and x 1.

So, when I – So, therefore, I write that. So, then I can use this and this understanding. And therefore, I come up with this expression for delta x and then we have eta; we define as a new similarity variable and another stream function. And then, using the stream function, of course, we write u and then that is all we need here. So, then we are just transforming this equation – the experimental equation into this similarity variable; so, to represent these derivatives using the similarity variable, which we have done here. So, let us see what we get in the end when we also do it for 3, 2 and 3, and also the quantitative equation. So, we will do that in the next two modules. So, I will stop for now.

Thank you.