Introduction to Boundary Layers
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Module – 02
Lecture –14
Derivation of Prandtl's Laminar BL equations-II

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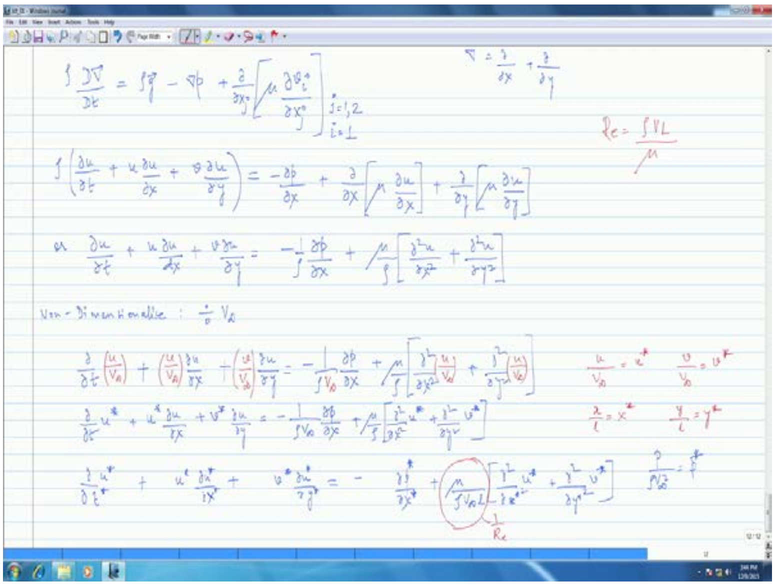

Hi, welcome back. So, we were trying to non-dimensionalize basically develop the boundary layer equations from the Navier-Stokes equation, when we started doing that. So, we kind of stopped here you know a very interesting junction where we had non-dimensionalize the velocities, the time, as well as the lengths which is x, y and so on and so forth.

So, now, what is interesting is that; so we can a stopped here, we will do just you know simple few things. Now if you see rho V infinity a square, so rho v infinity square is nothing but the dynamic pressure, twice the dynamic pressure. So, we can use this to another way of looking at that is just find out the dimensions of rho V infinity square (Refer Time: 01:11). So, see if that will match the same as pressures so that if I divide the pressure by rho infinity square then it becomes you know non-dimensionalize. And again l is here that can go down to l, so this term that we had the pressure term, so this also I can write as. So, basically all I am saying is that if I were to write this that P by rho V infinity square is p star. So, there therefore, I over take this of and this becomes P star by x star, this.

Now, I do not know if you already noticed or know that what is this mean; you know this term here that you see what is this, thus is ring a bell, what is this term. Now if you remember what is Reynolds number rho V L by mu. So, what is this, this is nothing but 1 by R e that is nothing but 1 by R e. So, we have come up with some equation you know which has the Reynolds number included and that is important to us. Because we did say that the boundary layer behavior depends on Reynolds number which is laminar, you know turbulent or even you know keeping close which Reynolds number less than one and so on and so forth. So, Reynolds number is now included is that a good thing or a bad thing, now we will find out very soon.

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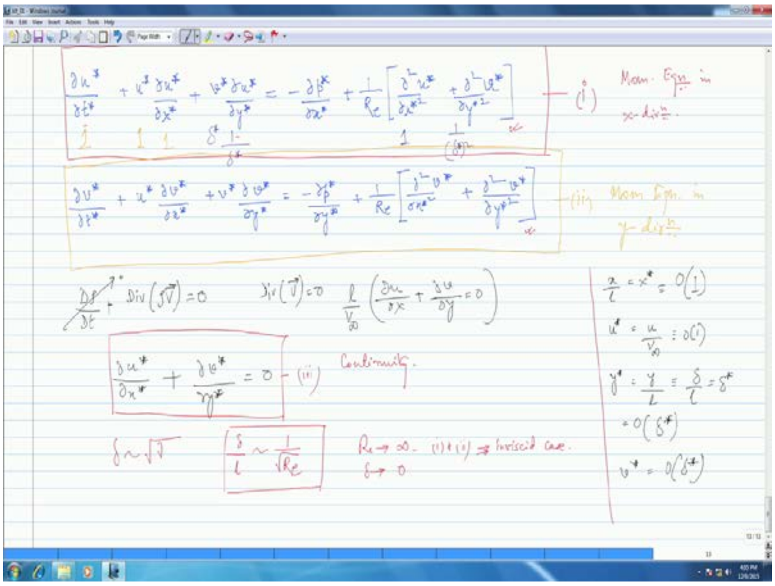

So having so let me therefore write down this equation, finally, so that is del. So, basically we are going from there to here. So, we going to write this as del u star del t star plus u star plus v star del u star is equal to minus del p star del x star plus 1 by R e del 2 del x star square, I do not know if you have noticed instead of the x I wrote del star there easily; u star plus del 2 del y star 2 v. So, this is essentially so this is nothing but the

Navier-Stokes equation or momentum equation in the x-direction. So, this is nothing but the momentum equation in the x direction. Let us call it one. Let us write it momentum equation in x-direction. So, we got this thing.

So, if you see the nature of this equation, we should be able to just look at this equation and write out the momentum equation in the y-direction as well. So, all we need to is change this u. So, if I want do that how would I do that, so that would be del v star del t star plus u star del v star del x star plus v star del v star del y, this thing is equal to minus del p star del y plus 1 by Re del 2 del square v star plus. I think I made a mistake here, sorry I wrote this as this term here as u by v, this is u star, it is u star; it is not v star, there is no v term here actually; u star and this is also u star as, forgive me for that.

This was the second equation. This has to be v here. So, this also has to be v. So, here it is v star, here it is v star. So, this equation is therefore, so this is the momentum equation in the y-direction. We will do something else as so along with this, you know we also need the continuity equation.

So, what is the continuity equation? As we know it. So, the continuity equation is nothing but d rho d t plus basically divergence of rho v is equal to 0. Now that this is let us take a study case, so then this term will go away. Let us take it incompressible, so that this rho term can be taken out of that. So, finally, what we are left out is basically divergence v is this, so which into d essentially means del u del x plus del v del y is 0. So, if I again normalize this, how will I normalize this, I would multiply this entire equation by l by v infinity. So, if I do that the u and the v can get normalized by v infinity and x in the y can get normalized by the l, so that we get del u star del x star plus del v star del y star is equal to 0. So, this is essentially the continuity equation. And let us call this and this. So, therefore, this is continuity.

So, now, what you see is that from the starting point, we come up with these equations; now what we will do is will you know take a breather and look at these equations so we will be able to do all this. We will be able to do all this you introduce the Reynolds number. What does this even mean? Now if you look at this, if you let us now just concentrate on these three equations. Now if you see here that now let us bring in a little bit of our knowledge of boundary layer so far and we have said that the boundary layer thickness, boundary layer thickness is a function of kinematic viscosity.

So, kinematic viscosity and delta by l is a function of 1 by root of Re. And of course, delta is much smaller than l. Now if you see this here now what is this tell you what is this tell you that here, so a boundary layer you know thickness is inversely propositional to the Reynolds number, this square root of the Reynolds number.

So, if the Reynolds number here is very large. So, say Reynolds number tends to this then what happens then what happens is that the boundary layer thickness tends to zero. So, then this means that boundary layer thickness tends to 0. Now boundary layer thickness tends to 0, and the boundary layer thickness depends on the kinematic viscosity. So, if you look the reason, I am saying all these things is that numerically like numerically, the viscosity effect into the equations is accounted for by these terms, these two terms, the second terms on the hand side of the two momentum equations. And what we are saying here, see we do not have a viscosity term here, we do not we got rid of the nu we do not have that any more, but we have a Reynolds number.

So, what we are saying is that if the Reynolds number, there are couple of ways of looking at this. So, you can straight way say what is that you talk so much I mean you know increase a Reynolds number, so anyway this will go true that is true, but I am trying to understand this more from a physical point of view also that is this. So, the point is if the Reynolds number is very large, then the boundary layer thickness is very small, nearly zero and that means, that which now boundary layer thickness also is you know directly propositional to the viscose to the kinematic viscosity which means that itself instead of it is not effecting the flow anymore. So, which essentially means that the large Reynolds number these two terms are going to vanish, what are going to be negligible you know. Or in the sense, it will not contribute to the equations in any way.

Now if that happens if you then look at these equations, so basically they have reduced to the in-viscid case. So, the whole you know lesson out of here is that at high Reynolds number the equations one and two reduce to the in-viscid case. And this is only from this consideration; this is from this consideration that we are talking about. So, which means that if that for very high Reynolds numbers, these equations the equations one and two - the non-dimensionalized, if you try to use this equations very close to the surface say surface of a plate, it is not going to make much of a difference because this can only account for the in-viscid case. So, it cannot take into account viscous effect. So, even if you use this. It is basically is going to mean that you know for very higher Reynolds numbers, this term is anyway going to go to 0. So, solution that you get will have no barring or will not be able to account for the boundary layer or any viscous effect at all.

Now, that is a problem you know because we were hoping to take these equations and bring them to a formula we can solve them in near the boundary layer. So, what do we do, we give up, we would not.

Let us see what else you know we can do to get around this problem. This is a problem we kind of brought upon ourselves by doing this now. So, we got a find ourselves way to get out of this. So, let us see. And basically all we have to do is make sure that at high Reynolds numbers, which is like the limiting case here that these terms do not disappear. So, what do I do, I do not want anything to disappear because if that disappears then my whole thought process falls to I cannot do anything with it, so that is my purpose. So, now what is important here?

So, when I say that disappears so in the sense that very in these here though there is something called order of magnitude order of magnitude. So, basically, if two things have the same order of magnitude essentially means that they are pretty much of the same size. You know and why do I do that because it helps me to compare values. For example, you have this instantaneous term here and this v star now convective term here, now these two values unless and until they are comparable then they have to be comparable to really make any sense to this entire equation.

Now, if one of these terms you know whichever does not matter is like so many (Refer Time: 16:54) so small so that that is does not contribute here it can it might us will be taken out. So, which we can do or what that means, is that then I cannot take care of the boundary layer, but then I set out what? I set out to do is to develop boundary layer equations, which is what I want to study. So, therefore, this is a problem. I did a nice job of you know non-dimensionalize everything and so on and so forth, but I cannot use this in the boundary layer.

So, let us see what we will do, the first things, first what we will do is that we will access the order of magnitude, order of magnitude of each term in this equation. Now, if you see here x star, you know what is x star. So, if I go here, I can use this space actually. Now, x star is essentially x by l. What you need to understand is that x that the boundary layer is developing along x. So, the maximum value of x can be l; that is the length of your plate. So, therefore, x by l which is equal to x star has an order of 1, it is of the order 1. So, in the sense, in another way of looking at that then x is you can think of x in terms of at the distance l, so this of the order 1. Then similarly, u star is nothing but u by V infinity. So, again this is also of the order 1, because u could take the value v infinity that is the that is the maximum that it can take, so it just the order of V infinity. So, therefore, it is order 1.

Now, another thing is if you look at this the Reynolds number, now as Reynolds number is OK, fine. So, u start, now another thing is y star. Now y star is basically y by l; y by l, but what we would like to think is that; what is the order of y star? what is your idea? what should be the value of what should be the order of y star? This should be the order so what so the way to look at this, what is the maximum value of y that we should be looking for because we trying to study the boundary layer. So, the maximum would be delta. So, delta by this so which means this is delta star so that means, the order is of delta star. So, y star is of the order delta star, y star is the order delta star so that now if that is so, now we have this term del v star del y star in the continuity equation.

Now if you see that as Reynolds number tends to infinity or Reynolds number is very large, the boundary layer thickness also tends to 0, which means the delta star tends to 0. If delta star tends to 0, then what happens to this derivative which is del y star because y star is of the order delta star? So, then the continuity equation itself becomes degenerate. So, we need to do something about that. So, numerically this is becoming degenerate. So, how do I tackle that well the way to you know one of ways to prevent this is that we also make v star is of the same order as y. So, we say that v star is also of the order of delta star, so that this to prevent the continuity equation from becoming degenerate. So, now, let us do one thing, let us write the order of magnitude of each term in the momentum equations.

So, let me write it here. So, if you see, so u star the order is 1, what about t star the order is 1, this is so that to order of this term is 1. Then the order of u star is 1; the order of del u star this is also 1. Now v star, now v star is of the order delta star and del u star order is 1 and y star is order delta star is that fine. And whenever you go there R e, and this is again u star. What about R e? Let us come to that now, u star the order is 1 and x star is also you know 1. So, this is one v star is order of delta star and no sorry this is u star, so that is one and y star is of the order delta star. So, this is delta star square. Now what about Re, so Re is rho v d by mu, so Re this and what should be the order of that.

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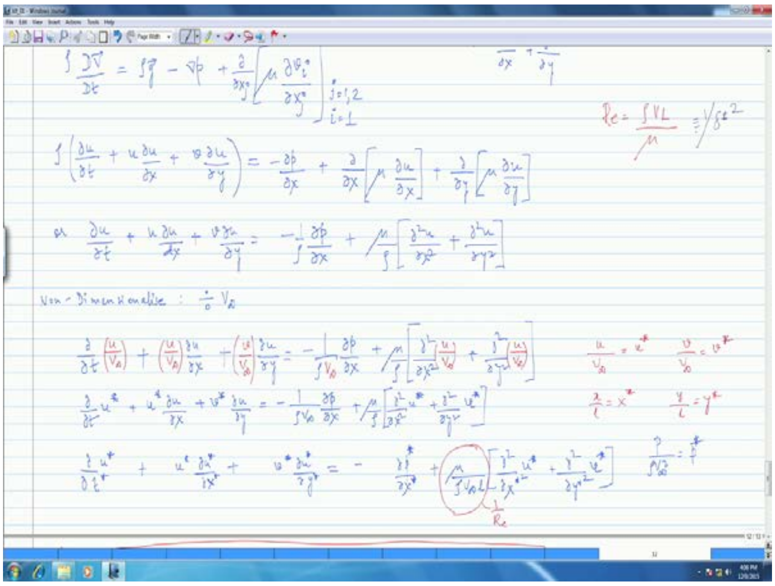

So, what should be the order of the Re, so in my opinion this should be of the order delta star square, you can cross check it. So, R e is basically actually so one by. So, essentially this if you want to look at that.

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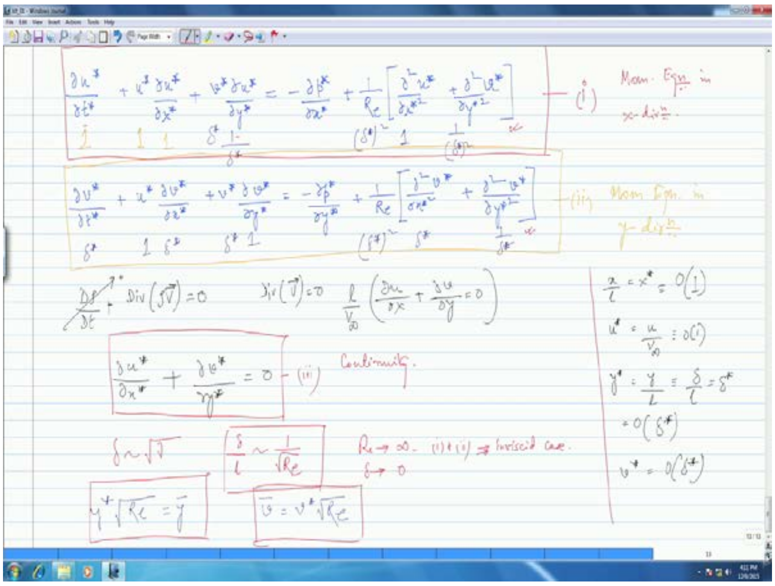

So that 1 by R e so the order is delta star square. So, if I choose that if I write that. Now let us go to this the second equation and see what we get. So, here v star, so this is the order is delta star, t star is one, u star is one, del v star del x star is del star because x star is one. Again, this is this del v star or v star is order of delta star y also of delta star, so this is 1. Then again this one is delta star, this is order of delta star, and this order of 1 by delta star well you know, correct and this is 1 by delta star. Now, if I have that so now you know of course, we have done this. So, what we going to do basically now, say that we need to do something we need a different you know so need to do something, so that we can retain the viscous terms of this equation.

Now, as Reynolds number tends you know to infinity or Reynolds number is very large, so y star is basically delta star. But then if y star of the order delta star, delta star also becomes very less with the increase in the Reynolds number. So, therefore, at high Reynolds number, we cannot use this description of y star, because y star also will become very less you know. So, what we going to do is that multiply this y star values by something, so that you know we can increase it, increases value give in the equation. So, what and the way we are going to do that is by using this transformation, so that y star I am going to multiply by under root of R e and v star because again multiply by under five. And what I get here is y bar and what I get here is v bar.

So, now if you see what is the order of y bar, right? Now, y star the order is delta star and Reynolds number is 1 by delta star, so here it will be 1 by delta. So, therefore, the order of y bar is 1, and similarly the order of v bar is also 1. In that case this is not going to this whole thing is not going to go you know degenerate any more, and also the viscous terms are not going to go away it is long as I multiply this by R e. So, if I do that; I am going to introduce this expression into these 1, 2 and 3. So, basically what I am going to do is therefore, you know take this nu non-dimensionalized form y star this y bar and v bar and in change the y star and v star to y bar and v bar. If I do that what I get finally, so I am going to do that for all the three equations, I am going to do that for all three equations 1, 2 and the continuity equation.

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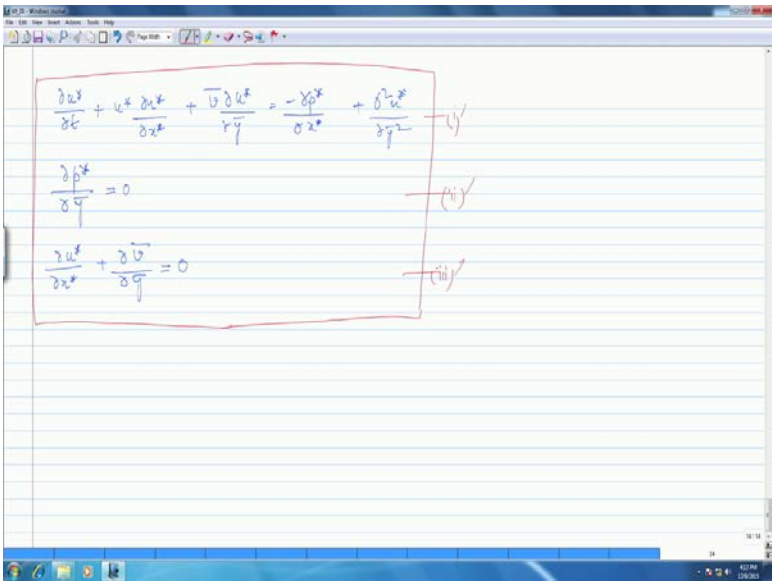

If I do that, what I get is this. So, I can write out the equation that plus we introduce v bar this that is the x momentum, then the y momentum completely reduces to that and the continuity. So, essentially so now we are going to stop here now and pick this up again in the next module. So, what we now been able to do is get this equations to a point where they look like this we did the non dimensionalize equations now look like this and we accounted for so here there is no chance of any term the viscous terms being neglected or anything like that. So, we are going to discuss this. So, this is basically let us say you know so let this be so essentially what we are seeing is that the y star and the v star basically being stretched by a power of Reynolds number which in this case is half, so because we have stretched it by root of R e.

So, let me stop here and we are going to pick this up again and see that what you know from where we go from here, we have not yet reached the boundary layer equations. And it is interesting if we can make further simplifications to this, and there are some you know inferences to be drawn from here which is interesting and the simplicity was start showing you know or rather the simplification to the Navier-Stokes equation and how it applies the boundary layer will start showing now. So, of course, they look much simpler compare to where we started with you can see that, but let us discuss this when we get to the next module.

Thank you.