Introduction to Boundary Layers
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Module – 01
Lecture –13
Derivation of Prandtl's Laminar BL equations-I

Hi, welcome. So, what we are going to start today? So long we have been set of handling boundary layers trying to understand how a boundary layer behaves, also trying to figure out what are the effects of a boundary layer and pretty much trying to figure out what is the purpose of this class, what are we learning here and whether this has some application in practical life. We are trying to do that now.

So far what you have seen, I do not think we really talked about any specific equations, which pertains to a boundary layer. We have talked about basic fundamental equations of fluid mechanics. We have also done the Reynolds transport theorem, but we have really not talked about any specific equation which is four boundary layers. What exactly will numerical equations do? So, after all these are equations.

Well, basically let us put it very simply. If you solve an equation numerically using your computer, using say mat lab, writing a Fortran code or all source of computing facilities that is available today, it shall give you knowledge of the velocity profile and therefore, that gives you big tool in hand to understand this and therefore, you go ahead and estimate the drag and hence, you use it for the application that you are looking for. So, that is basically why we need equations.

Both, they can be scaring looking sometimes, but I think if we understand that where each term is coming from or what each term means, it makes sense. It begins to make sense and then, it is not. So, that is what we are going to try and do and I hope you keep up with me and do not get bored or anything because it is very interesting as to how we will take the basic governing equations which is the Navier stokes equation, and then from there we will develop the equations for laminar boundary layer. We will keep up with laminar boundary layer for now and see what we are able to achieve by that and how do go from that to here.

Now, before I start this, one more question you can ask that you saying basically you will take the Navier stoke equation and develop boundary layer equations from there. Well, why cannot we just use the boundary layers equation as it is? Sorry, the Navier stoke equation as is well to make it simpler really. We do that quite a lot in fluid mechanics, where we specify problems, where we specify zones of problems, so that we can really make life easier for us.

Let me start doing this and hopefully by the end of this you should have an idea as to the final equations, we get compared to the equations that you are going to start with which is the Navier stokes equation. Any standard book will give you the statements of the Navier stoke equations and lot of people will write it differently. So, you know just if you hang on with it, you will kind of give a feel for what they mean. Sometimes they look a little different, but still it is the Navier stoke equation.

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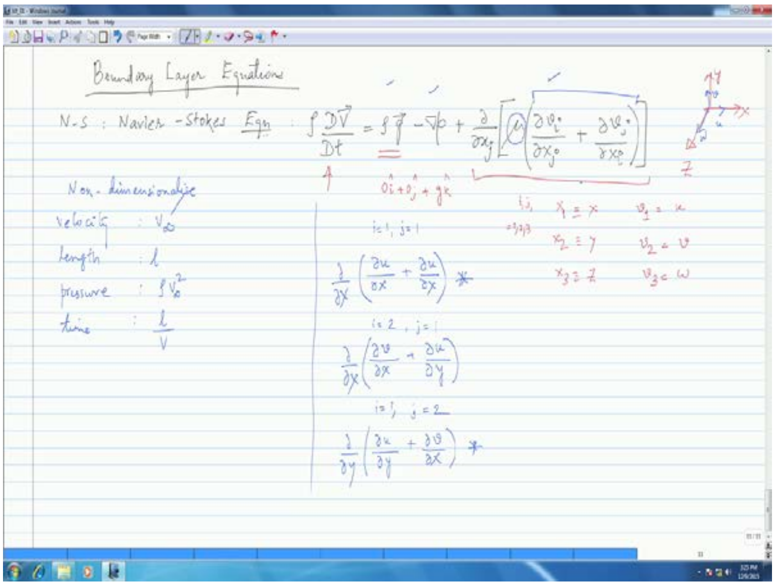

Let me start by that and let me say that this is what we are going to start doing and now is essentially boundary layer. So, that is what we are going to start doing now.

Now, what exactly are the Navier stokes? This is essentially Navier stokes equation; I think you are familiar with this and any standard book will basically tell you that. Now, to put it simply let me write this term first. Now, hopefully you recognize this. This term is basically the total derivatives of the velocity, which basically has the instantaneous component as well as the convective components. So, we are going to when we expand that, we will get that. Now, it is very important, then I always do this is that every time we say Navier stokes equation is not just a bunch of you know letters and symbols or whatever you want. You really need to understand what the equation actually means it makes life simpler.

Now, if you look at this term here, this is what Dv Dt, the total derivative or material derivative of the velocity and you multiply that by the rho which you say you know you could say you know it. Just think of this as a mass multiple that by a volume. If you do that, this is basically mass into acceleration. This is nothing, but mass flux. So, this is nothing, but a force. What you can see here is the due to the movement of the fluid. There is certain force which I am equating it to something here.

What is this something? It is this here g is nothing, but the acceleration due to gravity. Why do I write this arrow here? Well, it is that just to write this in a effective rotation, just simply as v, but we all know that this essentially is the i component, this j component is 0 and it is this g in the k direction. If we consider basically a normal access coordinate system, so again this is the gradient of pressure. Therefore this term here is very interesting and I would like to pull your attention to the fact that you know both, i and j can run from 1, 2, 3.

What this i j mean? You know these are basically nothing, but counters in this case. It is just counters. So, when I say j is so, say x1. X1 basically compare correspondence to this x, then x2 correspondence to y, then x3 correspondence to z. It is just a way is notation, right what you see here. Similarly when I say v i, it basically means u. i is do you know is this 1, sorry not i. So, i also run from 1, 2, 3, i and j they both run from 1, 2, 3. So, v1 is basically u, then v2 is basically v and v3 is basically w. What I am saying is that along the x component, so that is x1, x2 and x3, this is u, this is conventionally, this is v and this is w. This makes sense? So, that is what it is in if you notice carefully, what you have here is basically del v i, del l x j. This del v j del x i at this point, I would suggest that you revisit or just consult a standard book to the derivation of the Navier stokes equation itself. It should give you an idea over this is stoke here and this exactly means now.

Here is this term. I hope you recognize this; this is the coefficient of viscosity. What you see on the left hand side is basically force and what you see on the right hand side are three terms. This is basically what we are saying is the contribution of this force on fluid. It comes from gravitation because you got a mass. Then this is b due to the gravitation the grade into the pressure and then we have pressure forces. So, gravitation force is basically a body force. It is because of that and pressure is again with surface forces and the last term here is basically the surface force or viscous forces. You can see it is i and j. If I would just say j is say i is 1 and j is 1, what we will get there is del u del x plus del. J is also 1, del u del x. We will get that now say keep j1 make i is equal to 2 and then, what do we get then? What we get is, del v del x plus del u because j is to 1 del y so on and so forth. So, if you continue to do this, since I am doing this let me do this as well.

This i is equal to 2 and j is equal to 1. Let me do the other also. i is equal to 1 and j equal to 2. If I do that then what do I get is, i equal to 1. So, this is del u del y plus del v del x. That is what it basically boils down to. These terms that I have written, this basically within this bracket and for this one if I were to calculate the total, the stuff, it is basically then del x j which is x1 which is x. In this case again j is 1, then I look at del del x. In this case j is 2. So, this is del del y. That is what this basically boils down to and this again is the coefficient of viscosity. This is basically giving you an idea of the shear stresses. Hence, the shear force is resulting due to that, so basically we got gravitational forces, you got pressure forces and in a shear force which together forms the total force acting on the fluid.

Now, from here we are going to go and find out things about the boundary layer interesting. How shall we even begin to do that? Well, let us try. We are going to do something very interesting. What we are going to do is non-dimensionalise. So, this equation and with what, basically what I mean is that the velocity for example here v i, so u and v. What the question is? So, you ask that why are you taking only u and v? Where is w? Well, that is because I took 2d. So, what we are going to do here is, also write this in 2d. When we start with that, keep things simple that is all. When I say 2d in case of a fluid, you know a boundary layer. All I mean is that things are not changing in the third dimension which I have not considered. It is not like it is not there or anything it shows that things are constant and they are not changing. If I get a picture at one location of x, then y is the same picture which gets repeated along z. Therefore, I do not need to do that separately.

Now, I am going to non-dimensionalise. So, basically when I say non-dimensionalise, for example u. So, u is a velocity and it has a unit centimeter per second. Now, I need to non-dimensionalise that, so that it becomes something which does not have a dimension. What we are going to do is, we are going to non-dimensionalise the velocity with the free stream velocity. So, with the free stream velocity, then the lengths any lengths which is x y whatever, we are going to call it l, essentially this is any characteristic lengths. When I say characteristic length, what is that mean well for example, in the case of being on as I said is the chord. For a cylinder it could be the diameter. It really depends. If the characteristic length which defines the flow or which is the length of that body which is affecting the flow, so that is the characteristics length, pressure with rho v infinity square two types of dynamic pressure and the time with l by v.

What does that even mean? It is because if you see the unit of l is meter and this is meter per second. So this, both l by v is basically you know. I do not introduce another separate variable here. I take just take it from here, so that I can non-dimensionalise the time. This term here that you see as units of time seconds or minutes or whatever, now what I will do is, write this equation in 2d in x and y direction. So, basically your j shall go from 1 to 2 and I will go to 1. The first case that is what we will do. So, like j will go from 1 to 2 and i will be 1, so i will be 1 and j will be 2. What I will be using is essentially this term and this term if you want to a set of look at that. If I do that, if I do this, let us see what we get. Let us go to the next page.

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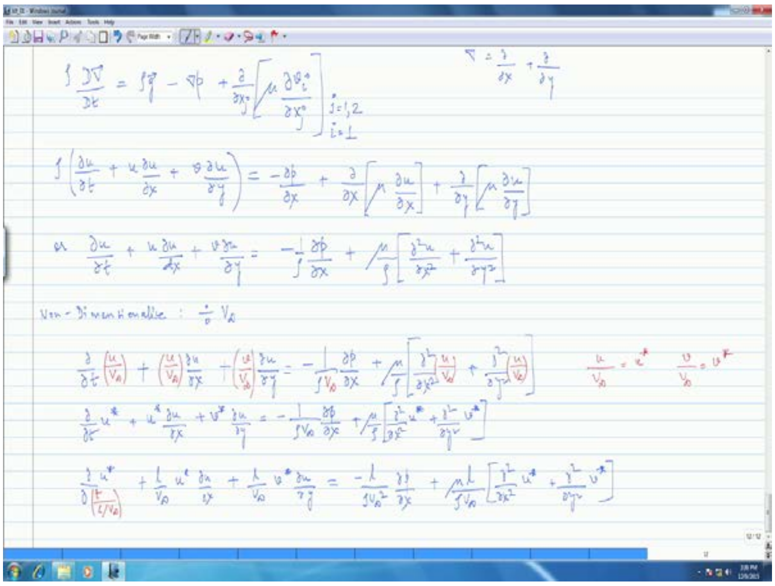

What we are saying is that we got this is equal to rho g grad of p. If I do this, so I can write this as it goes from 1 to 2 and i is 1 because that is we can write this first in the x direction. If I do that, let us expand the left hand side. What is the left hand side? It is basically, del u del t plus u del u del x plus v del u del y plus w del u del z, but we said that we are going to do this in 2d. So, we are going to get rid of this term. We would not have that. We will basically have this on the left hand side. This part is instantaneous and these two terms basically contribute to the convective expression. So, again this is equal to; now again here this is nothing, but rho g in the x direction which is 0.

We would not have a component from the gravitational force, we will have something from the pressure and say again this is gradient, this gradient is nothing, but del del x plus del del y. It is in 2d case. So, if I do that, this is nothing but del p del x. You could write this is basically in operator that is del p del x plus. So, i is equal to 1 and j is equal to 1 to 2. If I do that, then what I will get is del del x of mu del u del x plus del del y mu del u del y. If I do this, then I can write this out completely as or sum we divide by rho throughout. What I get is del u del t plus u del u del x plus v del u del y which is minus 1 by rho del p del x plus mu by rho. What you make of this? This is del 2 u del x 2 plus del 2 u del y 2. That is what we get. So, this is basically the equation in the x direction.

Now, if I non-dimensionalise this, what I will do is; I will divide by v infinity like we said we going to non-dimensionalise the velocity by v infinity. What we will do is, now what are we going to do is non-dimensionalise. So, to do that what we do is, we divide by v infinity throughout. If I do that, essentially what I get is, I could actually write it for you here itself. Then, let me not do that. Let del del t what happens now is it becomes u by v infinity then plus. So, let us just again, this is u by v infinity del u del x del x plus this is del u del y v by v infinity, is that right. We get minus 1 by rho rho v infinity or let us just say we going to put that we have nothing else there. Then this is del p del x plus mu by rho and this is del 2 del x 2 plus del 2 del y 2 u by v infinity u by v infinity. We get that. So, basically when I divide by v infinity, this is what I am doing.

Now, what we are going to do is that we are going to say that u by v infinity is equal to, we are going to just derive write in another denote this by another variable and v by v infinity is v start. These are non-dimensional quantities v star and v. If I do that, then what happens is all this term, let me write it. Therefore if I were to write this, then how does this become? So, this becomes del del t of u star plus u star del u del x plus v star del u del y is equal to minus 1 by rho v infinity del p del x plus mu by rho del 2 del x 2 u star is del 2 del y 2 v star. We need to non-dimensionalise this further.

We will non-dimensionalise the time. How will we do this? We can divide the time by l by v infinity. If I do that, what we going to do is we are going to say to dimensionalise that what we will get is del del. So, that is t by l by v infinity. So, if that is true, then I need to multiply this equation by something else, equation or I have to multiply this entire equation by that which is l by v infinity if you want to look at that. So, this is l by v infinity that will cancel out this thing that I am using here. So, if I do that, basically what I am doing is, I do that l by; I multiply by l by v infinity, here I use that. So, l by v infinity is something that I am going to use here throughout, l by v infinity. If I multiply l by v infinity, I do l by v infinity u star del u del x plus l by v infinity v star del u del y is equal to minus l by rho v infinity square del p del x plus nu l rho v infinity del 2 del x 2 u star plus del 2 del y 2 v star.

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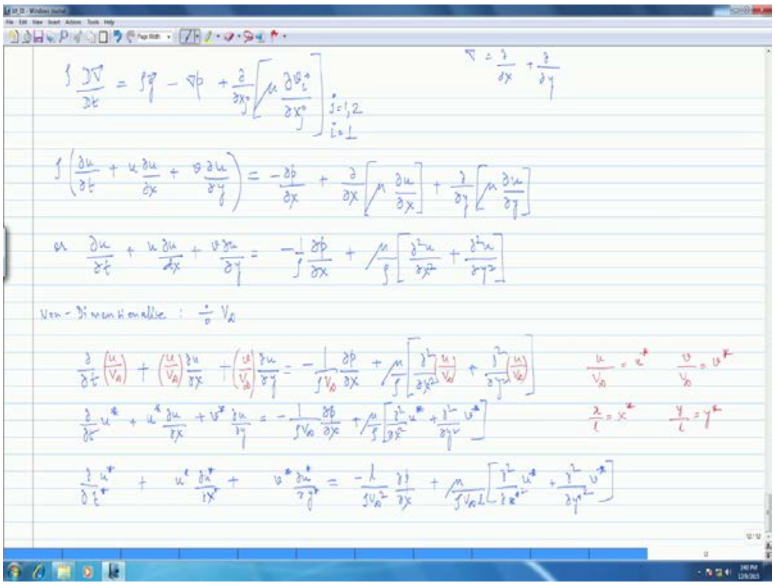

When I do that, what you can see here is that I will do without going to the next page is that this one I am going to use to non-dimensionalise this x. This v infinity I am going to use to non-dimensionalise u. What I will basically do is I am going to just write this term here. So, what I am saying is what we will get is this term here u star del del of u and then, becomes u by v infinity and x by l. This is u star and this is x star, so we are also going to say x by l is x star and y by l is also y star. If I do that, then let me erase that and let me also erase this and say this is u star del u star and del x star. Similarly, if I erase that del u star del x star, this is del y star del y star. We can just leave that in there and here again, well this bit is little tricky because here you need; also we can use the v infinity.

You have got u star already there, but to get x, x star, you need l square. I basically need l square. So, I can divide by l. So, what I will do is that I would multiply this by l square into l here, so that I still get l and then, I leave this l right here and take this l down to x here. You understand what I am doing? What that means is that I will erase this part that goes, but then that comes down here x by l, similarly y by l this. So, with that makes it del x star. So, that is what we are going to write. We will finish that. So, what we get here is del x star 2 and del y star 2. This is what you get nu by rho infinity and this part is also t star.

What I will do is, I will stop here and then, come back into the next module and take it up right from there and work on this a little more.

Thank you.