Foundation of Computational Fluid Dynamics
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Lecture - 09

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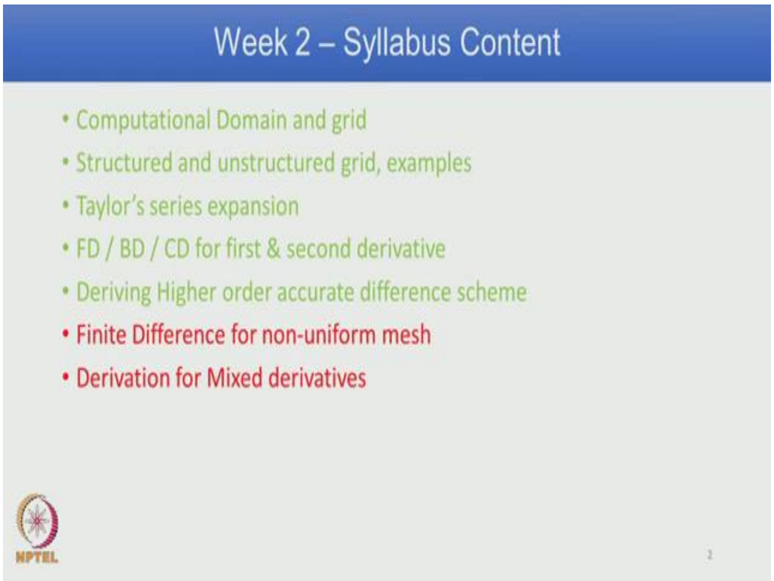

Greetings to you all, we are now onto module four of this course in this week. We have seen so far finite differences scheme and how to obtain forward, backward, central differences scheme for first derivative and same by to get higher order finite difference formula. In today's class, we will particularly find out how to derivation for non-uniform mesh, and how to get mixed derivative and how to obtain finite difference formula for higher derivatives.

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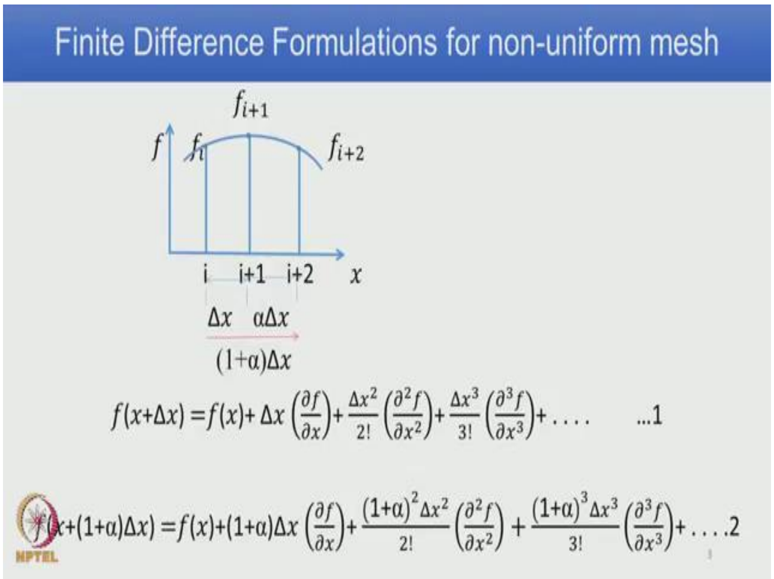

So, this is necessary in the sense you do not have always an options of getting a uniform mesh throughout the domain; you want have finer mesh at some location and coarser mesh at some other location. So, in such situation, you need to have a formula derived based on non-uniform spacing mesh. And this graphically shown here situation, where function and i is a point of interest, and i plus one and i plus two unlike other case, distance between i plus one and i is delta x, and distance between i plus two and i plus one is alpha times delta x. And if you look from i, i plus two is at a distance of one plus alpha delta x. So, we follow the same procedure, write down Taylor series expansion formula, but now with alpha as a coefficient for delta x, so that is also given here, this is already known f of x plus delta x as f of x plus delta x and so on – equation one.

Now, the new difference here is for f of for the second point, which is alpha delta x between i plus one and i plus two or with respect to i, it is at distance of one plus alpha delta x. So, if you write the expansion for the function at i with respect to i plus two then you get formula written here f of x plus one plus alpha delta x as this. The only difference between equation one and equation two is delta x is replaced by one plus alpha delta x; everywhere also, it is delta x square for example, is replaced by one plus alpha square and delta x square and so on. So, as we did in previous case, here also we are interested only the first derivative, so we consider only the first term and follow some arithmetic operation to get derivative. And if you consider higher term then you get corresponding either higher order derivatives, or higher derivatives difference formula.

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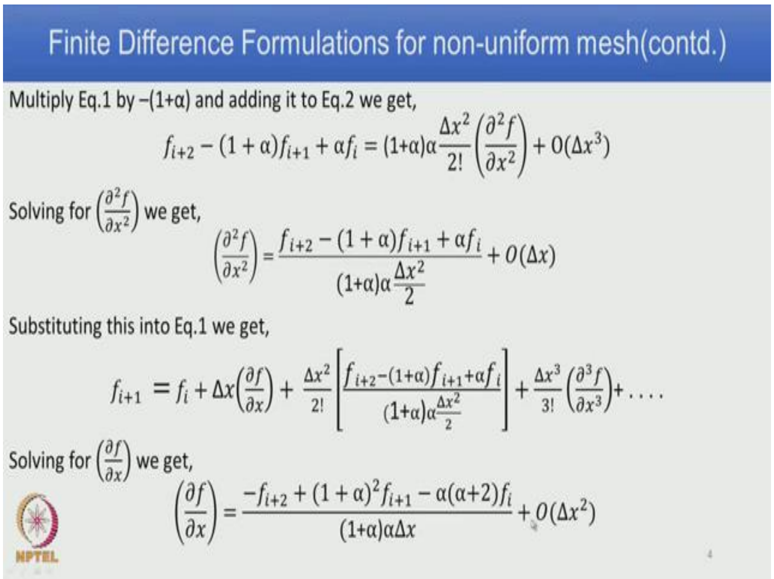

This equation one and this equation two by multiply equation one by one plus alpha and add it to equation two and we get the equation in this form f of i plus two minus one plus alpha f of i plus one plus alpha times f of i equal to this. And we are interested with second derivative here, so we retain that term, remaining terms are brought to the other side so that is given here as f of i plus two minus one plus alpha f of i plus one plus alpha f of i and this coefficient is brought to the denominator, and this is of order delta x cube and if you divide by delta x square then you get of order delta x. So, we obtain second derivative forward difference formula for non-uniform mesh, and this is the order of delta x. Now, if you substitute this expression into the equation one, which is basically equation for forward difference; after simplification, we get first derivative forward difference formula with non-uniform spacing and that is alpha is here. So, this of order del x square.

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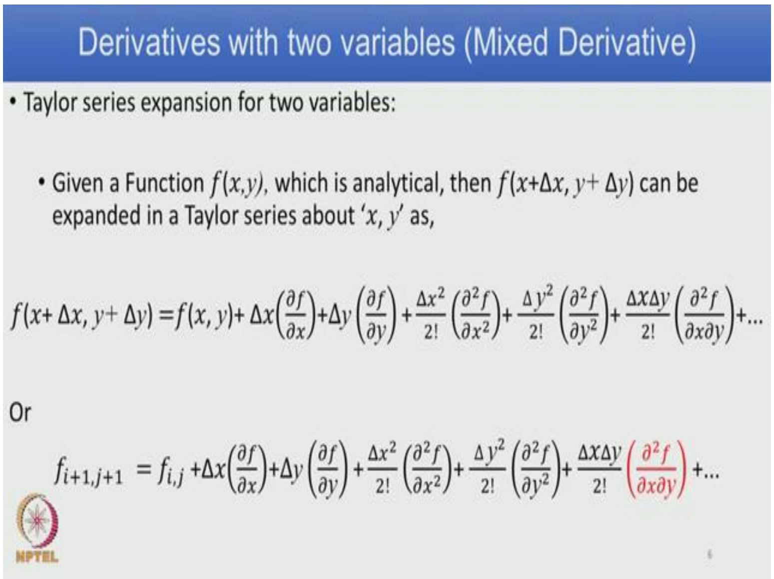

Now, you also have mixed derivative; that means, derivatives with two independent variables. So, for example, dou square u by dou x and dou y; in this case, u is the function of x and y. So, there should be a procedure to obtain finite difference formula for this mixed derivative also. Now, we are going to see how to get derivatives with two variables; otherwise, it is called mixed derivative. Again function f is dependent on x and y, so we do the expansion – Taylor series expansion with f of x plus delta x and y plus delta y. And we see here the Taylor series expansion formula written f of x and y delta x dou f by dou x with new term here for the other direction del y dou f by dou y, so this comes as a pair. Now, again next term, del x square by two factorial dou square f by dou x square del y square by two factorial dou square f by dou y square and other term.

We are actually interested to find out dou square f by dou x dou y, this is what is called mixed derivative or derivative in two direction. So, if you write in i form and j form, so i is for x-direction, and j is for y-direction and written here as del f of i plus one and j plus one corresponding to f of x plus delta x and y plus delta y. And we are actually interested in dou square f dou x and dou y. Now, we have written this expression for considering i plus one and j plus one.

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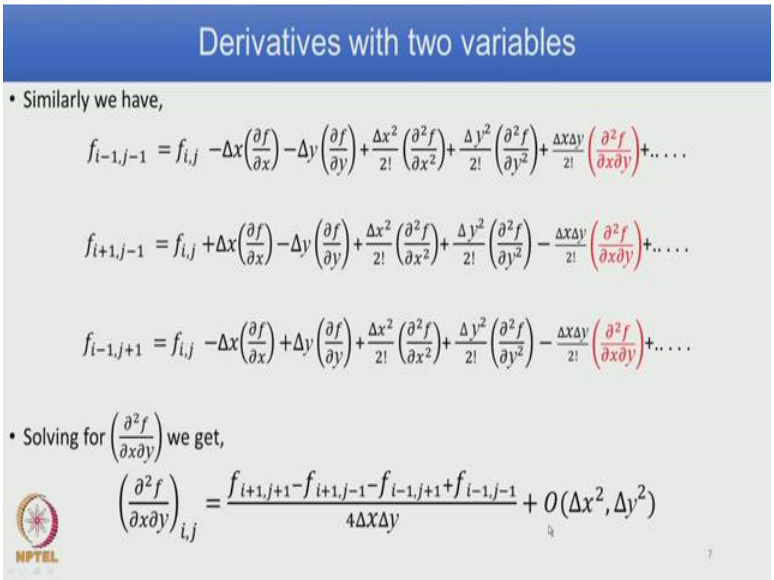

We can repeat this for different combination of i and j and that is what is given here, f of i minus one j minus one, f of i plus one, j minus one; and f i minus one, j plus one. So, in all the three expression here, and one expression in the previous slide, we have a mixed derivative term dou square f dou x dou y, and so we have four equations where the mixed derivative is appearing. So, we do simple arithmetic had all of them and take a average and you get finally, expression for mixed derivative as shown here. So, dou square f by dou x dou y which is evaluated at i comma j with function values consider from neighboring points that is f of i plus one j plus one i plus one j minus one i minus one j plus one i minus one j minus one, and four times del x and del y. And this is of order del x square as well as del y square. So, it is second order accurate in both x-direction as well as y-direction.

Once again, we have written only for a uniform mesh, it is possible to obtain mixed derivative expression also for non-uniform mesh, and for any combination of non-uniform mesh. In the sense, it maybe uniform in x-direction, non-uniform in y-direction vice versa is also true. And it maybe non-uniform in one way in x-direction and non-uniform in another way in y-direction, so any possible combination of mesh is arrangement is possible. And it is possible to obtain derivative for any arrangement of mesh.

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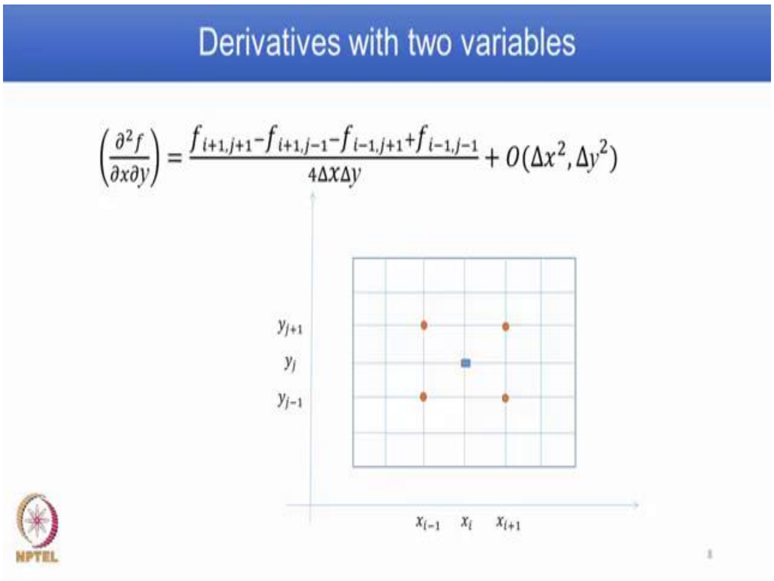

The last slide we had seen how to obtain different scheme for the mixed derivative. I rewrite that formula here again, dou square f by dou x dou y equal to f at i plus one j plus one minus f at i plus one j minus one minus f i minus one j plus one plus f i minus one j minus one divided by four delta x delta y, and this is of order delta x square and delta y square. Now, if you look at the tensile, what is shown here is graphical representation, the blue color is a point of interest, the nodal location x of i, and y of j. And it is influenced by neighboring nodes as shown here. For example, this node is at x i plus one and y j plus one; this node x i plus one, y j minus one; and here it is x i minus one, y j minus one; and here it is y j plus one x i minus one. So, it looks for this mesh arrangement, which is uniformly spaced in x-direction and uniformly spaced in y-direction. And also del x equal to del y that is why you get equal it appears as if the mixed derivative at this point has a equal weightage from surrounding mesh points.

It cannot be so in case of non-uniform mesh. So, we have obtained derivatives first derivative, second derivative, mixed derivative uniform mesh and non-uniform mesh. We have to see one more expression or one more expression for one more term that is higher derivatives.

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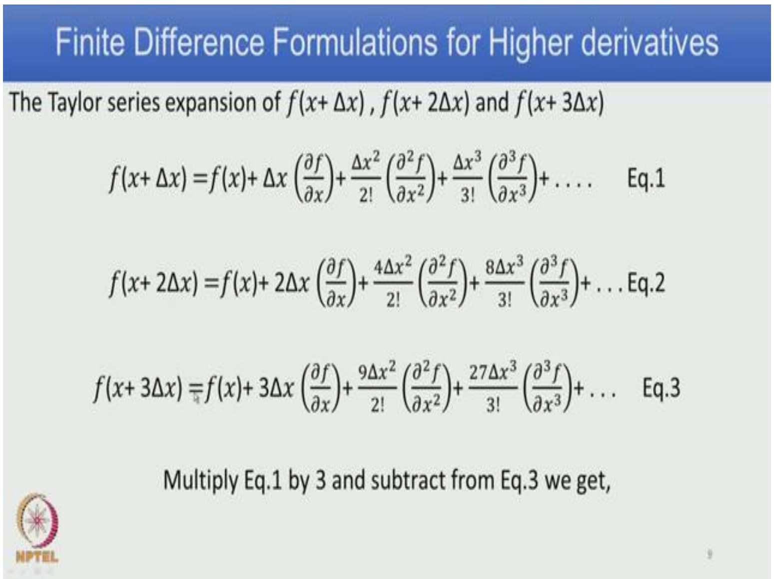

So, we extend the procedure, we got Taylor series expansion f of x plus delta x f of x plus two delta x, now we do one more I considering one more point that is x plus three delta x. So, all the three expressions are given here as equation one, two and equation three. Of course, now you understand the pattern, how to write Taylor series expansion in any direction forward or backward by considering one nodal point, two nodal point and three nodal points also for uniform mesh or non-uniform mesh. By this time, you are very much familiar and you are able to recognize each term, each expression in each term. So, equation one is for delta x and equation is for two delta x, equation three for three delta x. What is our goal, our goal is to get higher derivatives, so we have seen up to first derivative dou f by dou x, second derivative dou square f by dou x square, now we are interested to get third derivative as an example of higher derivative that is why we have consider three expressions. So, multiply equation one by three and subtract from equation three; so we are going to do two small arithmetic operations equation one by three, and equation subtracted from equation three.

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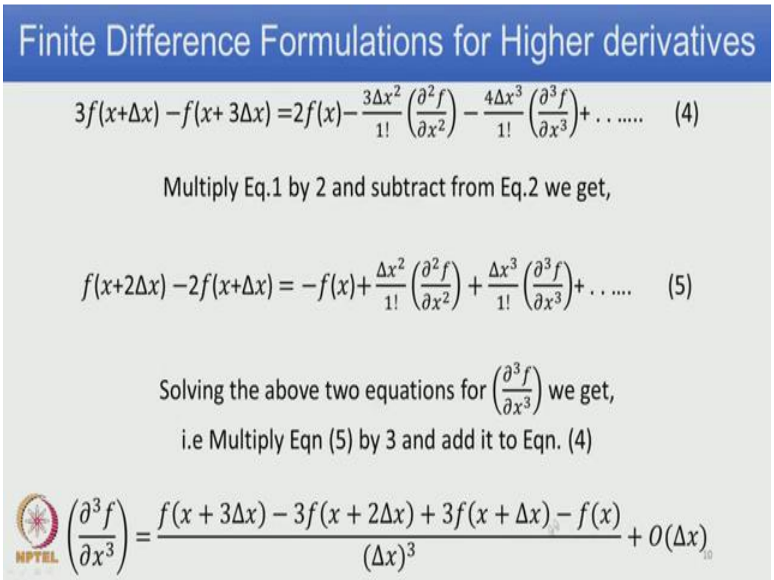

So, if you do that we get expression like this, three times f of x plus delta x minus f of x plus three delta x and on the left hand side; on the right hand side, we have the remaining terms. We name this as equation four. Then one more small operation multiply equation one by two and subtract it from two, then we get one more equation that is shown here f of x plus two delta x minus two times f of x plus delta x on the left hand side; on the right hand side, minus f of x and remaining term, and we name this equation as number five. Now, between four and five, again we have to do small arithmetic operation, we are only interested in third derivative term that is dou cube of f by dou x cube, and that is appearing in both equation four and five. This second derivative term appearing in equation four and five needs to be removed; so, we do a small arithmetic operation, solving the above two equation that is multiply equation five by three and add it to equation four, because the secondary derivative term that is here has coefficient one.

So, if you multiply this by three and if you add this corresponding term here, then you can observe that this term also this term gets cancelled that is why this equation five is multiplied by three and add it to four, so you get dou cube f by dou x cube as f of x plus three delta x minus three times f f of x plus two delta x plus three times f of x plus delta x by dou x cube and this of order delta x. So, in this class, what we have done, we have done how to obtain finite difference procedure on a non-uniform mesh, and how to obtain finite difference procedure or finite difference formula for mixed derivative that is derivative in two direction. And third important subject was how to obtain finite difference formula for higher derivative as on sample we took the third derivative dou cube x by dou x cube.

In next class, we will see how to obtain finite difference formula by other procedure that is we have seen getting all finite difference formula only from Taylor series expansion, there are also other procedures available, and we are going to see one such procedure in next class.

Thank you.